Contact and equivalence of submanifolds of homogeneous spaces

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1 Introduction

The problem of equivalence of submanifolds of homogeneous spaces of Lie groups was extensively treated by F. Cartan by his method of moving frames [2]. A basic idea of Cartan’s method is that for sufficiently high $k$, $G$-contact of order $k$ (see §4) implies $G$-equivalence. In other words, for each homogeneous space $M$ there exists an integer $k$, depending on the dimension $p$, such that if two submanifolds $S$ and $\overline{S}$ of same dimension $p$ have $G$-contact of order $k$ then, there exists $g \in G$ such that $gS = \overline{S}$. Cartan treated several important geometrical examples and proved in each case the existence of $k$.

Essentially, Cartan’s method of proving the existence of the element $g \in G$ consists in using the uniqueness of solution of a system of first order differential equations as in Frobenius theorem. Cartan’s theory has been the subject of attention of a great number of authors (see for example [4], [5]). However, they all reduce the proof of the existence of the element $g \in G$ to the uniqueness of solution of a first order differential system whereas it seems more natural and geometrical to deal directly with a higher order differential system.

The notion of contact element as defined by Ehresmann [3] allows a geometrical formulation of the theorem of existence and uniqueness of solution of higher order
completely integrable differential systems which is a straight forward generalization of Frobenius theorem (theorem 1). It is the uniqueness of this theorem that we use to solve the problem of $G$-equivalence. As a result, the regularity conditions on the submanifolds $S$ and $\overline{S}$, which are necessary for the theorem of equivalence to hold (theorem 3), can be given a simple and geometrical definition, valid in any homogeneous space $M$. Also, in the method of moving frames, the invariants of a submanifold $S$ of $M$ are defined attaching special higher order frames to the points of $S$, [2], [5]. These frames are constructed by subtle geometrical arguments valid for a fixed homogeneous space whereas we construct the invariants of $S$ as the elements of a complete set of invariants of the orbits of $G$ acting on a manifold of higher order contact elements.

The equivalence problem may be posed for two immersions $f, h : S \to M$ of a differentiable manifold $S$. $f$ and $g$ are equivalent if there exits $g \in G$ such that $h = L_g \circ f$ where $L_g(x) = gx$, $x \in S$. This fixed parametrization theorem has been treated by J.A. Vederesi [7] by means of a higher order differential system defined in a manifold of jets.

The paper ends with a necessary and sufficient condition for a submanifold $S$ of $M$ to be an open set of an orbit of a Lie subgroup $K$ of $G$.

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2 Contact elements

All manifolds and maps considered in this paper are assumed to be differentiable of class $C^\infty$. If $M$ and $N$ are manifolds and $f : M \to N$ is a map, the induced map on tangent spaces at points $a \in M$ and $b = f(a) \in N$ will be denoted by
Given integers \( p, k \geq 0 \), \( p \leq \dim M \), \( J^{k,p}M \) denotes the manifold of all \( k \)-jets of rank \( p \) whose source is the origin of \( \mathbb{R}^p \) and whose target is any point of \( M \). Let \( GL^k \mathbb{R}^p \) be the Lie group of invertible \( k \)-jets whose source and target are at the origin of \( \mathbb{R}^p \). By definition, a contact element of order \( k \) and dimension \( p \) of \( M \) is an equivalence class of \( J^{k,p}M \) under the equivalence relation: for \( X, Y \in J^{k,p}M \), \( X \sim Y \) if there exists \( Z \in GL^k \mathbb{R}^p \) such that \( Y = X \circ Z \). The set of contact elements of order \( k \) and dimension \( p \) of \( M \) is a differentiable manifold denoted by \( C^{k,p}(M) \). \( C^{0,p}(M) \) identifies naturally with \( M \).

For \( 0 \leq k' \leq k \) there is a natural projection \( \pi^k_{k'} : C^{k,p}(M) \rightarrow C^{k',p}(M) \). If \( k' = 0 \), we write \( \pi^k : C^{k,p}M \rightarrow M \) instead of \( \pi^k_0 \). The fiber of \( C^{k,p}M \) over \( a \in M \) is denoted by \( C^a_{k,p}M \). If \( p \) is the dimension of \( M \), \( C^a_{k,p}M \) has only one element which is denoted by \( C^a_{k}M \) and is called the contact element of order \( k \) of \( M \) at the point \( a \in M \).

Given a submanifold \( S \) of \( M \), \( S \subset M \), and an integer \( p \), \( 0 \leq p \leq \dim S \), there is a natural injection of \( C^{k,p}S \) into \( C^{k,p}M \). If \( p \) is the dimension of \( S \), composing the map \( a \in S \rightarrow C^a_{k,p}S \in C^{k,p}M \) with the injection \( C^{k,p}S \rightarrow C^{k,p}M \), we define an injection \( C^k : a \in S \rightarrow C^a_{k}S \in C^{k,p}M \). The image of this injection is denoted by \( C^kS \subset C^{k,p}M \). Two submanifolds \( S \) and \( \overline{S} \) of \( M \) of same dimension \( p \) have contact of order \( k \) at a common point \( a \) if \( C^a_{k}S = C^a_{k} \overline{S} \).

### 3 Completely integrable differential systems of higher order

A differential system of order \( k \geq 1 \) and dimension \( p \) defined over a manifold \( M \) is a submanifold \( \Omega^k \) of \( C^{k,p}M \) such that the projection \( \pi^k : \Omega^k \rightarrow M \) is of rank equal to the dimension of \( M \). An integral manifold of \( \Omega^k \) is a submanifold \( S \) of \( M \) of dimension \( p \) such that \( C^k_xS \in \Omega^k \) for all \( x \in S \). For \( X \in C^{k,p}M \), let \( F_X \) be the fiber
of $X$ by the projection $\pi_{k-1}^k : C^{k,p}M \to C^{k-1,p}M$. The symbol $\sigma(X)$ of $\Omega^k$ at the point $X \in \Omega^k$ is by definition, the vector space

$$\sigma(X) = T_X\Omega^k \cap T_XF_X.$$ 

Let $X^{k+1} \in C^{k+1,p}M$, $X^k = \pi_{k+1}^k(X)$, and let $S$ be a submanifold of $M$ such that $X^{k+1} = C_{\alpha}^{k+1} S$, $\alpha \in S$. Then, $C_{X^k}^1(C^kS)$ depends only on $X^{k+1}$ and not on the choice of $S$. Hence, there is a natural imbedding

$$\Lambda_{k,1} : C^{k+1,p}M \to C^{1,p}(C^{k,p}M)$$

which maps $X^{k+1}$ into $C_{X^k}^1(C^kS)$. By definition, the first prolongation of the differential system $\Omega^k$ is the subset $\Omega^{k,1}$ of $C^{k+1,p}M$ defined by

$$\Omega^{k,1} = (\Lambda_{k,1})^{-1}[C^{1,p}(\Omega^k) \cap \Lambda_{k,1}(C^{k+1,p}M)].$$

Since $\pi_{k+1}^k = \pi_0^1 \circ \Lambda_{k,1}^{-1}$, it follows that $\pi_{k+1}^k$ maps $\Omega^{k,1}$ into $\Omega^k$. If $S$ is an integral manifold of $\Omega^k$ then, $C_{x}^{k+1}S \in \Omega^{k,1}$ for every $x \in S$. Hence, a necessary condition for the existence of an integral manifold of $\Omega^k$ going through every point of $\Omega^k$ is that the projection $\pi_{k+1}^k : \Omega^{k,1} \to \Omega^k$ be surjective.

**Theorem 1:** Let $\Omega^k \subset C^{k,p}M$ be a differential system of order $k \geq 1$ and let $X \in \Omega^k$ be a contact element such that

1) $\sigma(X) = \{0\}$;

2) The image of $\Omega^{k,1}$ by the projection $\pi_{k+1}^k : \Omega^{k,1} \to \Omega^k$ is a neighborhood of $X$ in $\Omega^k$.

Then, there exists an integral manifold $S$ of $\Omega^k$ such that $X \in C^kS$. Moreover, if $S$ and $S'$ are integral manifolds of $\Omega^k$ such that $X \in C^kS \cap C^kS'$, there exists a set $W$ which is an open neighborhood of $X$ in $C^kS$ and $C^kS'$.
Theorem 1 is a geometrical version of the theorem of existence and uniqueness of solutions of completely integrable systems of partial differential equations of order \( k \geq 1 \). Taking suitable coordinates in \( C^{k+1,p}M \) and \( C^{k,p}M \), the existence of integral manifolds of \( \Omega^k \) reduces to the existence of solutions of a completely integrable system of partial differential equations [6].

4 Contact of submanifolds

Let \( G \) be a Lie group acting transitively on the manifold \( M \). Two submanifolds \( S \) and \( \overline{S} \) of \( M \) of same dimension \( p \), have \( G \)-contact of order \( p \) at points \( a \in S \) and \( \overline{a} \in \overline{S} \) if there exists \( g \in G \) such that \( ga = \overline{a} \) and \( gS \) and \( \overline{S} \) have contact of order \( k \) at the point \( \overline{a} \). \( S \) and \( \overline{S} \) have \( G \)-contact of order \( k \geq 0 \) if there exists a diffeomorphism \( \phi : S \to \overline{S} \) such that for all \( x \in S \), \( S \) and \( \overline{S} \) have contact of order \( k \) at points \( x \) and \( \phi(x) = g(x)x \). We say in this case that \( \phi \) makes contact of order \( k \) of \( S \) onto \( \overline{S} \). \( S \) and \( \overline{S} \) are \( G \)-equivalent at points \( a \in S \) and \( \overline{a} \in \overline{S} \) if there are open neighborhoods of \( a \) and \( \overline{a} \) in \( S \) and \( \overline{S} \) which are \( G \)-equivalent.

The action of \( G \) on \( M \) extends to an equivariant action on the manifold \( C^{k,p}M \) of contact elements of order \( k \) and dimension \( p \) of \( M \). For a point \( x \in M \), let \( C^k_xS \), \( G^k_x \) and \( d^k(x) \) denote respectively the contact element of order \( k \) of \( S \) at the point \( x \), the isotropy subgroup of \( G \) at the point \( C^k_xS \) and the dimension of \( G^k_x \). We call \( G^k_x \) the isotropy subgroup of order \( k \) of the point \( x \) of \( S \). Put \( X = C^k_xS \) and let \( h^k(x) \) be the dimension of the vector space \( T_X(GX) \cap T_XC^kS \) where \( C^kS \) is the submanifold of \( C^{k,p}M \) of all contact elements of order \( k \) of \( S \) and \( T_X(GX) \) and \( T_XC^kS \) are the tangent spaces of the orbit \( GX \) and of \( C^kS \) at the point \( X \).

For \( k' \leq k \), \( d^k(x) \leq d^{k'}(x) \) and \( h^k(x) \leq h^{k'}(x) \). Hence, there exists an integer
k \geq 1\) such that \(d^k(x) = d^{k-1}(x)\) and \(h^k(x) = h^{k-1}(x)\). We say that \(a \in S\) is a \(k\)-regular point of \(S\) under the action of \(G\) if there exists \(k \geq 1\) such that

1) \(d^k(a) = d^{k-1}(a)\) and \(h^k(a) = h^{k-1}(a)\);

2) \(d^k(x)\) and \(h^k(x)\) are constant for \(x\) varying in a neighborhood of \(a\) in \(S\).

The order of \(a\) is the least integer satisfying conditions above. If \(a\) is a \(k\)-regular point of \(S\) then \(ga\) is a \(k\)-regular point of \(gS\).

**Theorem 2:** Let \(S, \overline{S}\) be two submanifolds of \(M\) of same dimension \(p\). Let \(a \in S\) and \(\overline{a} \in \overline{S}\) be two points. Assume that \(\overline{a}\) is a \(k\)-regular point of \(\overline{S}\) and that there exists a continuous map \(\varphi : V \to G\), defined in a neighborhood \(V\) of \(a\) in \(S\), such that \(\varphi(a).a = \overline{a}\), \(\varphi(x).x \in S\) and \(\varphi(x).C^k_xS = C^k_{\varphi(x)}\overline{S}\) for all \(x \in V\). Then, there exist open neighborhoods \(W\) and \(\overline{W}\) of \(a\) and \(\overline{a}\) in \(S\) and \(\overline{S}\) which are \(G\)-equivalent.

The proof of theorem 2 is based on the uniqueness statement of theorem 1.

In theorems 3, 4, 5, 6, 8 below we assume that \(G\) is a compact Lie group and \(H\) is a closed subgroup of \(G\). Let \(L\) be the union of all \(G\)-orbits of \(C^{k,p}M\) of type \(H\) that is, orbits whose isotropy subgroups are conjugate to \(H\). Denote by \(L/G\) the quotient space of \(L\) by the orbits and by \(\pi : L \to L/G\) the natural projection. It is known [1] that \((L, L/G, \pi, G/H, G)\) is a differentiable fiber bundle with structural group \(G\) and standard fiber \(G/H\).

Let \(f : S \to \overline{S}\) be a diffeomorphism such that \(S\) and \(\overline{S}\) have \(G\)-contact of order \(k \geq 1\) at corresponding points \(x \in S\) and \(\overline{x} = f(x) \in \overline{S}\) and let \(a \in S\) and \(\overline{a} = f(a) \in \overline{S}\) be two points. Considering suitable cross sections of the fiber bundle \((L, L/G, \pi, G/H, G)\) one can prove the existence of a neighborhood \(U\) of \(a\) in \(S\) and of a differentiable map \(\varphi : V \to G\) such that \(\varphi(x).x = f(x)\) and \(\varphi(x).C^k_xS = C^k_{\varphi(x)}\overline{S}\).

Hence, theorem 2 can be restated as follows:
Theorem 3 Assume that $G$ is compact and that there exists $k \geq 1$ such that

1. $a \in S$ is a $k$-regular point.

2. The isotropy subgroups of $C^k_xS$ are conjugate in $G$ for all $x \in S$.

3. There exists a diffeomorphism $f : S \to \overline{S}$ such that $S$ and $\overline{S}$ have $G$-contact of order $k$ at corresponding points.

Let $a \in S$ be such that $f(a) = \overline{a}$. Then $S$ and $\overline{S}$ are locally $G$-equivalent at points $a$ and $\overline{a}$.

Theorem 4 Assume that $S$ and $\overline{S}$ are connected and that there exists an integer $k \geq 1$ such that:

1. $x \in S$ is a $k$-regular point of $\overline{S}$ and $h^k(x) = 0$ for all $x \in S$.

2. The isotropy subgroups of $C^k_xS$ are conjugate in $G$ for all $x \in S$.

3. There exists a diffeomorphism $f : S \to \overline{S}$ such that $S$ and $\overline{S}$ have $G$-contact of order $k$ at corresponding points.

Then, $f$ is the restriction to $S$ of the translation by an element $g$ of $G : f = L_g|S$.

Consider again the fiber bundle $(L, L/G, \pi, G/H, G)$. There exists a finite number of real valued differentiable functions $\tilde{\rho}_i, 1 \leq i \leq r$, defined in $L$, such that two contact elements $X, \overline{X} \in L$ are in the same fiber of $L$ if and only if $\tilde{\rho}_i(X) = \tilde{\rho}_i(\overline{X})$, $1 \leq i \leq r$. Given a submanifold $S$ of $M$ of dimension $p$, and assuming that the orbits of $C^k_xS$ are of type $H$ for all $x \in S$, one can pull back the functions $\tilde{\rho}_i$ by the map $\sigma^k : x \in S \to C^k_xS \in L$. The set of functions $\rho_i = \tilde{\rho}_i \circ \sigma^k, 1 \leq i \leq r$, is a complete set of $G$-invariants of order $k$ of the submanifold $S$ of $M$. Often the invariants can
be defined in a natural way and have deep geometrical meaning as for instance, the
curvature and torsion of curves and the principal curvatures of surfaces in $\mathbb{R}^3$.

Assuming that the isotropy subgroups of $C^k_xS$ and $C^k_\pi\overline{S}$ are of type $H$ for all
$x \in S$ and $\pi \in \overline{S}$, complete sets of invariants of order $k$, $\rho_i$ and $\overline{\rho}_i$ can be defined
in $S$ and $\overline{S}$. Condition $h^k(\pi) = 0$ in theorem 4 is then clearly equivalent to stating
that the rank of differentials $d\rho_i$, $1 \leq i \leq r$, is $p$ at every point $\pi \in \overline{S}$ One can then
restate theorems 3 and 4 in the following way.

**Theorem 5 :** Let $\pi \in \overline{S}$ be a $k$-regular point of $\overline{S}$, $k \geq 1$. Assume following
conditions are satisfied:

1. The isotropy subgroups of $C^k_xS$ and $C^k_\pi\overline{S}$ are conjugate for all $x \in S$ and $\pi \in S$.

2. There exists a diffeomorphism $f : S \rightarrow \overline{S}$ such that
$$\overline{\rho}_i = \rho_i \circ f, \ 1 \leq i \leq r.$$ 

Then, $S$ and $\overline{S}$ are locally $G$-equivalents at points $a = f^{-1}(\pi)$ and $\overline{\pi}$.

**Theorem 6 :** Let $S$, $\overline{S}$ be two submanifolds of $M$ and let $k \geq 1$ be such that

1. Every point $\pi \in \overline{S}$ is $k$-regular.

2. The isotropy subgroups of $C^k_xS$ and $C^k_\pi\overline{S}$ are conjugate for all $x \in S$ and $\pi \in S$.

3. There exists a diffeomorphism $f : S \rightarrow \overline{S}$ such that
$$\rho_i = \overline{\rho}_i \circ f, \ 1 \leq i \leq r.$$ 

4. The rank of differentials $d\overline{\rho}_i$, $1 \leq i \leq r$, is $p$ at every point $\pi \in \overline{S}$.
Then, \( f \) is the restriction to \( S \) of the left translation by an element of \( G \): \( f = L_g|S \).

Let us assume that \( S \) is an open set of an orbit of a Lie subgroup \( K \) of \( G \). Then, \( h^k(x) = p \) and the isotropy subgroups of \( C^k_x \) are conjugate for all \( x \in S \) and \( k \geq 0 \). Hence there exits \( k \geq 1 \) such that every \( x \in S \) is a \( k \)-regular point of \( S \). Conversely,

**Theorem 7:** A necessary and sufficient condition for a connected submanifold \( S \) of \( M \) to be an open set of an orbit of a Lie subgroup \( K \) of \( G \) is the existence of \( k \geq 1 \) such that for all \( x \in S \), \( x \) is a \( k \)-regular point of \( S \) and \( h^k(x) = p \).

Assuming \( G \) compact and the isotropy subgroups of order \( k \) of points of \( S \) conjugate, a complete set of invariants of order \( k \) can be defined on \( S \). Clearly, \( h^k(x) = p \) for every \( x \in S \) if and only if the invariants are constant on \( S \). Therefore, the following corollary to theorem 7 holds.

**Theorem 8:** Assume \( G \) compact and \( S \) connected. Assume also that for some integer \( k \geq 1 \), every point of \( S \) is \( k \)-regular and all isotropy subgroups of order \( k \) of points of \( S \) are conjugate. Then, a necessary and sufficient condition for \( S \) to be an open set of an orbit of a Lie subgroup of \( G \), is that the invariants of order \( k \) of \( S \) be constant.

**Bibliography**


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