VI - EBSD

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Simplectic Forms.

Let V be a vector space of finite dimension over $\mathbb{R}$ and let $\omega: V \times V \to \mathbb{R}$ be a bilinear map.

The map ω is skew-symmetric if

$$\omega(u,v) = -\omega(v,u) \quad ; \forall\, u,v \in V.$$

$\boxed{\text{Thm } ①}$ (Standard Form for Skew-symmetric Bilinear maps).

Let ω be a skew-symmetric bilinear map on V.

Then there is a basis

$\{u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that:

$$\omega(u_i, v) = 0 \qquad ; \forall\, i \in \{1,2,\dots,k\} \text{ and } v \in V$$

$$[1] \dots\dots\dots \omega(e_i, e_j) = 0 = \omega(f_i, f_j) \qquad ; \forall i,j \in \{1,2,\dots,n\}$$

$$[2] \dots\dots\dots \omega(e_i, f_j) = \delta_{ij} \qquad ; \forall\, i,j \in \{1,2,\dots,n\}$$

In matrix notation:

$$\omega(u,v) = [-u-] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \\ 0 & -\mathrm{Id} & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Definition ①. The map $\tilde{\omega}: V \to V^*$ is the linear map defined by:

$$\tilde{\omega}(u)(v) = \omega(u,v).$$

Consider the subspace:

$$\mathcal{U} = \{ u \in V / \omega(u,v) = 0 \; , \forall v \in V \}$$

Note that: $\text{Ker}(\tilde{\omega}) = \{ u \in V / \tilde{\omega}(u) = 0 \}$

$$= \{ u \in V / \tilde{\omega}(u)(v) = 0 \; ; \forall v \in V \}$$

$$= \{ u \in V / \omega(u,v) = 0 \; , \forall v \in V \}$$

$$= \mathcal{U}$$

Definition ② A skew symmetric bilinear map ω is symplectic (or nondegenerate) if

$\tilde{\omega}$ is bijective (i.e. $\mathcal{U} = \{0\}$)

The map ω is called a linear symplectic structure on V and (V,ω) is called a Symplectic Vector Space.

Observation. By Thm ①, if (V,ω) is a symplectic vector space then $k = \dim \mathcal{U} = 0$, so $\dim V = 2n$

Also (V,ω) has a basis:

$\{e_1, e_2, \ldots, e_n, f_1, \ldots, f_n\}$ satisfying [1] and [2]

such basis is called a symplectic basis of (V,ω)

With respect to symplectic basis, we have:

$$\omega(u,v) = [-u-] \begin{bmatrix} 0 & Id \\ -Id & 0 \end{bmatrix} \begin{bmatrix} | \\ v \\ | \end{bmatrix}$$

Example ① The prototype of a symplectic vector space is $(\mathbb{R}^{2n}, \omega_0)$ with ω_0 such that the basis

$$e_1 = (1,0,\ldots,0) \; , \ldots , e_n = (0,0,\ldots,0,\underset{\substack{\uparrow \\ n\text{-position}}}{1},0,\ldots,0) \; ,$$

$$f_1 = (0,0,\ldots,0,\underset{\substack{\downarrow \\ (n+1)\text{-position}}}{1},0,\ldots) \; , \ldots , f_n = (0,0,\ldots,1)$$

is a symplectic basis, this mean:

if $u=(a_1,\dots,a_n,b_1,\dots,b_n)$, $v=(c_1,\dots,c_n,d_1,\dots,d_n)\in\mathbb{R}^{2n}$

then: $$\omega_0(u,v)=\sum_{i=1}^{n} a_i d_i - \sum_{i=1}^{n} c_i b_i$$

Example ②. Let $(E,\langle,\rangle)$ be a vector space with $\langle,\rangle$ inner product. Consider the space $E\times E$

and the bilinear map:

$$\begin{aligned}\omega:(E\times E)\times(E\times E)&\longrightarrow\mathbb{R}\\ ((x_1,y_1),(x_2,y_2))&\longmapsto\langle y_1,x_2\rangle-\langle x_1,y_2\rangle\end{aligned}$$

Then $(E\times E,\omega)$ is a symplectic vector space.

Special Subspaces.

Let (V,ω) a symplectic vector space ($\dim V=2n$)

Given a linear subspace $Y\subset V$, its symplectic orthogonal Y^{ω} is the linear subspace defined by

$$Y^{\omega}:=\{v\in V \,/\, \omega(v,u)=0\,,\ \forall u\in Y\}$$

By considering the kernel of image of the map:

$$\begin{aligned}V&\longrightarrow Y^{*}\\ v&\longmapsto \omega(v,\cdot)\big|_{Y}\end{aligned}$$

we obtain: $$\dim Y+\dim Y^{\omega}=\dim V$$

By nondegeneracy of ω, we have that $(Y^{\omega})^{\omega}=Y$

ⓐ A subspace $Y\subset V$ is called symplectic if $\omega|_{Y\times Y}$ is non degenerate, this mean: $Y\cap Y^{\omega}=\{0\}$ or

$$V=Y\oplus Y^{\omega}$$

ⓑ a subspace $Y \subset V$ is called isotropic if $\omega|_{Y\times Y} = 0$

If Y is isotropic then: $\dim Y \leq \frac{1}{2} \dim V$

ⓒ a subspace $Y \subset V$ is called coisotropic if its symplectic orthogonal is isotropic. If Y is coisotropic then: $\dim Y \geq \frac{1}{2} \dim V$

Example ③ If $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is a symplectic basis of (V, ω), then:

(i) $\text{span}\{e_1, f_1\}$ is symplectic.

(ii) For $i \neq j$: $\text{span}\{e_i, e_j\}$ is isotropic.

(iii) $\text{Span}\{e_1, \dots, e_n, f_1, f_2\}$ is coisotropic.

ⓓ An isotropic subspace $Y \subset V$ is called Lagrangian when $\dim Y = \frac{1}{2} \dim V$

Note that:

Y is Lagrangian $\Leftrightarrow$ Y is isotropic and coisotropic

$\Leftrightarrow$ $Y = Y^{\omega}$

Hamiltonian Vector fields.

Let (V, ω) be a symplectic vector space and let $H : V \to \mathbb{R}$ be a smooth function. Its differential dH is a 1-form. By no degeneracy of ω there is a unique vector field X_H on V s.t.:

$$\omega(\cdot, X_H) = i_{X_H} \omega = dH$$

Definition ③ A vector field X_H as above is called the Hamiltonian Vector field with Hamiltonian function H.

Consider the euclidean space $(\mathbb{R}^{2n}, \omega_0)$ with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$. The curve $f_t = (x(t), y(t))$ is a integral curve for X_H if:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \dfrac{\partial H}{\partial y_i} \\[2ex] \dfrac{dy_i(t)}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases} \quad \longleftarrow \text{Hamilton Equations}$$

Jacobi Equation

Let $(E, \langle , \rangle)$ be a vector space with inner product $\langle , \rangle$

Consider $\quad End(E) = \{ A : E \to E \,/\, A \text{ is linear} \}$

$Sym(E) = \{ A \in End(E) \,/\, A \text{ is symetric} \}$

If $R : \mathbb{R} \to Sym(E)$ is a smooth C^∞-map (curvature operator), the linear differential equation:

$$\ddot{J}(t) + R(t) J(t) = 0$$

is called the Jacobi Eq. and its solutions $J : \mathbb{R} \to E$ are called Jacobi Fields. If $B : \mathbb{R} \to End(E)$ is a solution of the matrix eq.:

$$\ddot{B}(t) + R(t) B(t) = 0 \quad \longleftarrow \text{Jacobi Eq. Matrix}$$

we call B a Jacobi Tensor.

Note that; B is a Jacobi Tensor iff $B(t)x$ is a Jacobi Field for all $x \in E$.

Define for each $t \in \mathbb{R}$ the Hamiltonian:

$$\begin{aligned} H_t : E \times E &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto \frac{1}{2}\left(\langle y, y \rangle + \langle R(t)x, x \rangle \right) \end{aligned}$$

and we consider in $E\times E$ the linear symplectic structure:

$$\omega((x_1,y_1),(x_2,y_2)) = \langle y_1, x_2\rangle - \langle x_1, y_2\rangle$$

Then the Jacobi Eq. is equivalent to the Hamiltonian eq: $\underline{\underline{t \longmapsto (J_1(t), J_2(t))}}$

$$(\star)\dots\begin{cases} \dot{J}_1(t) = \dfrac{\partial H_t}{\partial y}(J_1(t), J_2(t)) = J_2(t) \\ \dot{J}_2(t) = -\dfrac{\partial H_t}{\partial x}(J_1(t), J_2(t)) = -R(t)J_1(t) \end{cases}$$

So, the Hamiltonian vector field $X_{H_t} = \left(\dfrac{\partial H_t}{\partial x}, -\dfrac{\partial H_t}{\partial y}\right)$

is also determined by the eq:

$$\omega(v, X_{H_t}(x,y)) = \left(i_{X_{H_t}(x,y)}\,\omega\right)(v) = dH_t(x,y)(v) \quad ; \forall v \in E\times E$$

The solutions of $(\star)$ generates for each $a\in\mathbb{R}$ a family:

$$\begin{aligned} \psi_{t,a} : E\times E &\longrightarrow E\times E \\ (x,y) &\longmapsto (J_1(t), J_2(t)) = (J(t), \dot{J}(t)) \end{aligned}$$

where $(J_1(t), J_2(t))$ is the sol. of $(\star)$ with initial conditions

$$(J_1(a), J_2(a)) = (x,y)$$

Note that: $\psi_{a,a} = Id_{E\times E}$.

Definition ④ If $A, B : [a,b] \to End(E)$ are C^1-curves on the interval $[a,b]$, the map:

$$\begin{aligned} : [a,b] &\longrightarrow End(E) \\ t &\longmapsto \Omega(A(t), B(t)) = B(t)^T \dot{A}(t) - \dot{B}(t)^T A(t) \end{aligned}$$

is called the Wronskian of A and B.

Note that, if $x, y \in E$ then:

$$\langle \Omega(A(t),B(t))x, y\rangle = \langle \left(B(t)^T \dot{A}(t) - \dot{B}(t)^T A(t)\right) x, y\rangle$$

$$= \langle B(t)^T \dot{A}(t) x, y\rangle - \langle \dot{B}(t)^T A(t) x, y\rangle$$

$$= \langle \dot{A}(t) x, B(t) y\rangle - \langle A(t) x, \dot{B}(t) y\rangle$$

$$= \omega\left((A(t)x, \dot{A}(t)x), (B(t)y, \dot{B}(t)y)\right) \quad \ldots\ldots (\star\star)$$

Lemma ⑤. Let $X, Y : \mathbb{R} \to End(E)$ be Jacobi Tensors.

Then: $\Omega(X(t), Y(t)) = cte \quad , \forall t \in \mathbb{R}$

Proof:

$$\frac{d}{dt}\left(\Omega(X(t),Y(t))\right) = \frac{d}{dt}\left(Y(t)^T \dot{X}(t)\right) - \frac{d}{dt}\left(\dot{Y}(t)^T X(t)\right)$$

$$= \left[\dot{Y}(t)^T \dot{X}(t) + Y(t)^T (\ddot{X}(t))\right]$$

$$- \left[(\ddot{Y}(t)^T) X(t) + \dot{Y}(t)^T \dot{X}(t)\right]$$

$$= Y^T(-R(t)X(t)) + [R(t)Y(t)]^T X(t)$$

$$= 0$$

Lemma ⑥. Let $A, B \in End(E)$. and

$$\mathcal{L} = \{(Ax, Bx) / x \in E\} \subset E \times E$$

Then $\mathcal{L}$ is isotropic iff $A^T B = B^T A$. If additionally the function:

$$\alpha : E \longrightarrow E \times E$$
$$x \longmapsto (Ax, Bx)$$

is injective then $\mathcal{L}$ is Lagrangian

Remark: (i) By Lemma ⑥ above, the graph:

$$\mathcal{L} = \{(x, Bx) / x \in E\} \subset E \times E$$

is Lagrangian iff B is symmetric

(ii) $\mathcal{L} = \{(Ax, Bx)/x \in E\}$ is a graph iff

A is nonsingular.

Proof (Lemma ⑥). By (★★) we have:

$$\omega((Ax, Bx), (Ay, By)) = \langle (A^T B - B^T A)x, y \rangle$$

$$\Rightarrow \quad \omega((Ax, Bx), (Ay, By)) = 0 \Leftrightarrow \quad A^T B = B^T A$$

If additionally the function $\mathcal{L}(x)$ is injetive then

$$\dim(\mathcal{L}) = \dim(\operatorname{Im}(\mathcal{L})) = \dim E \qquad ▨$$

Definition ⑦. Let $R: \mathbb{R} \to Sym(E)$ be a smooth map.

A family of Lagrangians $\mathcal{L}_t \subset E \times E$, $t \in [a,b]$

with $\mathcal{L}_t := \Psi_{t,a}(\mathcal{L}_a)$, is called a

Lagrangian Deformation of $\mathcal{L}_a$ induced by R.

Remark (A) If $\mathcal{L}_a = \{(Ax, Bx)/x \in E\}$ then

$$\mathcal{L}_t = \Psi_{t,a}(\mathcal{L}_a) = \{(A(t)x, \dot{A}(t)x)/x \in E\}$$

where $A(a) = A$ and $\dot{A}(a) = B$

So, the family $(\mathcal{L}_t)_{t \in [a,b]}$ are isotropic iff

$$0 = \Omega(A(a), A(a)) = A(a)^T \dot{A}(a) - \dot{A}(a)^T A(a)$$

and $(\mathcal{L}_t)_{t \in [a,b]}$ is isotropic and $\dim \mathcal{L}_a = \dim E$

then $(\mathcal{L}_t)_{t \in [a,b]}$ is Lagrangian.

If additionally $A(t)$ is invertible for all $t \in [a,b]$ then

$\mathcal{L}_t$ is a Lagrangian graph.

Definition ⑧ Let $R: \mathbb{R} \to Sym(E)$ be a C^∞-map.

We say that the Jacobi Equation:

$$\ddot{J}(t) + R(t)J(t) = 0$$

has no conjugate points on $[a,b]$ if for all Jacobi fields $J(t)$ with $J(a)=0$ and $\dot{J}(a)\neq 0$, we have that

$$J(t)\neq 0 \ ; \forall t\in (a,b]$$

Remark (ΔΔ) The Jacobi Eq. has no conjugate points on $[a,b]$ iff the Jacobi Tensor $A(t)$ with $A(a)=0$, $\dot{A}(a)=Id$ is non singular for all $t\in (a,b]$. In particular, $\mathcal{L}_t=\{(A(t)x, \dot{A}(t)x)/x\in E\}$ is a Lagrangian graph for $t\in(a,b]$.

Lemma ⑨. Let $R:\mathbb{R}\to Sym(E)$ be a curvature operator and for $a\in\mathbb{R}$ let $\mathcal{L}_t=\Psi_{t,a}(\mathcal{L}_a)$ be a Lagrangian graph for all $t\in[a,b]$. Then the Jacobi equation associated to R has no conjugate points on $[a,b]$

Proof: Since $\mathcal{L}_t$ is a Lagrangian graph on $[a,b]$, it can be written as:

$$\mathcal{L}_t=\{(B(t)x, \dot{B}(t)x)/x\in E\}$$

where $B:[a,b]\to End(E)$ is a non singular Jacobi tensor on $[a,b]$ and $\Omega(B(a), B(a))=0$ (isto pela Remarca (Δ))

Let A be a Jacobi tensor with $A(a)=0$ and $\dot{A}(a)=Id$

The Wronskian of A and B is given by:

$$\Omega(A(t),B(t)) = B(t)^T\dot{A}(t)-\dot{B}(t)^T A(t)$$

as A and B are Jacobi tensors then by Lemma ⑤

$$cte = \Omega(A(t), B(t)) \quad \ldots\ldots\ldots\ldots\ldots \quad (✡)$$

as $$\Omega(A(a), B(a)) = B(a)^T \cancelto{Id}{\dot{A}(a)} - \dot{B}(a)^T \cancelto{0}{A(a)}$$

$$= B(a)^T$$

then on (✡): $$\Omega(A(t), B(t)) = B(a)^T$$

As $\mathcal{L}_t$ is isotropic then by Lemma ⑥:

$$B(t)^T \dot{B}(t) = \dot{B}(t)^T B(t)$$

as $B(t)$ is invertible then $B(t)^T$ is also invertible, thus the last equality we obtain:

$$\dot{B}(t)\, B(t)^{-1} = \left[\dot{B}(t)\, B(t)^{-1}\right]^T$$

Hence $\dot{B}(t)\, B(t)^{-1}$ is symmetric.

So, $$\frac{d}{dt}\left(B(t)^{-1} A(t)\right) = -B(t)^{-1} \dot{B}(t)\, B(t)^{-1} A(t) + B(t)^{-1} \dot{A}(t)$$

$$= -B(t)^{-1} \left[B(t)^T\right]^{-1} \dot{B}(t)^T A(t) + B(t)^{-1} \left[B(t)^T\right]^{-1} B(t)^T \dot{A}(t)$$

$$= B(t)^{-1} \left[B(t)^T\right]^{-1} \left(\underbrace{B(t)^T \dot{A}(t) - \dot{B}(t)^T A(t)}_{}\right)$$

$$= B(t)^{-1} \left[B(t)^T\right]^{-1} B^T(a)$$

Integrating:

$$A(t) = B(t) \int_a^t \left[B(s)^T B(s)\right]^{-1} B(a)^T\, ds$$

▨