VI - EBSD

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Index Lemma.

Definition.① Let $R:\mathbb{R}\to \mathrm{Sym}(E)$ be a curvature operator and:

$$V:=\{X:[a,b]\to E \,/\, X \text{ is piecewise differentiable}\}$$

be the vector space of continuous and piecewise differentiable curves in E.

The symmetric bilinear form: (Index form)

$$I:=I_{[a,b]}:V\times V\to\mathbb{R}$$

inner product. $(E,\langle,\rangle)$

$$\rightarrow I_{[a,b]}(X,Y)=\int_a^b\left(\langle\dot X(t),\dot Y(t)\rangle-\langle R(t)X(t),Y(t)\rangle\right)dt$$

as $X,Y\in V$ then there exist a subdivision $a=t_0<t_1<\dots<t_k=b$. Then using integration by parts:

$$[1]\ldots\ I_{[a,b]}(X,Y)=\underbrace{\sum_{i=1}^{k}\langle\dot X,Y\rangle\Big|_{t_{i-1}}^{t_i}}-\int_a^b\langle\overbrace{\ddot X(t)+R(t)X(t)}^{\text{Eq: Jacobi}},Y(t)\rangle dt$$

Observation.② On [1] If $X,Y\in V$ with X a Jacobi field and Y with $\underline{Y(a)=0=Y(b)}$

then

$$\rightarrow I(X,Y)=\sum_{i=1}^{k}\langle\dot X,Y\rangle\Big|_{t_{i-1}}^{t_i}$$

$$\rightarrow\ =\langle\dot X(t),Y(t)\rangle\Big|_{t_0}^{t_1}+\langle\dot X(t),Y(t)\rangle\Big|_{t_1}^{t_2}+\dots+\langle\dot X(t),Y(t)\rangle\Big|_{t_{k-1}}^{t_k}$$

$$=\langle\dot X(t_k),\overset{Y(b)}{\cancel{Y(t_k)}}\rangle-\langle\dot X(t_0),\overset{Y(a)}{\cancel{Y(t_0)}}\rangle$$

$$=0$$

Conversely, if $X\in V$ s.t. $I(X,Y)=0$ for all $Y\in V$ with $Y(a)=0=Y(b)$

then, let $\underline{f:[a,b]\to\mathbb{R}}$, be a smooth function s.t.

$\underbrace{f(t_i)}=0$, $\forall i=0,1,\dots,k$ and positive elsewhere.

Taking: $Y(t)=\underline{f(t)}\left(\ddot{X}(t)+R(t)X(t)\right)\in V$

and note that: $\forall i=0,1,\dots,k$: $\underline{Y(t_i)=0}$

$$\Rightarrow \underline{\overset{\text{Hyp}}{0}=I(X,Y)}=-\int_a^b\langle \ddot{X}(t)+R(t)X(t), f(t)(\ddot{X}(t)+R(t)X(t))\rangle dt$$

$$=-\int_a^b f(t)\,\|\underline{\ddot{X}(t)+R(t)X(t)}\|^2\,dt$$

$$=-\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\underline{f(t)}\,\underline{\|\ddot{X}(t)+R(t)X(t)\|^2}\,dt$$

$$\Rightarrow \forall i=1,2,\dots,k:\ \boxed{\ddot{X}(t)+R(t)X(t)=0}\ ;\ \underline{\forall t\in[t_{i-1},t_i]}$$

this implies that: $\underline{X_i=X\big|_{[t_{i-1},t_i]}}$, $\forall i=1,2,\dots,k$.

are Jacobi fields.

We will show that: $\boxed{\dot{X}_j(t_j)=\dot{X}_{j+1}(t_j)}$, $\forall j=1,2,\dots,k-1$

Indeed, choose a smooth vector field W with:

$\boxed{W(a)=0=W(b)}$ and : $W(t_j)=\dot{X}_j(t_j)-\dot{X}_{j+1}(t_j)$

for all $j=1,2,\dots,k-1$

$$\overset{\text{Hyp.}}{\Rightarrow}\ 0=I(X,W)=\sum_{j=1}^{k}\langle\dot{X},W\rangle\Big|_{t_{j-1}}^{t_j}$$

$$=\langle\dot{X}_1(t),W(t)\rangle\Big|_{t_0}^{t_1}+\langle\dot{X}_2(t),W(t)\rangle\Big|_{t_1}^{t_2}$$

$$+\dots+\langle\dot{X}_k(t),W(t)\rangle\Big|_{t_{k-1}}^{t_k}$$

$$=\sum_{j=1}^{k-1}\|W(t_j)\|^2$$

$$\Rightarrow \forall j=1,2,\dots,k-1:\ \boxed{0=W(t_j)}.$$

Hence we show that:

Proposition ③ $X\in V$ is a Jacobi field

$\Updownarrow$

$$\to \quad I(X,Y)=0 \; ; \; \forall\, Y\in V, \text{ with } Y(a)=0=Y(b)$$

Consider the following subspace:

$$\to \quad V^{\circ} := \{ X\in V \,/\, X(a)=0=X(b) \}$$

Lemma ④ (Index Lemma) Assume that the Jacobi equation has no conjugate points on $[a,b]$. Then for all $X\in V^{\circ}$, $X\neq 0$ we have:

$$I_{[a,b]}(X,X) > 0.$$

(i.e.: $I_{[a,b]}$ is positive define on the subspace V°)

Proof. Suppose that $X\in V^{\circ}$ is smooth.

Consider the Jacobi tensor $A(t)$ with initial conditions $A(a)=0$, $\dot{A}(a)=\mathrm{Id}$.

$$\left(\begin{array}{l}\text{Jacobi Eq. has no}\\ \text{conjugate points}\end{array}\right) \underset{\text{(non-singular)}}{\Longrightarrow} \left(\begin{array}{l}\exists\, Y\in V \text{ s.t.}\\ X(t)=A(t)\,Y(t)\end{array}\right)$$

As $\quad 0=X(b)=\overbrace{A(b)}\,Y(b) \;\Rightarrow\; \boxed{Y(b)=0}$

Also: $\quad \ddot{X} = \ddot{A}Y + \dot{A}\dot{Y} + \dot{A}\dot{Y} + A\ddot{Y}$

$$= -RAY + 2\dot{A}\dot{Y} + A\ddot{Y} \quad \ldots\ldots\ldots [2]$$

On the other hand, as $A(t)$ is a Jacobi tensor then

$$\to \quad 0 = A^{T}\dot{A} - \dot{A}^{T}A \quad \text{(Wronskiano etc)}$$

$$\Rightarrow \quad \dot{\overline{A^{T}AY}} = 2\dot{A}^{T}AY + A^{T}\dot{Y} \quad \ldots\ldots\ldots [3]$$

thus: $\quad \langle \dot{Y}, A^{T}AY\rangle' = \langle \ddot{Y}, A^{T}AY\rangle + \langle \dot{Y}, \dot{\overline{A^{T}AY}}\rangle$

$$\Rightarrow \quad \cancelto{0}{\langle \dot{Y}, A^{T}AY\rangle\big|_a^b} = \int_a^b \langle \ddot{Y}, A^{T}AY\rangle dt + \int_a^b \langle \dot{Y}, \dot{\overline{A^{T}AY}}\rangle dt$$

$$\Rightarrow \quad \int_a^b \langle \ddot{Y}, A^{T}AY\rangle dt = -\int_a^b \langle \dot{Y}, \dot{\overline{A^{T}AY}}\rangle dt$$

$$\Rightarrow \quad \int_a^b \langle \ddot{Y}, A^{T}AY\rangle dt \overset{[3]}{=} -\int_a^b \langle \dot{Y}, 2\dot{A}^{T}AY + A^{T}\dot{Y}\rangle dt \quad \ldots\ldots [4]$$

Hence:

$$\to \quad I(X,X) = \langle \dot{X}, X\rangle\big|_a^b - \int_a^b \langle \ddot{X} + R(t)X, X\rangle dt$$

$$\overset{[2]}{=} -\int_a^b \langle 2\dot{A}\dot{Y} + A\ddot{Y}, AY\rangle dt$$

$$\to \quad = -2\int_a^b \langle \dot{Y}, \dot{A}^{T}AY\rangle dt - \int_a^b \langle \ddot{Y}, A^{T}AY\rangle dt$$

$$\overset{\{1\}}{=} \int_a^b \langle \dot{Y}, A^T A \dot{Y} \rangle \, dt$$

$$\geq 0$$

For general case, we can obtain the same inequality above by integrating by parts.

Now,

If $\underline{I(X,X)=0} \Rightarrow \langle A\dot{Y}, A\dot{Y} \rangle = 0 \quad ; \forall t \in [a,b]$

$$\Rightarrow A\dot{Y} = 0 \quad ; \forall t \in [a,b]$$

$$\Rightarrow \dot{Y}(t) = 0 \quad ; \forall t \in [a,b]$$

$Y(t) = cte$ and as $Y(b)=0$

$$\Rightarrow X(t) = 0 \quad ; \forall t \in [a,b].$$

▨

Corollary ⑤ (Minimizing property of Jacobi fields).

Assume that the Jacobi Eq:

$$\ddot{J}(t) + R(t)J(t) = 0$$

has no conjugate points on $[a,b]$. Let J be a Jacobi field and $X \in V$ be a piecewise differentiable field with $X(a) = J(a)$, $X(b) = J(b)$. Then:

$$\rightarrow \quad I(J,J) \leq I(X,X)$$

above the inequality is strict unless $\boxed{J = X.}$

Proof:

$$0 \leq I(J-X, J-X)$$

$$= I(J,J) - 2I(J,X) + I(X,X)$$

$$= -I(J,J) + I(X,X)$$

▨

Rauch estimates.

For $k \in \mathbb{R}$, denote by $S_k(t) \in \mathcal{C}^\infty(\mathbb{R})$ the solution of the Diff. Eq:

$$\begin{cases} \ddot{S}(t) + k S(t) = 0 \\ S(0) = 0, \quad \dot{S}(0) = 1 \end{cases}$$

thus:

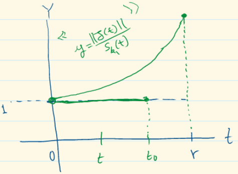
$$S_k(t) = \begin{cases} \frac{1}{\sqrt{-k}} \operatorname{senh}(\sqrt{-k}\, t) & ; k < 0 \\ t & ; k = 0 \\ \frac{1}{\sqrt{k}} \operatorname{sen}(\sqrt{k}\, t) & ; k > 0 \end{cases}$$

Notation: For $A, B \in \mathrm{Sym}(E)$ we denote.

$$\boxed{A \leq B} \Longleftrightarrow \boxed{\langle Ax, x\rangle \leq \langle Bx, x\rangle} \; ; \forall x \in E$$

Proposition ⑥ (Rauch) Let $R : \mathbb{R} \to \mathrm{Sym}(E)$ be a $\mathcal{C}^\infty$-map. and J a Jacobi field with $J(0) = 0$ and $\|\dot{J}(0)\| = 1$.

$\langle R(t)x, x\rangle \leq k_1 \langle x, x\rangle$

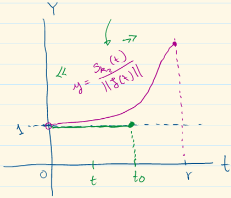


ⓐ Assume that $R(t) \leq k_1 \cdot Id$ and $\boxed{r \leq \frac{\pi}{\sqrt{k_1}}}$ if $\boxed{k_1 > 0}$.

Then $\left(\frac{\|J(t)\|}{S_{k_1}(t)}\right)$ is monotone increasing on $(0, r]$ and bounded below by 1

If for some $t_0 > 0$, we have $\|J(t_0)\| = S_{k_1}(t_0)$ then

$\boxed{J(t) = S_{k_1}(t) \cdot \dot{J}(0)}$ for all $t \in [0, t_0]$



ⓑ Assume that $\boxed{k_2 \cdot Id \leq R(t)}$ and that the Jacobi Eq. associated to R has no conjugate points on $[0, r]$

Then $\left(\frac{\|J(t)\|}{S_{k_2}(t)}\right)$ is monotone decreasing on $(0, r]$ and bounded above by 1. If for some $\boxed{t_0 > 0}$, we have $\boxed{\|J(t_0)\| = S_{k_2}(t_0)}$ then $\boxed{J(t) = S_{k_2}(t) \cdot \dot{J}(0)}$ for all $t \in [0, t_0]$.

Proof. ⓐ It is enough to show :

$$\frac{d}{dt}\left(\frac{\|J(t)\|^2}{S_{k_1}^2(t)}\right)\Bigg|_{t=t_0} \geq 0 \quad ; \forall t_0 \in (0, r)$$

differentiating $\Updownarrow$

$$\langle \dot{J}(t_0), J(t_0)\rangle S_{k_1}^2(t_0) \geq \|J(t_0)\|^2 S_{k_1}(t_0) \dot{S}_{k_1}(t_0)$$

For $t_0 \in (0, r)$ fixed and $\boxed{\tilde{R} = k_1 \cdot Id.}$ Consider:

$$X(t) = S_{k_1}(t) J(t_0) \quad \text{and} \quad Y(t) = S_{k_1}(t_0) J(t)$$

Note that:

$$\ddot{X} + \underset{\tilde{R}}{\underset{\shortparallel}{k_1}} X = 0$$

$$\ddot{X}(t) = \ddot{S}_{k_1}(t) J(t_0) = -k_1 S_{k_1}(t) J(t_0) = -k_1 X(t) \; ; \text{and}$$

$$\ddot{Y}(t) = S_{k_1}(t_0) \ddot{J}(t) = S_{k_1}(t_0)(-R(t) J(t)) = -R(t) Y(t) \rightarrow \ddot{Y} + RY = 0$$

with $\boxed{X(0) = 0 = Y(0)}$ and $\boxed{X(t_0) = S_{k_1}(t_0) J(t_0) = Y(t_0)}$

By hypothesis the Jacobi Eq. with respect to $\tilde{R}$ has no conjugate points on $[0,r]$ and $X(t)$ is a Jacobi field with respect to $\tilde{R}$, then by Corollary ⑤

$$\underbrace{\dot{s}_{k_1}(t_0)s_{k_1}(t_0)\|Y(t_0)\|^2} = \underbrace{I^{\tilde{R}}_{[0,t_0]}(X,X)} \leq \underbrace{I^{\tilde{R}}_{[0,t_0]}(Y,Y)}$$

$$= \underbrace{\langle \dot{Y}(t_0), Y(t_0)\rangle} - \int_0^{t_0} \langle \ddot{Y}+\tilde{R}(t)Y, Y\rangle dt$$

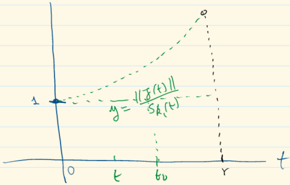
$R(t) \geq k_1 Id$

$$\leq s_{k_1}^2(t_0)\langle \dot{J}(t_0), J(t_0)\rangle - \int_0^{t_0} \langle \cancelto{0}{\ddot{Y}+R(t)Y}, Y\rangle dt$$

$$= \underbrace{s_{k_1}^2(t_0)\langle \dot{J}(t_0), J(t_0)\rangle}$$

On the other hand, by L'Hospital's Rule :

$$\lim_{t\to 0^+}\left(\frac{\|J(t)\|}{s_{k_1}(t)}\right) = 1$$



Now, if $\boxed{\|J(t_0)\| = s_{k_1}(t_0)}$ for some $t_0 \in (0,r)$

$\Rightarrow \boxed{\|J(t)\| = s_{k_1}(t)}$; $\forall\, t \in \underline{[0,t_0]}$

differentiating

$\Rightarrow \|J(t)\|^2 = s_{k_1}^2(t)$

$\Rightarrow \langle \dot{J}(t_0), J(t_0)\rangle = \dot{s}_{k_1}(t_0)\, s_{k_1}(t_0)$

$\Rightarrow \underbrace{\dot{s}_{k_1}(t_0)\, s_{k_1}(t_0)\, \|J(t_0)\|^2} = \underbrace{s_{k_1}^2(t_0)\langle \dot{J}(t_0), J(t_0)\rangle}$

$\Rightarrow \boxed{I^{\tilde{R}}_{[0,t_0]}(X,X) = I^{\tilde{R}}_{[0,t_0]}(Y,Y)}$

Corollary ⑤

$\Rightarrow \boxed{X(t) = Y(t)}$, $\forall\, t \in [0,t_0]$

$\Rightarrow \boxed{\ddot{Y}(t) = \ddot{X}(t) = -k_1 X(t) = -k_1 Y(t)}$

and: $\boxed{J(t) = s_{k_1}(t)\cdot \dot{J}(0)}$ ▨

Corollary. ⑦ Let $R: \mathbb{R} \to \mathrm{Sym}(E)$ be a curvature operator s.t. its Jacobi Eq. has no conjugate points on $[0,r]$.

If $-\beta^2 Id \leq R(t)$ for $\boxed{\beta > 0}$ and A is the Jacobi tensor with $A(0)=0$, $\dot{A}(0) = Id$, then:

$$-\beta \operatorname{cotgh}(\beta(r-t)) \leq \dot{A}(t)A(t)^{-1} \leq \beta \operatorname{cotgh}(\beta t),$$

for all $t \in (0,r)$. $\quad (\langle \dot{A}(t)A(t)^{-1}x, x\rangle \leq \beta \operatorname{cotgh}(\beta t)\langle x, x\rangle)$

Proof. By Rauch's estimate (Lemma 6), the function

$$t \longmapsto \frac{\overset{= \|J(t)\|^2}{\langle A(t)x, A(t)x\rangle}}{\frac{1}{\beta}\operatorname{senh}^2(\beta t)} \quad ; \forall x \in E \setminus \{0\}$$

$J(t) = A(t)x.$

$\hookrightarrow s_{k_2}(t)$ — com $k_2 = -\beta^2$

is monotone decreasing.

differentiating

$$\Rightarrow \quad 2\langle \dot{A}(t)x, A(t)x\rangle \frac{1}{\beta}\operatorname{senh}^2(\beta t) - 2\|A(t)x\|^2 \operatorname{senh}(\beta t)\cosh(\beta t) \leq 0$$

Given $\boxed{y \in E}$, one define $\boxed{x = A(t)^{-1}y}$, thus:

$$(\ast) \ldots\ldots\ldots \langle \dot{A}(t)A(t)^{-1}y, y\rangle \leq \beta\|y\|^2 \operatorname{cotgh}(\beta t) \quad ; \forall t \in (0,r)$$

On the other hand, consider $D(t)$ the solution of the Jacobi Eq.

$$\begin{cases} \ddot{D}(t) + R(t)D(t) = 0 \\ D(r) = 0, \ \dot{D}(r) = -Id \end{cases}$$

Define $C(t) = D(r-t)$ and note that:

$$\ddot{C}(t) = \ddot{D}(r-t) = -R(r-t)D(r-t) = -R(r-t)C(t)$$

$$\Rightarrow \quad \ddot{C}(t) + R(r-t)C(t) = 0$$

e $\quad C(0) = D(r) = 0 \quad$ e $\quad \dot{C}(0) = -\dot{D}(r) = Id.$

Applying $(\ast)$ to $C(t)$

$$\Rightarrow \quad \underset{\langle -\dot{D}(r-t)D(r-t)^{-1}y, y\rangle}{\underset{\parallel}{\langle \dot{C}(t)C(t)^{-1}y, y\rangle}} \leq \beta\|y\|^2 \operatorname{cotgh}(\beta t) \quad ; \forall t \in (0,r)$$

$$\Rightarrow \quad \langle \dot{D}(t)D(t)^{-1}y, y\rangle \geq -\beta\|y\|^2 \operatorname{cotgh}(\beta(r-t)) \quad ; \forall t \in (0,r)$$

$\langle \dot{A}(t)x, x\rangle$

Applying the Index Lemma (Lemma ④) to the curve:

$$J(s) = \begin{cases} A(s)A^{-1}(t)x & ; 0 \leq s \leq t \\ D(s)D^{-1}(t)x & ; t \leq s \leq r \end{cases}$$

we get:

$$0 \leq I_{[0,r]}(J, J) = \langle \dot{A}(t)A(t)^{-1}x, x\rangle - \langle \dot{D}(t)D(t)^{-1}x, x\rangle$$

$\boxtimes$