A note on the Karlsson-Margulis Theorem *†

Heric Corrêa
correaheric2@gmail.com

Lamartine Medeiros
lamartinecmrj4702@gmail.com

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§ 1 Motivation

The following quote is a piece of the celebrated and classical work of Harry Furstenberg [Fur63, Noncommuting random products, Trans. AMS, 1963]

> Let $X_1, \dots, X_n$ be a sequence of independent real valued random variables with a common distribution function $F(x)$, and consider the sums $X_1 + X_2 + \dots + X_n$. A fundamental theorem of classical probability theory is the strong law of large numbers which asserts that with probability one, $X_1 + X_2 + \dots + X_n \sim n \int x dF(x)$, provided that $\int |x| dF(x)$ is finite. It is natural to inquire whether there exist laws governing the asymptotic behavior of products $X_n X_{n-1} \dots X_1$, where the X_j are now identically distributed independent random variables with values in an arbitrary group.

Let us rewrite this inquiry from the point of view of the ergodic theory. Consider (Ω, μ) a probability space, $T\colon \Omega \circlearrowleft$ a measure preserving transformation, and G a group (or semi-group) non-necessarily commutative. Given a function $g\colon \Omega \to G$, we can define, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, the following *Birkhoff products*:

$$\overrightarrow{g}_n(\omega) := g(\omega) g(T\omega) \dots g(T^{n-1}\omega) \quad \text{and} \quad \overleftarrow{g}_n(\omega) := g(T^{n-1}\omega) \dots g(T\omega) g(\omega).$$

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The map $\overrightarrow{g}\colon \mathbb{N}\times\Omega\to G$ (resp. $\overleftarrow{g}$) is called *right cocycle* on G (resp. *left cocycle*). In fact, these maps satisfy the respective *cocycle identities*, that is,

$$\overrightarrow{g}_{m+n}(\omega)=\overrightarrow{g}_m(\omega)\overrightarrow{g}_n(T^{m-1}\omega)\quad\text{and}\quad \overleftarrow{g}_{m+n}(\omega)=\overleftarrow{g}_n(T^{m-1}\omega)g_m(\omega). \tag{1.1}$$

A classical situation occurs when $G=(\mathbb{R},+)$. In this case, if $g\colon\Omega\to\mathbb{R}$ is integrable, Birkhoff proved in the 30's that

$$\frac{1}{n}\overrightarrow{g}_n(\omega)=\frac{1}{n}\overleftarrow{g}_n(\omega)=\frac{1}{n}\sum_{j=0}^{n-1}g\circ T^j\omega\xrightarrow{n\to+\infty}\int g(\omega)d\mu(\omega),\quad\text{for }\mu\text{-a.e. }\omega\in\Omega.$$

It is instructive to compare this result with the strong law of large numbers in the probability theory, as it was written in the quote above. The map $(n,\omega)\mapsto g_n(\omega)=g(\omega)+\ldots+g(T^{n-1}\omega)$ is called an *additive cocycle*. Kingman made an important generalization of this result in the 60's. A function $f\colon\mathbb{N}\times\Omega\to\mathbb{R}$, $(n,\omega)\mapsto f_n(\omega)$ is called a *subadditive cocycle* with respect to $T\colon(\Omega,\mu)\circlearrowleft$ if for every $\omega\in\Omega$ and $m,n\in\mathbb{N}$

$$f_{m+n}(\omega)\leqslant f_m(\omega)+f_n(T^{m-1}\omega). \tag{1.2}$$

Theorem 1.1 ([Kin68])**.** *Consider a subaddtive cocycle $f\colon\mathbb{N}\times\Omega\to\mathbb{R}$ and assume that T is ergodic. If $f_n\colon\Omega\to\mathbb{R}$ is measurable for each $n\in\mathbb{N}$ and f_1^+ is integrable, then there exists $c>-\infty$ such that*

$$\frac{f_n(\omega)}{n}\xrightarrow{n\to\infty}c,\quad\textit{for }\mu\textit{-almost }\omega\in\Omega.$$

Moreover, $c=\inf_n\frac{1}{n}\int f_n d\mu$.

Our main goal in this note is to understand to what extend these results. In particular, we prove a theorem due A. Karlsson and G. Margulis which provides a deep and elegant result for the asymptotic behavior of the Birkhoff Products in some *nice* group actions which applies to many different situations. Also, we will prove Kingman's and Birkhoff's ergodic theorems as a by-product.

§ 2 Formulation of the main theorem

2.1. Drift of a cocycle of semi-contractions

Let (H,d) be a metric space and consider G the group of *semi-contractions* of H. That is, for each $g\in G$ and $p,q\in H$

$$d(gp,gq)\leqslant d(p,q). \tag{2.1}$$

Given a mensurable map $g\colon\Omega\to G$, the map $(n,\omega)\in\mathbb{N}\times\Omega\mapsto\overrightarrow{g}_n(\omega)$ is called a *cocycle of semi-contraction* of H. We will fix some notations

- Choose a point $p_0\in H$, we call it *origin* or *base point*.
- Each $p_n(\omega):=\overrightarrow{g}_n(\omega)p_0\in H$ is called the *n-steps*. The distance of $p_n(\omega)$ to the origin is denoted by $d_n(\omega)$, that is, $d_n(\omega)=d(p_0,p_n(\omega))$.

- We say that $\overrightarrow{g}$ has *finite first moment* (or satisfies the integrability hypothesis) if

$$\int_\Omega d_1(\omega)\, d\mu(\omega) < \infty. \tag{2.2}$$

Exercise 2.1. *Show that the map $(n,\omega) \mapsto d_n(\omega)$ defined above is a subbaditive cocycle, that is, inequality (1.2) holds.*
Exercise 2.2. *Show that the integrability hypothesis does not depend on the choice of the origin.*

Proposition 2.1 (Drift). *Let $\overrightarrow{g}: \mathbb{N} \times \Omega \to G$ be a mensurable cocycle of semi-contractions of H with finite first moment. Then there exists $D \geqslant 0$ such that*

$$\frac{d_n(\omega)}{n} \xrightarrow{n\to+\infty} D, \quad \textit{for } \mu\textit{-almost everywhere } \omega \in \Omega.$$

The number D is called drift *of the cocycle $\overrightarrow{g}$.*

Proof. It is an immediate consequence of the Exercise 2.1 and the Theorem 1.1. □

Exercise 2.3. *The statement of Proposition 2.1 suggests that D does not depend on the choice of the origin. Prove this.*

2.2. Karlsson-Margulis metric space

A *geodesic path* with velocity $v \geqslant 0$ joining $p, q \in H$ (or, more briefly, a *geodesic* from p to q) is a map $\gamma\colon [0,\tau] \subset \mathbb{R} \to H$ such that $\gamma(0) = p$, $\gamma(\tau) = q$ and

$$d(\gamma(t), \gamma(s)) = v|t-s|, \quad \forall t, s \in [0,\tau].$$

In particular, $d(p,q) = v\tau$. If $v = 1$ we say that it is a *unitary speed geodesic*. The image of γ is called *geodesic segment* joining p and q and it is denoted by $[p;q]_\gamma$ or simply $[p;q]$ if there is no risk of confusion. A *geodesic ray* in H starting in p with velocity v is a map $\gamma\colon [0,\infty) \to H$ such that $d(\gamma(t),\gamma(s)) = v|t-s|$ for all $t, s \geqslant 0$ and $\gamma(0) = p$. The metric space (H,d) is said to be a *(uniquely) geodesic space* if every two points in H are joined by a (unique) geodesic segment.

A subset $C \subset H$ is called *convex* if every pair of points $p, q \in C$ can be joined by a geodesic γ and $[p;q]_\gamma \subset C$. In particular, if (H,d) is a geodesic space, so is convex.

Remark 2.1. If (H,d) is a complete metric space, then (H,d) is a geodesic space if and only if for each pair $p, q \in H$ there exists a *midpoint* in H. That is, there exists $m_{pq} \in H$ such that

$$d(p, m_{pq}) = d(m_{pq}, q) = \frac{1}{2} d(p,q).$$

Indeed, given a unitary geodesic $\gamma\colon [0,\tau] \to H$ joining p and q, the point $m_{pq} := \gamma(\tau/2)$ satisfies the desired conclusion. On the other hand, we can construct a function $\eta\colon \mathbb{Q}_2 \cap [0,\tau] \to H$ where $\mathbb{Q}_2 = \{\frac{k}{2^m} : k, m \in \mathbb{Z}_+\}$ is the dyadic rational number by putting:

$$\begin{cases} \eta(\tau/2) = \text{ midpoint of } p \text{ and } q; \\ \eta(\tau/4) = \text{ midpoint of } p \text{ and } \eta(\tau/2); \\ \eta(3\tau/4) = \text{ midpoint of } \eta(\tau/2) \text{ and } q; \\ \vdots \end{cases}$$

Constructing η recursively by this way, we obtain a Lipschitz function on $\mathbb{Q}_2 \cap [0,\tau]$. Since this set is dense in $[0,\tau]$, the completeness of (H,d) implies that we can extend η to a unique Lipschitz map $\gamma\colon [0,\tau] \to H$.

If H is a Banach space, Clarkson introduced in the 30s the concept of *uniformly convexity*. More especifically, the Banach space H is said to be *uniformly convex* if

$$\forall \varepsilon > 0 \; \exists\, \delta > 0 \text{ such that } \quad \left.\begin{array}{r} p,q \in H, \|p\| = \|q\| = 1 \\ \|p-q\| \geqslant \varepsilon \end{array}\right\} \implies \|m_{pq}\| \leqslant 1-\delta. \tag{2.3}$$

where m_{pq} stands for midpoint of p and q.

Definition 2.1. We say that H is *uniformly convex* if it is convex and there is a continuous strictly decreasing function $h\colon [0,1] \circlearrowleft$ with $h(1) = 0$ such that

$$\left.\begin{array}{r} p_0, p, q \in H \\ m = \text{ midpoint of } p,q \end{array}\right\} \implies \frac{d(m,p_0)}{R} \leqslant h\left(\frac{d(p,q)}{2R}\right), \tag{2.4}$$

where $R := \max\{d(p_0,p), d(p_0,q)\}$.

EXERCISE: Compare the definitions (2.3) and (2.4). What can you say about it? How to interpret geometrically these implications? make a drawing to convince yourself.

Proposition 2.2. *If (H,d) is a complete and uniformly convex metric space, then for every $p,q \in H$ there exists a unique midpoint between p and q. In particular, (H,d) is uniquely geodesic.*

Proof. We argue by contradiction. Assume that there are $m_1, m_2 \in H$ midpoints of some $p,q \in H$. Let m' be the midpoint of m_1 and m_2. Now, putting $R = d(p,q)/2$, by uniformly convexity, it follows that

$$\frac{d(p,m')}{R} \leqslant h\left(\frac{d(m_1,m_2)}{2R}\right) \implies d(p,m') < R.$$

Analougsly, $d(m',q) < R$. Then, $d(p,q) \leqslant d(p,m') + d(m',q) < 2R$ yields a contradiction. □

Many examples satisfy the definition above: Euclidean spaces, hyperbolic spaces and symmetric spaces of noncompact type such as $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R})$, or more generally CAT(0)-space (e.g. $\mathbb{R}$-trees).

Definition 2.2. A convex metric space (H,d) is said to be *nonpositively curved in the sense of Busemann* if

$$\text{for every } p_0, p, q \in H, \quad d(m_{p_0 p}, m_{p_0 q}) \leqslant \frac{1}{2} d(p,q). \tag{2.5}$$

Definition 2.3. A metric space (H,d) is called *Karlsson-Magulis space* if it is complete, uniformly convex (hence uniquely geodesic) and nonpositively curved in the sense of Busemann.

Theorem 2.1 (Main theorem, [KM99])**.** *Let $T\colon (\Omega,\mu) \circlearrowleft$ an ergodic transformation and (H,d) a Karlsson-Margulis space. Consider $g\colon \Omega \to G$ a cocycle of semi-contractions and $p_0 \in H$. Then, for μ-almost every $\omega \in \Omega$ there exists a unique geodesic ray $\gamma_\omega\colon [0,\infty) \to H$ starting in p_0 with velocity D (the drift) such that*

$$d(\gamma_\omega(n), p_n(\omega)) = o(n).$$

Remark 2.2. If $D = 0$ the geodesic is the constant funcion equals to p_0.

The proof will be divided into two parts, a geometric and an ergodic one.

§ 3 Proof of the main theorem

3.1. Geometric tools

Consider (H,d) a *Karlsson-Margulis* metric space. We note that the condition of nonpositive curvature implies that $t \mapsto d(\gamma_1(t),\gamma_2(t))$ is a convex function for any two geodesics γ_1 and γ_2. In particular, for two rays γ_1 and γ_2 with same origin, we have that

$$t \longmapsto \frac{1}{t} d(\gamma_1(t),\gamma_2(t)) \quad \text{is non-decreasing.} \tag{3.1}$$

Exercise 3.1 *Don't trust us. Prove that (3.1) holds (or you can simply look for the answer in [J. Jost, p. 46, Nonpositive Curvature: Geometric and Analytic Aspects, 1997]).*

Let $(\gamma_n)_n$ be a sequence of unitary speed rays with same origin such that $(\gamma_n(t))_n$ is a Cauchy sequence in H for every t. By completeness of (H,d) we can define a function

$$t \mapsto \gamma(t) := \lim_{n\to\infty} \gamma_n(t).$$

It is immediate that γ is a unitary speed ray starting at the same origin as any of the rays γ_n. We say that γ is the *limit geodesic* of γ_n.

The next lemma is a fundamental part of the proof of the Theorem 2.1. It basically says that if a triangle is "thin", so one of its vertices is next to the opposite edge. We suggest you make your own drawings along the following proof for your best understanding.

Lemma 3.1 (Geometric lemma)**.** *There exists a function $\delta = \delta(\varepsilon)$ on $[0,1]$ with $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ such that, for each $\varepsilon \in [0,1]$*

$$p,q,r \in H, \quad d(q,p) + d(p,r) \leqslant d(q,r) + \varepsilon d(q,p) \implies d(\tilde{p},p) \leqslant \delta(\varepsilon) d(q,p) \tag{3.2}$$

where $\tilde{p}$ is the point in $[q;r]$ such that $d(q,\tilde{p}) = d(q,p)$.

Proof. Provide $p,q,r \in H$, let m be the midpoint of $\tilde{p}$ and p. Uniform convexity implies that

$$d(m,r) \leqslant \max\{d(\tilde{p},r), d(p,r)\}.$$

By definition of $\tilde{p}$ we have $d(m,r) = d(q,r) - d(q,p)$ and by hypothesis that $d(p,r) \leqslant d(q,r) - d(q,p) + \varepsilon d(q,p)$ we get

$$d(m,r) \leqslant d(q,r) - d(q,p) + \varepsilon d(q,p).$$

Hence it follows, by $\triangle$-inequality, that

$$d(q,m) \geqslant d(q,p) - \varepsilon d(q,p) = (1-\varepsilon)R \tag{3.3}$$

where $R := d(q,p) = \max\{d(q,p); d(q,\tilde{p})\}$. Again by uniform convexity, we have

$$\frac{d(m,q)}{R} \leqslant h\left(\frac{d(\tilde{p},p)}{2R}\right).$$

From the inequality (3.3) and since h is decreasing we get

$$h^{-1}(1-\varepsilon) \geqslant \frac{d(\tilde{p},p)}{2R}.$$

Recalling that $R = d(q,p)$ and letting $\delta(\varepsilon) := 2h^{-1}(1-\varepsilon)$ we obtain the desired conclusion. □

3.2. Ergodic tools

F. Riesz [Rie45] introduced a new short proof of the so-called maximal ergodic theorem on which Birkhoff's ergodic theorem is based. A. Karlsson [Kar17] presents a proof of the subbadtive ergodic theorem (Theorem 1.1) as an extension of F. Riesz's approach to the Birkhoff ergodic theorem. In both works, they use a combinatorial lemma which we state as follows.

Consider a finite sequence of real number $(a_1, a_2, \ldots, a_n)$. We say that ℓ is a *good index* if

$$S_j(a_\ell) := a_\ell + a_{\ell+1} + \ldots + a_{\ell+j} \geqslant 0, \quad \text{for all } \ 0 \leqslant j \leqslant n-\ell.$$

Otherwise, if there exists some j for which $S_j(a_\ell)$ is negative, we say that ℓ is a *bad index*. Each element a_ℓ with a bad index is called a *leader*.

Lemma 3.1 (Riesz's combinatorial lemma)**.** *The sum of the leaders of any finite sequence of real numbers is nonpositive (by convention, an empty sum is 0). In the other words,*

$$\sum_{\ell \text{ bad index}} a_\ell \leqslant 0.$$

Proof. We will argue by induction on the number of elements of the sequence. That is, if it holds for all finite sequences with k elements for which $k < n$, we will conclude that it holds for all finite sequences with n elements. The conclusion is clear for $k = 1$. Assume now, that is holds for any integer smaller than n. So provide a sequence $(a_1, \ldots, a_n)$, let's split in two cases.

Case 1: $\ell = 1$ is a good index. Then all the bad indices are also bad indices in the shifted sequence $(a_2, \ldots, a_n)$ and hence, by induction hypothesis, the sum of the leaders is nonpositive.

Case 2: $\ell = 1$ is a bad index. Then consider m the smaller index for which $a_1 + \ldots + a_m$ is negative. We claim that $2, \ldots, m$ are also bad indices. Indeed, if one of them, say i, were a good index, then $a_i + \ldots + a_m$ would be nonnegative and hence $a_1 + \ldots + a_{i-1}$ negative, contrary to the choice of m. The leaders $a_1, a_2, \ldots, a_m$ thus have negative sum; the remaining leaders (if there are any) have nonpositive sum because they lead in sequence shorter than n. This completes the proof. □

Henceforth in this section, consider a subbaditive cocycle $f\colon \mathbb{N}\times\Omega\to\mathbb{R}$ such that

$$\int f_1^+(\omega)\, d\mu(\omega) < +\infty.$$

So we have that $F_n := \int f_n(\omega)d\mu(\omega) < +\infty$ for all n (but it is possible $F_n = -\infty$). Note that $(F_n)_n$ is a real subbaditive sequence, that is, $F_{n+m} \leqslant F_n + F_m$. The standard Fekete's lemma implies that

$$F := \lim \frac{1}{n} F_n < +\infty.$$

Consider the functions $\underline{f}, \overline{f}\colon \Omega \to \mathbb{R}$, defined by

$$\underline{f}(\omega) := \liminf_{n\to\infty} \frac{f_n(\omega)}{n}, \qquad \overline{f}(\omega) := \limsup_{n\to\infty} \frac{f_n(\omega)}{n}. \tag{3.4}$$

Lemma 3.2. *The functions $\underline{f}$ and $\overline{f}$ are T^k-invariant for every $k\in\mathbb{N}$. That is,*

$$\underline{f} = \underline{f}\circ T^k, \qquad \overline{f} = \overline{f}\circ T^k \qquad \mu\text{-almost everywhere.}$$

In particular, if T is ergodic, these functions are constant almost everywhere.

Proof. Let us consider only $\underline{f}$, the argument for $\overline{f}$ is analogous. By subadditivity we have that $f_n\circ T^k \geqslant f_{n+k} - f_k$. Hence for each ω

$$\frac{1}{n} f_n(T^k\omega) \geqslant \frac{n+k}{n}\frac{1}{n+k} f_{n+k}(\omega) - \frac{1}{n} f_k(\omega) \implies \underline{f}\circ T^k \geqslant \underline{f}.$$

Which implies $\int \underline{f}\circ T^k - \underline{f}\, d\mu = 0$. □

Fix some $\ell \geqslant 1$ and define

$$\underline{f}^{\ell}(\omega) := \liminf_{k\to\infty} \frac{f_{k\ell}(\omega)}{k\ell}, \qquad \overline{f}^{\ell}(\omega) := \limsup_{k\to\infty} \frac{f_{k\ell}(\omega)}{k\ell}. \tag{3.5}$$

The following lemma have been used by Karlsson in [Kar17] to prove the Kingmann theorem (Theorem 1.1).

Lemma 3.3. *Consider $f\colon \mathbb{N}\times\Omega\to\mathbb{R}$ a nonpositive subbaditive cocycle, that is, $f_n(\omega)\leqslant 0$ everywhere. Then for each $\ell\geqslant 1$ we have*

$$\underline{f}(\omega) = \underline{f}^{\ell}(\omega), \qquad \overline{f}(\omega) = \overline{f}^{\ell}(\omega) \qquad \text{for every } \omega.$$

Proof. Since $\{k\ell\}$ is subsequence of $\{k\}$ we get $\underline{f}^{\ell}(\omega) \geqslant \underline{f}(\omega)$ and $\overline{f}^{\ell}(\omega) \leqslant \overline{f}(\omega)$ for each ω. On the other hand, for each $n\in\mathbb{N}$ we can find some $k = k(n)\in\mathbb{N}$ and $0\leqslant r(n) < \ell$ such that $n = k\ell + r$. By subadditivity and nonpositivity we have

$$f_{(k+1)\ell}(\omega) \leqslant f_n(\omega) + f_{\ell-r}(T^n\omega) \leqslant f_n(\omega) \tag{3.6}$$
$$f_n(\omega) \leqslant f_{k\ell}(\omega) + f_r(T^{k\ell}\omega) \leqslant f_{k\ell}(\omega). \tag{3.7}$$

Hence letting $n \to +\infty$

$$\cancelto{1}{\frac{(k+1)\ell}{n}}\frac{1}{(k+1)\ell} f_{(k+1)\ell}(\omega) \leqslant \frac{1}{n} f_n(\omega) \leqslant \cancelto{1}{\frac{k\ell}{n}}\frac{1}{k\ell} f_{k\ell}(\omega) \implies \begin{cases} \underline{f}^{\ell}(\omega) \leqslant \underline{f}(\omega) \\ \overline{f}^{\ell}(\omega) \geqslant \overline{f}(\omega) \end{cases}$$

□

EXERCISE 3.2 *Prove that the conclusion of the Lemma 3.3 still holds (almost everywhere) if we omit the hypothesis of non-positivity. (See, if necessary, [Con13, Lemma 7.8 and Proposition 7.7])*

The following two propositions are the unique "ergodic tools" used in the proof of Theorem 2.1. They can be considered ergodic versions of Pliss Lemma (see, e.g. [Mn87, Chapter 4, lemma 11.8]).

Proposition 3.1 (Weak ergodic Pliss lemma). *Let $T\colon (\Omega,\mu) \circlearrowleft$ be a measure preserving transformation and $F > 0$. Consider the set*

$$E_1 := \left\{ \omega \in \Omega \;\middle|\; \exists \text{ infinitely many } n \text{ such that } \begin{array}{r} f_n(\omega) - f_{n-k}(T^k\omega) \geqslant 0 \\ \forall\, k:\ 1 \leqslant k \leqslant n. \end{array} \right\}.$$

Then, $\mu(E_1) > 0$.

Proof. For every $i \in \mathbb{N}$, consider

$$M_i := \{\omega \in \Omega | \exists k \;:\; 1 \leqslant k \leqslant i \text{ and } f_i(\omega) - f_{i-k}(T^k\omega) < 0\}.$$

We must show that the set of $\omega \in \Omega$ for which $\omega \notin M_i$ for infinitely many i has a positive measure. For this purpose, consider the function

$$\varphi_i(\omega) := N_i(\omega) - f_{i-1}(T\omega).$$

We note that

$$f_n(\omega) - f_{n-k}(T^k\omega) = \sum_{j=0}^{k-1} \varphi_{n-j}(T^j\omega). \tag{3.8}$$

and, in particular,

$$f_n(\omega) = \sum_{j=0}^{n-1} \varphi_{n-j}(T^j\omega). \tag{3.9}$$

In view of (3.8), if $T^k\omega \in M_{n-k}$ then for some j, $k \leqslant j \leqslant n-1$ we have

$$\varphi_{n-k}(T^k\omega) + \ldots + \varphi_{n-j}(T^j\omega) < 0.$$

From this and the combinatorial lemma (Lemma 3.1, letting $a_\ell := \varphi_{n-\ell}(T^\ell\omega)$) we deduce that for every $\omega \in \Omega$ and $n \in \mathbb{N}$,

$$\sum_{\substack{0 \leqslant k \leqslant n-1 \\ T^k\omega \in M_{n-k}}} \varphi_{n-k}(T^k\omega) \leqslant 0. \tag{3.10}$$

Using the T-invariance of μ, we get from the last inequality that

$$\begin{aligned}\sum_{j=1}^{n}\int_{M_j}\varphi_j(\omega)d\mu(\omega) &= \sum_{0\leqslant k\leqslant n-1}\int_{M_{n-k}}\varphi_{n-k}(\omega)d\mu(\omega)\\ &= \sum_{0\leqslant k\leqslant n-1}\int_{T^{-k}M_{n-k}}\varphi_{n-k}(T^k\omega)d\mu(\omega) \\ &= \int_{\Omega}\Bigg(\sum_{\substack{0\leqslant k\leqslant n-1\\ T^k\omega\in M_{n-k}}}\varphi_{n-k}(T^k\omega)\Bigg)d\mu(\omega)\leqslant 0.\end{aligned} \tag{3.11}$$

On the other hand, recalling $F_n=\int f_n(\omega)d\mu(\omega)$ and (3.9), from the T-invariance we get

$$F_n=\sum_{j=1}^{n}\int_X\varphi_j(\omega)d\mu(\omega). \tag{3.12}$$

By hypothesis we have that $\lim F_n/n=F>0$, so there exists a number N such that

$$F_n>\frac{2F}{3}n,\quad \text{for all } n>N. \tag{3.13}$$

Let $M_i^c:=\Omega\backslash M_i$, then, in view of (3.11), (3.12) and (3.13) and the inequality $\varphi_i(\omega)=f_i(\omega)-f_{i-1}(T\omega)\leqslant f_1(\omega)\leqslant f_1^+(\omega)$, we obtain

$$\sum_{j=1}^{n}\int_{M_j^c}f_1^+(\omega)d\mu(\omega)\geqslant\sum_{j=1}^{n}\int_{M_j^c}\varphi_j(\omega)d\mu(\omega)>\frac{2F}{3}n \tag{3.14}$$

for all $n>N$. Let $\chi_n:=\sum_{j=1}^{n}\mathbb{1}_{M_j^c}$, where $\mathbb{1}_A$ denotes the indicator function of a set $A\subset\Omega$. Let $F_1^+:=\int f^+(\omega)d\mu(\omega)$ and

$$B_n:=\left\{\omega\in\Omega\;\middle|\;n\geqslant\chi_n(\omega)>\frac{F}{3F_1^+}n\right\}.$$

Since

$$B_n^c=\left\{\omega\in\Omega\;\middle|\;\frac{F}{3F_1^+}n\geqslant\chi_n(\omega)\geqslant 0\right\},$$

we have that

$$\begin{aligned}\sum_{j=1}^{n}\int_{M_j^c}f_1^+(\omega)d\mu(\omega) &= \int_{\Omega}f_1^+(\omega)\chi_n(\omega)d\mu(\omega)\\ &= \int_{B_n}f_1^+(\omega)\chi_n(\omega)d\mu(\omega)+\int_{B_n^c}f_1^+(\omega)\chi_n(\omega)d\mu(\omega)\\ &\leqslant n\int_{B_n}f_1^+(\omega)d\mu(\omega)+\frac{F}{3F_1^+}n\int_{B_n^c}f_1^+(\omega)d\mu(\omega)\\ &\leqslant n\int_{B_n}f_1^+(\omega)d\mu(\omega)+\frac{F}{3}n.\end{aligned}$$

combining this inequality and the inequality (3.14) we get that

$$\int_{B_n} f_1^+(\omega)d\mu(\omega) > \frac{F}{3}. \tag{3.15}$$

for all $n > N$.

The condition $\int f_1^+ d\mu < \infty$ implies the existence of $\delta > 0$ such that

$$\mu(C) < \delta \implies \int_C f_1^+(\omega)d\mu(\omega) < \frac{F}{3}.$$

Hence it follows from (3.15) that $\mu(B_n) \geqslant \delta$ for every $n > N$. Let

$$C_n := \left\{\omega \in \Omega \mid \omega \in M_i^c \text{ for ate least } \frac{F}{3F_1^+}n \text{ positive integers } i\right\},$$

so $B_n \subset C_n$ and $C_{n+1} \subset C_n$. Therefore, the measure of the set

$$\bigcap_{n\geqslant 1} C_n = \{\omega \in \Omega \mid \omega \in M_i^c \text{ for infinitely many } i\}$$

is greater than or equal to $\delta > 0$. Now recalling the definition of M_i we get the desired statement. □

Proposition 3.2 (Ergodic Pliss Lemma). *Assume that T is ergodic and $F > -\infty$. For any $\varepsilon > 0$, let*

$$E_\varepsilon := \left\{\omega \in \Omega \,\middle|\, \exists\, \tilde{k} = \tilde{k}(\varepsilon) \text{ and a infinite set } N_\varepsilon \subset \mathbb{N} \text{ s.t. } \begin{array}{c} f_n(\omega) - f_{n-k}(T^k\omega) \geqslant (F-\varepsilon)k \\ \forall\, n \in N_\varepsilon, \ \text{ and } k \,:\, \tilde{k} \leqslant k \leqslant n. \end{array}\right\}.$$

Consider $E = \bigcap_{\varepsilon>0} E_\varepsilon$, then $\mu(E) = 1$.

Proof. For any $\varepsilon > 0$, let $\psi_n(\omega) := f_n(\omega) - (F-\varepsilon)n$. Then ψ is a subadditive cocycle and, by definition of F,

$$\lim \frac{1}{n}\int_\Omega \psi_n(\omega)d\mu(\omega) = \varepsilon > 0.$$

Note also that

$$f_n(\omega) - f_{n-k}(T^k\omega) \geqslant (F-\varepsilon)k$$

is equivalent to

$$\psi_n(\omega) - \psi_{n-k}(T^k\omega) \geqslant 0.$$

Therefore, Proposition 3.1 applied to ψ gives us that $\mu(E_\varepsilon) > 0$.

By definition of subadditivity,

$$f_n(T^i\omega) - f_{n-k}(T^{k+i}\omega) \geqslant f_{n+i}(\omega) - f_{(n+i)-(k+i)}(T^{k+i}\omega) - f_i(\omega).$$

It follows that $T^iE_\varepsilon \subset E_{2\varepsilon}$ for all $i \geqslant 0$. Then, $\mu(E_{2\varepsilon}) = 1$. □

We note that Lemma 3.2 does not rely on Birkhoff's (Additive) or Kingmann's (Subadditive) Ergodic Theorems. So we present a simple proof of Kingman's result (hence of the

Birkhoff's one) as a corollary of the results of this section.

Proof of Theorem 1.1. Subadditivity and Proposition 3.2 implies that for almost every ω, we have that for each $\varepsilon > 0$ there exists $\tilde{k}(\varepsilon)$ such that for every $k > \tilde{k}(\varepsilon)$ there exists $n = n(\varepsilon) > k$ such that

$$(F-\varepsilon)k \leqslant f_n(\omega) - f_{n-k}(T^k\omega) \leqslant f_k(\omega) \tag{3.16}$$

Hence

$$\liminf_{k\to\infty} \frac{1}{k} f_k(\omega) \geqslant F - \varepsilon. \tag{3.17}$$

Moreover, since ε is take arbitrarily, we have

$$\underline{f}(\omega) \geqslant F. \tag{3.18}$$

If f is an additive cocycle, then in particular $-f$ is a subadditive cocycle such that

$$\lim \frac{1}{n}\int -f_n d\mu = -\frac{1}{n}\int f_n d\mu = -F \in \mathbb{R}.$$

Then by the same arguments as in (3.16)

$$\frac{-f_k(\omega)}{k} \geqslant -F-\varepsilon \quad \text{for large enough } k \implies \limsup_{k\to\infty} \frac{f_k(\omega)}{k} \leqslant F+\varepsilon.$$

Hence

$$\overline{f}(\omega) = \underline{f}(\omega). \tag{3.19}$$

This prove the convergence almost everywhere in case of additive cocycle.
EXERCISE. *Convince yourself that, at this point, we have proved Birkhoff's theorem as stated in* §1.

In the case of a general subadditive cocycle f we can assume, without lost of generality, that $f_n(\omega) \leqslant 0$ for every n and ω. Indeed, if were not, for each $\ell \geqslant 1$ consider the additive cocycle

$$(n,\omega) \mapsto a_n^\ell(\omega) := \sum_{j=0}^{n-1} f_\ell \circ T^{j\ell}\omega$$

and so, note that $f - a^1 \leqslant 0$ everywhere. Moreover,

$$f_n^\ell(\omega) := f_{n\ell}(\omega) - a_n^\ell(\omega).$$

is a nonpositive subadditive cocycle.

EXERCISE 3.3. *(a) Again, don't trust us. Show that in fact f^ℓ is a nonpositive subadditive cocycle. (b) prove that if the conclusion of the theorem holds for f^1 then it holds for f, where f is an arbitrary subadditive cocycle.*

Now, fix an $\varepsilon > 0$ and take $\ell = \ell(\varepsilon)$ large enough such that

$$\frac{1}{\ell}\int f_\ell(\omega) d\mu(\omega) \leqslant F + \varepsilon. \tag{3.20}$$

We claim that

$$0 \geqslant \liminf_{k\to\infty} \frac{f_k^\ell(\omega)}{k\ell} \geqslant -\varepsilon.$$

Indeed, consider $F^\ell := \lim \frac{1}{k}\int f_k^\ell d\mu$ the drift associated to the cocycle $k \mapsto f_k^\ell$. By (3.20) we have

$$\begin{aligned}\frac{1}{k\ell}\int f_k^\ell d\mu &= \frac{1}{k\ell}\int [f_{k\ell} - a_n^\ell]d\mu = \frac{1}{k\ell}\int f_{k\ell}d\mu - \frac{1}{k\ell}\sum_{j=0}^{k-1}\int f_\ell \circ T^{j\ell}d\mu \\ &\geqslant \frac{1}{k\ell}\int f_{k\ell}d\mu - \frac{1}{k\ell}\sum_{j=0}^{k-1}(F+\varepsilon)\ell = \frac{1}{k\ell}F_{k\ell} - F - \varepsilon.\end{aligned}$$

Since $(\frac{1}{k\ell}F_{k\ell})$ is a subsequence of $(\frac{1}{k}F_k)$ which converge to F, we obtain $\frac{1}{\ell}F^\ell \geqslant -\varepsilon$. By nonpositive of the cocycle and (3.18) we get

$$0 \geqslant \liminf_{k\to\infty} \frac{1}{k\ell} f_k^\ell(\omega) \geqslant \frac{1}{\ell}F^\ell > -\varepsilon.$$

as we desired and the claim is proved.

From this claim, the nonpositivity and subadditivity of f, Lemma 3.3 and the convergence for additive cocycles, it follows that

$$\begin{aligned}\overline{f}(\omega) - \underline{f}(\omega) &= \limsup_{k\to\infty}\frac{1}{k\ell}f_{k\ell}(\omega) - \liminf_{k\to\infty}\frac{1}{k\ell}f_{k\ell}(\omega) \\ &= \limsup_{k\to\infty}\frac{1}{k\ell}f_k^\ell(\omega) - \liminf_{k\to\infty}\frac{1}{k\ell}f_k^\ell(\omega) \leqslant -\liminf_{k\to\infty}\frac{1}{k\ell}f_k^\ell(\omega) \leqslant \varepsilon.\end{aligned}$$

Therefore $\lim_{n\to\infty}\frac{1}{n}f_n(\omega)$ exists and it is constant almost everywhere since T is assumed ergodic. Finally, we note that this constant is exactly F. Since $f_n \leqslant 0$ for every, the dominated convergence theorem assures that

$$\lim_{n\to\infty}\frac{1}{n}f_n(\omega) = \int \lim_{n\to\infty}\frac{1}{n}f_n(\omega)d\mu(\omega) = \lim_{n\to\infty}\frac{1}{n}\int f_n(\omega)d\mu(\omega) = F.$$

This proves the theorem. □

3.3. Proof of the main theorem

Unicity. Assume there exists two different of such geodesic rays γ_1 and γ_2. Then, the following $\triangle$-inequality shows a contradiction.

$$0 \leqslant \underbrace{\frac{d(\gamma_1(k),\gamma_2(k))}{k}}_{\text{never deacrease}} \leqslant \cancelto{0}{\frac{d(\gamma_1(k),p_k)}{k}} + \cancelto{0}{\frac{d(\gamma_2(k),p_k)}{k}}.$$

□

Existence. If $D = 0$ we can consider by convention $\gamma_\omega \equiv p_0$ and the result follows by Theorem 1.1. Assume then $D > 0$. Let E be the set defined in Proposition 3.2. Consider $\omega \in E$ in the full measure set where Kingmann theorem holds. From now on, ω will frequently be supressed in the notations.

For each $i \in \mathbb{N}$, consider $\varepsilon_i > 0$ small enough such that

$$\delta\left(\frac{2\varepsilon_i}{D-\varepsilon_i}\right) \leqslant 2^{-i} \quad \text{and} \quad \varepsilon_i \xrightarrow{i\to\infty} 0, \tag{3.21}$$

where δ is the function given by Lemma 3.1 (geometric Lemma). For each $i \in \mathbb{N}$, consider $N_i := N_{\varepsilon_i}$ and r_i such that

$$(D-\varepsilon_i)k \leqslant d_k(\omega) \leqslant (D+\varepsilon_i)k \tag{3.22}$$

for all $k \geqslant r_i$ and

$$(D-\varepsilon_i)k \leqslant d_n(\omega) - d_{n-k}(T^k\omega) \tag{3.23}$$

for all $n \in N_i$ and $\tilde{k}(\varepsilon_i) \leqslant k \leqslant n$.

Consider $K_i := r_i \vee \tilde{k}(\varepsilon_i)$, then (3.22) and (3.23) holds simultaneuosly for every $n \in N_i$ and k such that $K_i \leqslant k \leqslant n$. Consider $(n_i)_i$ defined recursively as follows*

$$n_1 := \min\{n \in F_1 : n > K_1 \vee K_2\}, \quad n_{i+1} := \min\{n \in F_{i+1} : n > K_{i+2} \vee n_i\}.$$

Remark 3.1. Consider $K_0 := \min_i K_i$, so

$$\bigcup_{i\in\mathbb{N}} [K_i, n_i] = [K_0, \infty). \tag{3.24}$$

Then, joining (3.22) and (3.23), we have for each i and $k \in [K_i, n_i]$ that

$$d_{n_i}(\omega) - d_{n_i-k}(T^k\omega) + (D+\varepsilon_i)k \geqslant (D-\varepsilon_i)k + d_k(\omega) \tag{3.25}$$

$$d_{n_i}(\omega) + 2\varepsilon_i k \geqslant d_k(\omega) + d_{n_i-k}(T^k\omega) \tag{3.26}$$

Recall the definition of d and the semi-contraction property to observe that

$$\begin{aligned} d_{n_i-k}(T^k\omega) &= d\big(p_0, g(T^k\omega)\dots g(T^{n_i-1}\omega)p_0\big) \\ &\geqslant d\big(\overrightarrow{g}_k(\omega)p_0, \overrightarrow{g}_k(\omega)g(T^k\omega)\dots g(T^{n_i-1}\omega)p_0\big) \\ &= d\big(p_k(\omega), p_{n_i}(\omega)\big). \end{aligned} \tag{3.27}$$

Note that for (3.27) holds it is important that the cocycle is a *right* cocycle. Then, (3.26) and (3.27) implies that

$$\begin{aligned} d_k(\omega) + d\big(p_k(\omega), p_{n_i}(\omega)\big) &\leqslant d_{n_i}(\omega) + 2\varepsilon_i k \\ &\leqslant d_{n_i}(\omega) + \tilde{\varepsilon}_i d_k(\omega) \end{aligned} \tag{3.28}$$

where $\tilde{\varepsilon}_i := \frac{2\varepsilon_i}{D-\varepsilon_i}$. We can rewrite (3.28) by

$$d\big(p_0, p_k\big) + d\big(p_k, p_{n_i}\big) \leqslant d\big(p_{n_i}, p_0\big) + \tilde{\varepsilon}_i d\big(p_0, p_k\big). \tag{3.29}$$

This means that the origin and the k and n_i-step form a “thin” triangle. Consider γ_i the

*Recall the notation $a \vee b := \max\{a, b\}$ for each $a, b \in \mathbb{R}$

unitary speed geodesic joining p_0 and p_{n_i}, then the Lemma 3.1 implies

$$d(\gamma_i(d_k), p_k) \leqslant \delta(\tilde{\varepsilon}_i) d_k.$$

Claim 1. For each $t \geqslant 0$, the sequence $(\gamma_i(t))_i$ is Cauchy in H.

Proof. First note that, since $D > 0$, we have $\lim_{i\to\infty} d_{n_i} = +\infty$. Fixing $t \geqslant 0$, consider i large enough such that $d_{n_i} > t$. Since $K_{i+1} \leqslant n_i \leqslant n_{i+1}$ we have that

$$d\big(\gamma_{i+1}(d_{n_i}), \gamma_i(d_{n_i})\big) = d\big(\gamma_{i+1}(d_{n_i}), p_{n_i}\big) \leqslant \delta(\tilde{\varepsilon}_{i+1}) d_{n_i}.$$

Then, the condition (3.1) implies

$$d\big(\gamma_{i+1}(t), \gamma_i(t)\big) \leqslant \delta(\tilde{\varepsilon}_i) t \leqslant 2^{-i} t.$$

Hence, for each $m \in \mathbb{N}$

$$d\big(\gamma_{i+m}(t), \gamma_i(t)\big) \leqslant \sum_{j=1}^{m} 2^{-i-j} t \leqslant 2^{-i} t \longrightarrow 0.$$

Concluding the claim. □

Consider γ the limit geodesic of $(\gamma_i)_i$, we must reparametrize to obtain a geodesic with velocity D, $\tilde{\gamma}(t) := \gamma(tD)$ for all $t \geqslant 0$. Rest to show that $\tilde{\gamma}$ is the geodesic we are looking for. That is, $\frac{1}{k} d(\gamma(k), p_k) \to 0$. Indeed, in view of (3.24) for $k > K_0$ we can take $i = i(k)$ such that $K_i \leqslant k \leqslant n_i$. It's clear that $i \to \infty$ as $k \to \infty$. Then, we have

$$\begin{aligned} d\big(\gamma(kD), p_k\big) &\leqslant d\big(\gamma(kD), \gamma_i(kD)\big) + d\big(\gamma_i(kD), \gamma_i(d_k)\big) + d\big(\gamma_i(d_k), p_k\big) \\ &\leqslant 2^{-i} kD + |kD - d_k| + \delta(\varepsilon_i) d_k \\ &\leqslant 2^{-i} kD + \varepsilon_i k + \delta(\varepsilon_i)(D + \varepsilon_i) k \\ &= \big(2^{-i} D + \varepsilon_i + \delta(\varepsilon_i)(D + \varepsilon_i)\big) k. \end{aligned}$$

That is,

$$\limsup_{k\to+\infty} \frac{d\big(\tilde{\gamma}(k), p_k\big)}{k} \leqslant \lim_{k\to\infty} \big(2^{-i} D + \varepsilon_i + \delta(\varepsilon_i)(D + \varepsilon_i)\big) = 0$$

since $k \to \infty$ implies $i \to \infty$ which implies $\varepsilon_i \to 0$ which implies $\delta(\varepsilon_i) \to 0$. This proves the theorem □

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