Ruelle Operator

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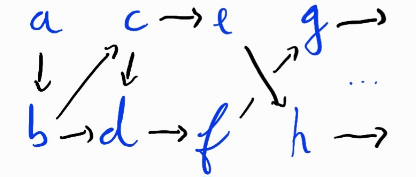
Contents

Topological Markov Shifts

- Let S be a countable set and $\mathbb{A} = (t_{ij})_{S \times S}$ be a matrix of zeroes and ones with no columns or rows which are all zeroes.
- Out of this one can construct a directed graph with set of vertices S and set of edges $\{a \to b; t_{ab} = 1\}$.
- The set of all one-sided infinite allowed paths on the graph is called a topological Markov shift.



Definition

The *topological Markov shift* (TMS) with set of *states* S and *transition matrix* $\mathbb{A} = (t_{ij})_{S \times S}$ is the set

$$X := \left\{ x \in S^{\mathbb{N}_0}; t_{x_i x_{i+1}} = 1, \forall i \geq 0 \right\}$$

equipped with the topology generated by *cylinders*

$$[a_0, ..., a_{n-1}] := \{x \in X; x_i = a_i, 0 \leq i \leq n-1\} \;\; (a_0, ..., a_{n-1} \in S),$$

and endowed with the *left shift* map $T : (x_0, x_1, ...) \mapsto (x_1, x_2, ...)$.

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- ◇ The topology of a TMS is metrizable, and it is a complete and separable metric space.
- ◇ A TMS with a finite set of states is called a *subshift of finite type* (SFT).
- ◇ The metric space X is often denoted by Σ_A^+.

Topological properties of a TMS

- An *admissible word* in X is an element $\underline{x} = (a_0, ..., a_{n-1}) \in S^n$, $n \in \mathbb{N}$, such that $[a_0, ..., a_{n-1}]$ is a non-empty cylinder in X.
- Write $a \xrightarrow{n} b$ if there is an admissible word of length $n+1$ which starts at a and ends at b, e.g., $[a, x_1, ..., x_{n-1}, b]$.

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Proposition

Let X be a topological Markov shift with set of states S and transition matrix $\mathbb{A} = (t_{ab})_{S\times S}$.

1. X is topologically transitive iff for all $a, b \in S$ there is an n s.t. $a \xrightarrow{n} b$.
2. X is topologically mixing iff for all $a, b \in S$ there is a number N_{ab} s.t. for all $n \geq N_{ab}$, $a \xrightarrow{n} b$.

Proof

Suppose that for all $a, b \in S$ there is a number N_{ab} s.t. for all $n \geq N_{ab}$, $a \xrightarrow{n} b$. Given open sets $U := [u_0, ..., u_{k-1}]$ and $V := [v_0, ..., v_{l-1}]$ in X, $k \geq l$, there is $N_{u_{k-1} v_0}$ s.t. for $n \geq N_{u_{k-1} v_0}$, $u_{k-1} \xrightarrow{n} v_0$.

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Therefore, in U, there is such admissible words

$$\begin{array}{rcl} \underline{x} & = & (u_0, ..., u_{k-1}, x_1, ..., x_n, v_0, ..., v_{l-1}) \\ \underline{y} & = & (u_0, ..., u_{k-1}, y_1, ..., y_{n+1}, v_0, ..., v_{l-1}) \\ & \vdots & \end{array}$$

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Suppose that for all $a, b \in S$ there is a number N_{ab} s.t. for all $n \geq N_{ab}$, $a \xrightarrow{n} b$. Given open sets $U := [u_0, ..., u_{k-1}]$ and $V := [v_0, ..., v_{l-1}]$ in X, $k \geq l$, there is $N_{u_{k-1}v_0}$ s.t. for $n \geq N_{u_{k-1}v_0}$, $u_{k-1} \xrightarrow{n} v_0$.
Therefore, in U, there is such admissible words

$$\begin{aligned}
\underline{x} &= (u_0, ..., u_{k-1}, x_1, ..., x_n, v_0, ..., v_{l-1}) \\
\underline{y} &= (u_0, ..., u_{k-1}, y_1, ..., y_{n+1}, v_0, ..., v_{l-1}) \\
&\vdots
\end{aligned}$$

So, for all $n \geq N_{u_k v_0} + k$, $T^n U \cap V \neq \varnothing$. Thus, X is topologically mixing.

Example: Cayley graphs

- Suppose G is a finitely generated group with a finite set of generators R.
- The *Cayley graph TMS* associated to R is the TMS with set of states G and transition matrix

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- The Cayley graph TMS of $(\mathbb{Z}, +)$ with $R = \{\pm 1\}$ is given by the transition matrix $\mathbb{A} = (t_{ij})_{\mathbb{Z}\times\mathbb{Z}}$

$$t_{ij} = 1 \iff |i - j| = 1.$$

- Generally, Cayley graphs TMS are always topologically transitive, because for any pair of states $a, b \in G$ one can find an admissible word which starts at a and ends at b by expanding $a^{-1}b = r_1 \cdots r_n$ and setting $(a, ar_1, ar_1 r_2, ..., b)$.

Example: Expanding Markov Interval Maps

Let $J = [0, 1]$ and $f : J \to J$ be a map for which there exists a finite or countable collection of pairwise disjoint open intervals $\{J_a\}_{a \in S}$ s.t.

1. $J = \bigcup_{a \in S} \overline{J_a}$;
2. $f|_{J_a}$ extends to a C^1 monotonic map on an open n'hood of $\overline{J_a}$;
3. *Uniform expansion*: There are constants $N \in \mathbb{N}$ and $\lambda > 1$ s.t. $|(f^N)'| > \lambda > 1$ on $\bigcup_{a \in S} \overline{J_a}$;

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4. *Markov partition*: For every $a, b \in S$, if $f(J_a) \cap J_b \neq \varnothing$, then $f(J_a) \supseteq J_b$;
5. For every $a \in S$, there are $b, c \in S$ s.t. $f(J_a) \supseteq J_b$ and $f(J_c) \supseteq J_a$.

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Let X denote the TMS with set of states S and transition matrix $(t_{ab})_{S \times S}$ given by $t_{ab} = 1 \iff f(J_a) \supseteq J_b$.

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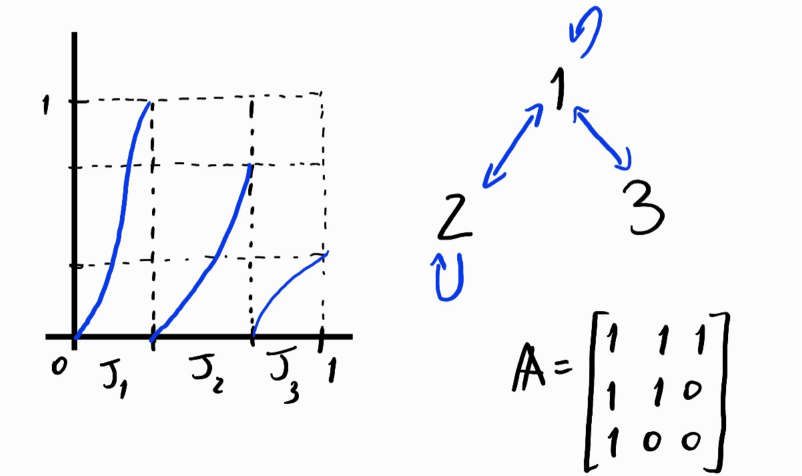

An example of a SFT given by a expanding Markov interval map.

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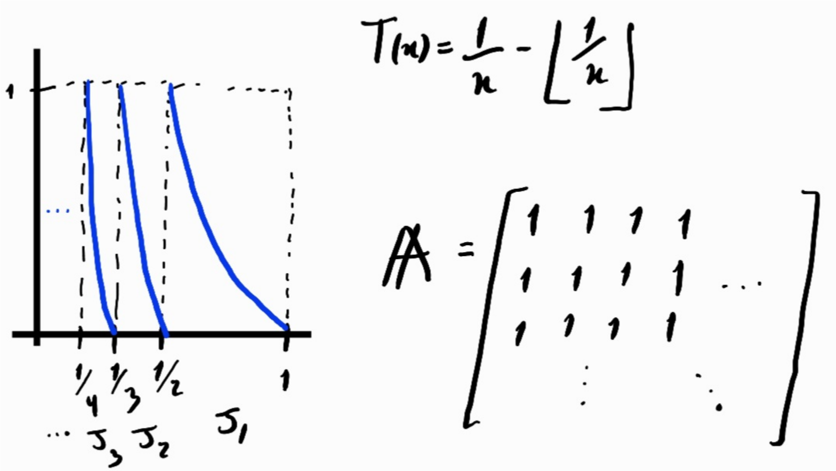

An example of a countable TMS given by the Gauss map.

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The following proposition gives us a topological conjugacy between the TMS generated by the interval maps and the expanding Markov function f in a big subset of $[0, 1]$.

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Proposition

Set $\mathcal{N} := \bigcup_{n\geq 0} f^{-n}\left(\partial J \cup \bigcup_{a\in S} \partial J_a\right)$, there is a Hölder continuous map $\pi : X \to I$ with the following properties:

1. The image of π contains $(0,1) \setminus \mathcal{N}$;
2. Every $t \in (0,1) \setminus \mathcal{N}$ has a unique pre-image $(x_0, x_1, \ldots) \in X$ and this pre-image is determined by $f^n(t) \in J_{x_n}, n \in \mathbb{N}$;
3. If $x \in X$ and $\pi(x) \in (0,1) \setminus \mathcal{N}$, then $(\pi \circ T)(x) = (f \circ \pi)(x)$.

Induced map and the full shift

- ◇ Suppose that $T : X \to X$ is a measurable map on a measurable space $(X, \mathcal{B})$.
- ◇ If $A \in \mathcal{B}$, set $A' := \{x \in A; T^n(x) \in A \text{ infinitely often}\}$. The *induced map* on A is the map $T_A : A' \to A'$, $T_A(x) = T^{\varphi_A(x)}(x)$, where $\varphi_A(x) := \inf\{n \geq 1; T^n(x) \in A\}$.

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- For X a TMS and $A = [a]$, for some state $a \in S$, then T_A is topologically conjugate to a full shift on a countable alphabet.
- The domain of the induced map is $A' := \{x \in [a]; x_i = a \text{ for infinitely many } i \in \mathbb{N}\}$.

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- Define:
  1. $\overline{S} := \{[a, \xi_1, ..., \xi_k]; k \geq 0, \xi_i \in S \setminus a, [a, \xi_1, ..., \xi_k, a] \neq \emptyset\}$;
  2. $\overline{X} := \overline{S}^{\mathbb{N}}, \overline{T} : \overline{X} \to \overline{X}$ is the left shift;
  3. $\pi : \overline{X} \to A, \pi([a, \underline{\xi}^1], [a, \underline{\xi}^2], ...) = (a, \underline{\xi}^1, a, \underline{\xi}^2, ...)$.
- Then $\pi \circ \overline{T} = T_A \circ \pi$.

Functions on TMS

- Consider X a fixed TMS. The *variations* of $\phi : X \to \mathbb{R}$ are

  $$\mathrm{var}_n(\phi) := \sup\{|\phi(x) - \phi(y)|; x, y \in X, x_i = y_i, 0 \leq i \leq n-1\}$$

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- We say that ϕ is *weakly Hölder continuous* with parameter θ if there is $A > 0$ and $\theta \in (0,1)$ such that for all $n \geq 2$, $\text{var}_n(\phi) \leq A\theta^n$.

- We say that ϕ has *summable variations* if $\sum_{n=2}^{\infty} \text{var}_n(\phi) < \infty$.

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- Each of these conditions implies that ϕ is uniformly continuous.
- If $|S| = \infty$, then these conditions do not imply that ϕ is bounded.

Ruelle Operator (a wrong definition)

Let (M, d) be a proper metric space and $T : X \to X$ be a proper continuous map. Given a proper continuous function $\phi : X \to \mathbb{R}$ (the potential function), we can define an operator $\mathcal{L}_\phi : C(X) \to C(X)$ as

$$\mathcal{L}_\phi f(x) = \sum_{y \in T^{-1}x} e^{\phi(y)} f(y).$$

This operator is called the Ruelle operator.

Riesz-Markov theorem

Let M a compact metric space. Consider a linear functional $\Phi : C(M) \to \mathbb{C}$. Then there exists a unique complex measure μ on M such that

$$\Phi(\varphi) = \int \varphi d\mu \text{ for all } \varphi \in C(M)$$

and $|\mu|(X) := \|\mu\| = \|\Phi\|$.

Eigenfunctions and Eigenmeasures

We sey that $h : M \to \mathbb{C}$ is an eigenfunction if there exists $\lambda \neq 0$ such that $\mathcal{L}_\varphi(h) = \lambda h$.
We say that a Borel measure on M is a eigenmeasure if there exists $\lambda \neq 0$ such that $\mathcal{L}^*_\varphi(\mu) = \lambda\mu$.
Where $\mathcal{L}^*_\varphi$ is the dual operator of $\mathcal{L}_\varphi$.

Ruelle's Perron-Frobenius Theorem

Let X be topologically mixing and S finite, ϕ weakly Hölder.
There are $\lambda > 0$, $h \in C(X)$ with $h > 0$ and $\nu \in \mathcal{M}\left(\Sigma_A^+\right)$ for which $\mathcal{L}h = \lambda h, \mathcal{L}^*\nu = \lambda\nu$, $\nu(h) = 1$,

$$\lim_{m\to\infty} \left\|\lambda^{-m}\mathcal{L}^m g - \nu(g)h\right\| = 0 \quad \text{for all} \quad g \in C(X)$$

and

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- $\mu = \nu h$ is a Gibbs measure.
- $\mu = \nu h$ is the unique equilibrium state.
- λ is the radius of $\mathcal{L}_\varphi$
- $e^{P(T,\varphi)} = \lambda$

Guveric Pressure

For every $a \in S, n \in \mathbb{N}$ set $Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$ where $\phi_n = \sum_{k=0}^{n-1} \phi \circ T^k$. If X is topologically mixing and ϕ is locally Hölder continuous then the limit

$$P_G(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, a)$$

exists, is independent of a and belongs to $(-\infty, \infty]$.

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exists, is independent of a and belongs to $(-\infty, \infty]$.
If $\|L_\phi 1\|_\infty < \infty$, this limit is finite and satisfies

$$P_G(\phi) = \sup \left\{ h_\mu(T) + \int \phi d\mu : \mu \in \mathcal{P}_T(X), \mu(-\phi) < \infty \right\}$$

where $\mathcal{P}_T(X)$ denotes the set of all invariant Borel probability measures. $P_G(\phi)$ is called the Gurevic Pressure of ϕ.

Hölder continuous Potentials

Set

$$Z_n(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a}} e^{\phi_n(x)}; \quad Z_n^*(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a:; x_1, \ldots, x_{n-1} \neq a}} e^{\phi_n(x)}.$$

Let X be topologically mixing and ϕ be locally Hölder continuous with finite Gurevic pressure $\log \lambda$. ϕ is called:

- recurrent if for some (hence all) $a \in S, \sum_{n<\infty} \lambda^{-n} Z_n(\phi, a) = \infty$; and transient otherwise;
- positive recurrent if it is recurrent and for some (hence all) $a \in S$, $\sum_{n<\infty} n\lambda^{-n} Z_n^*(\phi, a) < \infty$
- null recurrent if it is recurrent and for some (hence all) $a \in S$, $\sum_{n<\infty} n\lambda^{-n} Z_n^*(\phi, a) = \infty$.

Theorem, Sarig

Let X be topologically mixing and ϕ locally Hölder continuous with finite Gurevic pressure. ϕ is recurrent iff there exist $\lambda > 0$, a measure ν, finite and positive on cylinders, and a positive continuous function h such that $L_\phi^* \nu = \lambda \nu$ and $L_\phi h = \lambda h$. In this case $\lambda = \exp P_G(\phi)$ and $\exists a_n \nearrow \infty$ such that for every cylinder $[\underline{a}]$ and $x \in X$

$$\frac{1}{a_n} \sum_{k=1}^{n} \lambda^{-k} \left(L_\phi^k 1_{[\underline{a}]} \right)(x) \underset{n\to\infty}{\longrightarrow} h(x)\nu[\underline{a}],$$

where $\{a_n\}_n$ satisfies $a_n \sim \left(\int [a] h d\nu \right)^{-1} \sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, a)$ for every $a \in S$. Furthermore,

- if ϕ is positive recurrent then $\nu(h) < \infty, a_n \sim n.$ const, and for every $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]/\nu(h)$ uniformly on compacts;
- if ϕ is null recurrent then $\nu(h) = \infty, a_n = o(n)$, and for every $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on cylinders.