Dimension and Lyapunov exponents in conformal non-hyperbolic dynamics

1. Multifractal analysis of uniformly hyperbolic systems

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Będlewo, 2013
Multifractal analysis – general

$f : \Lambda \to \Lambda$ continuous map of a compact metric space.

- Birkhoff averages of continuous $\phi : \Lambda \to \mathbb{R}$,
- joint Birkhoff averages of vector valued functions
- local dimension of probability $\mu$
- local entropy of probability $\mu$
- recurrence rates and hitting times
- Rényi spectrum, HP (correlation) spectrum for dimension

Study “complexity” of level sets $B(\alpha)$ in terms of

$$\dim_H B(\alpha), \quad \dim_{\text{box}} B(\alpha), \quad h(f|B(\alpha)).$$
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Multifractal analysis – general

Multifractal analysis – local dimension

Define lower and upper local dimension of Borel probability $\mu$ at $x$

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\underline{d}_\mu(x) \overset{\text{def}}{=} \liminf_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}, \quad \overline{d}_\mu(x) \overset{\text{def}}{=} \limsup_{r \to 0} \frac{\log \mu(B_r(x))}{\log r}
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If $\underline{d}_\mu(x) = \overline{d}_\mu(x) \overset{\text{def}}{=} d_\mu(x)$, we call this value local dimension of $\mu$ at $x$.

If $\underline{d}_\mu(x) = \overline{d}_\mu(x) = d$ almost everywhere, we call $\mu$ exact dimensional.
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examples of measures which are not exact dimensional [Cutler ’90, Ledrappier-Misiurewicz ’85], typical Borel probability is not exact dimensional [Genyuk ’97]
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- expanding ergodic measure 1) rational maps [Przytycki ’85, Mañé ’88], 2) under piecewise monotone interval map $T$ with bounded variation derivative [Hofbauer-Raith ’92],

$$d_\mu(x) = d_\mu \quad \mu\text{-a.e. } x \quad \text{and} \quad d_\mu = \frac{h(\mu)}{\lambda(\mu)}$$
Multifractal analysis – local dimension

Not necessarily $d_{\mu}(\cdot)$ is constant everywhere, even if $\mu$ is exact dimensional.

Lemma

$$\text{ess sup } d_{\mu} = \inf \{ \dim_{H} Y : \mu(Y) = 1 \} \overset{\text{def}}{=} \dim_{H} \mu.$$  

Lemma (Frostman, mass distribution principle)

Suppose that $X$ is bounded.

- If $0 < \nu(X)$ and for all $x \in X$ we have $D \leq d_{\nu}(x)$ then $D \leq \dim_{H} X$.
- If $\nu(X) < \infty$ and for all $x \in X$ we have $d_{\nu}(x) \leq D$ then $\dim_{H} X \leq D$.  

(Będlewo, 2013)  
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Proof.

If for $c < D$ and $r$ small we have $\log \mu(B_r(x)) < c \log r$.
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**Proof.**

If for \( c < D \) and \( r \) small we have \( \mu(B(x, r)) < r^c \)
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By [Falconer, Mattila]

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\mu(X) \leq \mathcal{H}^c(X),
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where $\mathcal{H}^c(\cdot)$ denotes the $c$-dimensional Hausdorff measure.
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If for \( c < D \) and \( r \) small we have \( \mu(B(x, r)) < r^c \) then \( \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^c} < 1 \). By [Falconer, Mattila]

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$$0 < \mu(X) \leq \mathcal{H}^c(X),$$

where $\mathcal{H}^c(\cdot)$ denotes the $c$-dimensional Hausdorff measure. So, $D \leq \dim_H X$. 

(Będlewo, 2013)
Given $d \geq 0$, study the level set

$$\mathcal{D}_\mu(d) \overset{\text{def}}{=} \{x : d_\mu(x) = d\}$$

as well as

$$\mathcal{D}_\mu^{\text{irr}} \overset{\text{def}}{=} \{x : d_\mu(x) \text{ not well-defined, i.e. } d_\mu(x) < \bar{d}_\mu(x)\}.$$ 

Study “size” of level sets

$$\dim_H \mathcal{D}_\mu(d).$$
Mixing expanding conformal repeller

\[ T : \Lambda \rightarrow \Lambda \text{ is an mixing expanding conformal repeller if} \]

- \[ T \text{ topologically mixing} \]
- \[ C > 0, \lambda > 1 \text{ s.t. } \|dT^n(x)\| \geq C\lambda^n \text{ for every } n \geq 1 \text{ and } x \in \Lambda \]
- \[ dT_x \text{ multiple of an isometry for every } x \in \Lambda \]
- \[ \Lambda = \bigcap_{n \geq 0} T^{-n}(U) \text{ for some neighborhood } U \text{ of } \Lambda \]

Examples: one-dimensional Markov maps, conformal toral endomorphisms, expanding rational maps of Riemann sphere

- admits a Markov partition of arbitrarily small diameter
- \[ T|_{\Lambda} \text{ semi-conjugate to } \sigma|_{\Sigma_A^+} \text{ by some Hölder continuous map} \]
  \[ \chi : \Sigma_A^+ \rightarrow \Lambda, \]
  \[ \chi \text{ injective at every point which never hits Markov partition boundaries} \]
Basics in ergodic theory — entropy, Lyapunov exponent, pressure

Use cylinders $[\omega_1 \ldots \omega_n] = \{(i_1 i_2 \ldots) \in \Sigma_A^+: i_k = \omega_k, k = 1, \ldots, n\}$ and

$$C_{\omega_1 \ldots \omega_n} = \chi([\omega_1 \ldots \omega_n]) = C_n(x) \quad \text{for } \chi^{-1}(x) = (\omega_1 \omega_2 \ldots)$$
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For a $T$-invariant probability measure $\mu$ define entropy of $\mu$ by

$$h(\mu) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \sum_{(\omega_1 \ldots \omega_n)} -\mu(C_{\omega_1 \ldots \omega_n}) \log \mu(C_{\omega_1 \ldots \omega_n})$$
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and the Lyapunov exponent of \(\mu\) by

\[ \lambda(\mu) \overset{\text{def}}{=} \int \log |T'| \, d\mu . \]
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By [Hofbauer-Raith ’92], if $\mu$ ergodic and $\lambda(\mu) > 0$ then

$$h(\mu) = d(\mu) \lambda(\mu).$$
Basics in ergodic theory — entropy, Lyapunov exponent, pressure

Let $\psi: \Lambda \to \mathbb{R}$ be a continuous potential. We will often use the Birkhoff sum $S_n\psi = \psi + \psi \circ T + \cdots + \psi \circ T^{n-1}$. The pressure of $\psi$ (w.r.t. $T|_\Lambda$) is

$$P(\psi) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \sum_{(\omega_1 \ldots \omega_n)} \sup_{x \in C_{\omega_1 \ldots \omega_n}} e^{S_n\psi(x)}$$
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$$= \sup_{\mu \in \mathcal{M}} \left( h(\mu) + \psi(\mu) \right), \text{ where } \psi(\mu) \overset{\text{def}}{=} \int \psi \, d\mu.$$
Basics in ergodic theory — normalized potential

For $\psi$ Hölder continuous, there exists unique $\psi$-Gibbs measure $\mu$

$$\mu(C_n(x)) \asymp e^{S_n\psi(x)} e^{-nP(\psi)} \text{ for all } n \geq 1 \text{ and all } x \in \Lambda.$$
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Convenient to work instead with $\psi$ with the "normalized" potential $\phi = \psi - P(\psi)$ that satisfies $P(\phi) = 0$ and has very same Gibbs measure

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1) affine cookie cutter, piecewise constant potential

\( C_0, C_1 \subset [0, 1] \) disjoint intervals, \( T_i: C_i \rightarrow [0, 1] \) bijective \( C^1 \) with \( |T_i'| \overset{\text{def}}{=} r_i^{-1} > 1 \)
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$C_0, C_1 \subset [0, 1]$ disjoint intervals, $T_i: C_i \to [0, 1]$ bijective $C^1$ with $|T_i'| \overset{\text{def}}{=} r_i^{-1} > 1$,

$\phi: C_0 \cup C_1 \to \mathbb{R}$ locally constant $\phi|_{C_i} = \phi_i$ with $e^{\phi_0} + e^{\phi_1} = 1$

cconsider the $(e^{\phi_0}, e^{\phi_1})$-Bernoulli measure $\mu$
Multifractal analysis – local dimension:

1) affine cookie cutter, piecewise constant potential

\( C_0, C_1 \subset [0, 1] \) disjoint intervals, \( T_i : C_i \rightarrow [0, 1] \) bijective \( C^1 \) with 
\[ |T_i'| \overset{\text{def}}{=} r_i^{-1} > 1, \]
\( \phi : C_0 \cup C_1 \rightarrow \mathbb{R} \) locally constant \( \phi|_{C_i} = \phi_i \) with \( e^{\phi_0} + e^{\phi_1} = 1 \)

consider the \((e^{\phi_0}, e^{\phi_1})\)-Bernoulli measure \( \mu \)

\[
\mu(C_{\omega_1...\omega_n}) = e^{\phi_{\omega_1} + ... + \phi_{\omega_n}}
\]

\[
|C_{\omega_1...\omega_n}| = r_{\omega_1} \cdot ... \cdot r_{\omega_n}, \quad \text{where } r_i = |T_i'|^{-1}
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Multifractal analysis – local dimension:

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for $q \in \mathbb{R}$ and $\beta = \beta(q)$ with $\sum_{i=0,1} (e^{\phi_i})^q r_i^\beta = 1$ define $\nu_q(l) \overset{\text{def}}{=} \mu(l)^q \cdot |l|^\beta$

$$
d_{\nu_q}(\omega) \sim \frac{\log \nu_q(C_n(\omega))}{\log |C_n(\omega)|}
$$

(Będlewo, 2013)
Multifractal analysis – local dimension:

1) affine cookie cutter, piecewise constant potential

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Multifractal analysis – local dimension:

1) affine cookie cutter, piecewise constant potential

\( C_0, C_1 \subset [0, 1] \) disjoint intervals, \( T_i : C_i \to [0, 1] \) bijective \( C^1 \) with \( |T'_i| \overset{\text{def}}{=} r_i^{-1} > 1 \),

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d_{\nu_q} (\omega) \sim \frac{\log \nu_q(C_n(\omega))}{\log |C_n(\omega)|} = \frac{\log \mu(C_n(\omega))}{\log |C_n(\omega)|} q + \frac{\log |C_n(\omega)|}{\log |C_n(\omega)|} \beta \sim q \, d_{\mu} (\omega) + \beta
\]
Multifractal analysis – local dimension:

1) affine cookie cutter, piecewise constant potential

for \( q \in \mathbb{R} \) and \( \beta = \beta(q) \) with
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Lemma

\( \beta : \mathbb{R} \to \mathbb{R} \) is real analytic, decreasing, convex (strict if \( \phi_0 / \log r_0 \neq \phi_1 / \log r_1 \))
Multifractal analysis – local dimension:

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Lemma

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Recalling \( d\nu_q(\omega) = q d\mu(\omega) + \beta \) and observing

\[ \nu_q\left( \mathcal{D}_\mu(-\beta'(q)) \right) = 1, \]
Multifractal analysis – local dimension:

1) affine cookie cutter, piecewise constant potential

for $q \in \mathbb{R}$ and $\beta = \beta(q)$ with $\sum_{i=0,1} (e^{\phi_i})^q r_i^\beta = 1$ define $\nu_q(I) \overset{\text{def}}{=} \mu(I)^q \cdot |I|^{\beta}$

Lemma

$\beta : \mathbb{R} \to \mathbb{R}$ is real analytic, decreasing, convex (strict if $\phi_0 / \log r_0 \neq \phi_1 / \log r_1$)

Recalling $d_{\nu_q}(\omega) = q d_\mu(\omega) + \beta$ and observing

$$\nu_q\left(\mathcal{D}_\mu(-\beta'(q))\right) = 1,$$

the Mass Distribution Principle implies

$$\dim_H \mathcal{D}_\mu(\alpha) = -\beta'(q) q + \beta(q) \text{ for } -\alpha = \beta'(q).$$

that is, $\dim_H \mathcal{D}_\mu(\cdot)$ is the (modified) Legendre-Fenchel transform of $\beta(\cdot)$.
Let $\beta : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be convex. Its Legendre-Fenchel transform is defined by

$$\beta^*(\alpha) \overset{\text{def}}{=} \sup_{q \in \mathbb{R}} (\alpha q - \beta(q)),$$

it is convex on its domain $I^* = \{\alpha : \beta^*(\alpha) < \infty\}$. In particular, convex $\beta$ is differentiable at all but at most countably many points and

$$\beta^*(\alpha) = \beta'(q) q - \beta(q) \quad \text{for} \quad \alpha = \beta'(q)$$

On the class of strictly convex functions, transform is involutive $\beta^{**} = \beta$. 
Multifractal analysis – local dimension:
Legendre-Fenchel transform

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Formally, the function $f_\mu(\alpha) \overset{\text{def}}{=} - \dim_H D_\mu(-\alpha)$ is the Legendre-Fenchel transform of $\beta$, but it is a common practice to address $D_\mu$ by this name

$$\dim_H D_\mu(\alpha) = \inf_{p \in \mathbb{R}} (\alpha p + \beta(p)) = -\beta'(q) q + \beta(q) \quad \text{for } \alpha = -\beta'(q).$$