Groups of Hölder diffeomorphisms.

1 - Overview. Let $X$ be a smooth, compact manifold. For all $k \in \mathbb{N}$ and for all $\alpha \in [0, 1]$, let $\text{Diff}^{k,\alpha}(X)$ denote the space of homeomorphisms of $X$ which are of Hölder class $(k, \alpha)$. Trivially, this is a group except when $k + \alpha \notin [0, 1]$. However, for almost all values of $k$ and $\alpha$, it is not a topological group. Indeed, it is straightforward to show that the operations of composition and inversion are only continuous when $k = \alpha = 0$. The purpose of this note is to show that continuity of these operations is recovered when $\text{Diff}^{k,\alpha}(X)$ is replaced by the space $\text{diff}^{k,\alpha}(X)$, defined to be the closure of $\text{Diff}^{\infty}(X)$ in $\text{Diff}^{k,\alpha}(X)$.

2 - Definitions. We recall the notation and terminology of Hölder norms and seminorms. Consider first two metric spaces $X$ and $Y$. For all $f : X \to Y$ and for all $\alpha \in [0, 1]$, the Hölder seminorm of $f$ of order $\alpha$ is defined by

$$[f]_\alpha := \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)^\alpha}. \quad (1)$$

In particular, $[f]_0$ is the total variation of $f$ and $[f]_1$ is its Lipschitz seminorm. The following log-concavity property will prove useful.

**Lemma 2.1**

For all $f : X \to Y$, and for all $\alpha, \beta, t \in [0, 1]$,

$$[f]_{t\alpha+(1-t)\beta} \leq [f]_\alpha^t[f]_\beta^{1-t}. \quad (2)$$

**Proof:** Indeed,

$$[f]_{t\alpha+(1-t)\beta} = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)^{(t\alpha+(1-t)\beta)}}$$

$$= \sup_{x \neq y} \left( \frac{d(f(x), f(y))}{d(x, y)^\alpha} \right)^t \left( \frac{d(f(x), f(y))}{d(x, y)^\beta} \right)^{(1-t)}$$

$$\leq [f]_\alpha^t[f]_\beta^{1-t},$$

as desired. □

Suppose now that $Y$ is a normed vector space. For all $f : X \to Y$, the uniform norm of $f$ is defined by

$$\|f\|_{C^0} := \sup_x \|f(x)\|. \quad (3)$$

In particular, it is trivially related to the total variation of $f$ by

$$[f]_0 \leq 2\|f\|_{C^0}. \quad (4)$$

For all $f : X \to Y$ and for all $\alpha \in [0, 1]$, the Hölder norm of $f$ of order $(0, \alpha)$ is defined by

$$\|f\|_{C^{0,\alpha}} := \|f\|_{C^0} + [f]_\alpha. \quad (5)$$
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Suppose finally that $X$ is an open subset of some normed vector space. The concept of derivatives of functions from $X$ into $Y$ can then be introduced. For all $k \in \mathbb{N}$, for all $\alpha \in [0, 1]$, and for every $k$-times differentiable function $f : X \to Y$, the Hölder norm of $f$ of order $(k, \alpha)$ is defined by

$$\|f\|_{C^k,\alpha} := \sum_{m=0}^{k} \|D^m f\|_{C^0} + [D^k f]_\alpha.$$  \hspace{1cm} (5)

In particular, the Hölder norms satisfy the following inductive formula

$$\|f\|_{C^{k+1},\alpha} = \|f\|_{C^0} + \|Df\|_{C^k,\alpha}.$$

For all $(k, \alpha)$, the Hölder space of order $(k, \alpha)$, denoted by $C^{k,\alpha}(X,Y)$, is defined to be the space of all $k$-times differentiable functions $f : X \to Y$ such that $\|f\|_{C^{k,\alpha}} < \infty$. It is straightforward to show that this space is non-separable and that $C^{\infty}(X,Y)$ is not a dense subset even when $X$ is compact. For this reason, for all $(k, \alpha)$, the little Hölder space of order $(k, \alpha)$, denoted by $c^{k,\alpha}(X,Y)$, is defined to be the closure of $C^{\infty}(X,Y)$ in $C^{k,\alpha}(X,Y)$. In the special case where $\alpha = 1$, we have, for all $k$,

$$c^{k,1}(X,Y) = C^{k+1}(X,Y).$$

3 - Multilinear maps. In this section, $E_1, \ldots, E_m$ and $F$ will be normed vector spaces, and $\mu : E_1 \oplus \ldots \oplus E_m \to F$ will be a bounded, multilinear map. We first suppose that $X$ is a metric space.

Lemma 3.1

The map

$$C^0(X, E_1) \oplus \ldots \oplus C^0(X, E_m) \to C^0(X, F); (f_1, \ldots, f_m) \mapsto \mu(f_1, \ldots, f_m)$$

is a continuous, multilinear map of norm bounded by $\|\mu\|$.

Proof: Indeed, for all $f_1, \ldots, f_m$,

$$\|\mu(f_1, \ldots, f_m)\|_{C^0} = \sup_{x \in X} \|\mu(f_1, \ldots, f_m)(x)\|$$

$$= \sup_{x \in X} \|\mu(f_1(x), \ldots, f_m(x))\|$$

$$\leq \sup_{x \in X} \|\mu\| \|f_1(x)\| \ldots \|f_m(x)\|$$

$$\leq \|\mu\| \|f_1\|_{C^0} \ldots \|f_m\|_{C^0},$$

and the result follows. □
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Lemma 3.2

For all $\alpha \in [0, 1]$, the map

$$C^{0,\alpha}(X, E_1) \oplus \ldots \oplus C^{0,\alpha}(X, E_m) \to C^{0,\alpha}(X, F); (f_1, \ldots, f_m) \mapsto \mu(f_1, \ldots, f_m)$$

is a continuous, multilinear map of norm bounded by $\|\mu\|$.

Proof: It suffices to consider the case where $m = 2$. For all $f_1, f_2$, and for all $\alpha \in [0, 1]$,

$$[\mu(f_1, f_2)]_\alpha = \sup_{x \neq y} \frac{\|\mu(f_1, f_2)(x) - \mu(f_1, f_2)(y)\|}{d(x, y)^\alpha}$$

$$= \sup_{x \neq y} \frac{\|\mu(f_1(x), f_2(x)) - \mu(f_1(y), f_2(y))\|}{d(x, y)^\alpha}$$

$$\leq \sup_{x \neq y} \frac{\|\mu(f_1(x), f_2(x)) - \mu(f_1(y), f_2(x))\|}{d(x, y)^\alpha}$$

$$+ \sup_{x \neq y} \frac{\|\mu(f_1(x), f_2(x)) - \mu(f_1(y), f_2(y))\|}{d(x, y)^\alpha}$$

$$\leq \|\mu\|[f_1]_\alpha ||f_2||_{C^0} + \|\mu\| ||f_1||_{C^\alpha} [f_2]_\alpha,$$

and since

$$\|\mu(f_1, f_2)||_{C^0} \leq \|\mu\| ||f_1||_{C^\alpha} ||f_2||_{C^0},$$

it follows that

$$\|\mu(f_1, f_2)||_{C^{0,\alpha}} \leq \|\mu\| ||f_1||_{C^{0,\alpha}} ||f_2||_{C^{0,\alpha}},$$

as desired. □

Suppose now that $X$ is an open subset of a normed vector space.

Lemma 3.3

For all $k \in \mathbb{N}$ and for all $\alpha \in [0, 1]$, the map

$$C^{k,\alpha}(X, E_1) \oplus \ldots \oplus C^{k,\alpha}(X, E_m) \to C^{k,\alpha}(X, F); (f_1, \ldots, f_m) \mapsto \mu(f_1, \ldots, f_m)$$

is a continuous, multilinear map of norm bounded by $m^k \|\mu\|$.

Proof: It suffices to consider the case where $m = 2$. We prove this result by induction on $k$. The case where $k = 0$ follows from Lemma 3.2. Denote by $E$ the normed vector space in which $X$ is contained and define the continuous bilinear maps $\mu_1 : \text{Lin}(E, E_1) \oplus E_2 \to \text{Lin}(E, F)$ and $\mu_2 : E_1 \oplus \text{Lin}(E, E_2) \to \text{Lin}(E, F)$ by

$$\mu_1(A, V)(U) := \mu(A(U), V),$$

and

$$\mu_2(U, A)(V) := \mu(U, A(V)).$$
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Consider now \( f_1 \in C^{k+1,\alpha}(X, E_1) \) and \( f_2 \in C^{k+1,\alpha}(X, E_2) \). By the chain rule,
\[
D\mu(f_1, f_2) = \mu_1(Df_1, f_2) + \mu_2(f_1, Df_2),
\]
so that, by the inductive hypothesis,
\[
\|D\mu(f_1, f_2)\|_{C^k,\alpha} \leq 2^k \|\mu\| \left( \|Df_1\|_{C^k,\alpha} \|f_2\|_{C^k,\alpha} + \|f_1\|_{C^k,\alpha} \|Df_2\|_{C^k,\alpha} \right).
\]

However, since \( \|\mu(f_1, f_2)\|_{C^0} \leq \|\mu\| \|f_1\|_{C^0} \|f_2\|_{C^0} \),

it follows that
\[
\|\mu(f_1, f_2)\|_{C^{k+1,\alpha}} \leq 2^{k+1} \|\mu\| \|f_1\|_{C^{k+1,\alpha}} \|f_2\|_{C^{k+1,\alpha}},
\]
as desired. \(\square\)

4 - Composition. In this section, \( X \) will be a metric space. We suppose first that \( Y \) and \( Z \) are also metric spaces and that \( Y \) is locally compact.

Lemma 4.1

The composition map
\[
C^0(X, Y) \times C^0(Y, Z) \to C^0(X, Z); (f, g) \mapsto g \circ f
\]
is continuous.

Proof: We prove this result using the compact-open topology. Consider an element \((f, g)\) of \( C^0(X, Y) \times C^0(Y, Z) \). Let \( U \) be an open subset of \( Z \) and let \( K \) be a compact subset of \( X \) such that \((g \circ f)(K) \subseteq U\). In particular, \( f(K) \subseteq g^{-1}(U) \). Since \( f(K) \) is compact and since \( Y \) is locally compact, there exists a relatively compact, open subset \( V \) of \( Y \) such that \( K \subseteq V \) and \( g(V) \subseteq U \). Define now the neighbourhoods \( \mathcal{U} \) and \( \mathcal{V} \) of \( f \) and \( g \) respectively by
\[
\mathcal{U} := \{ f' \in C^0(X, Y) \mid f(K) \subseteq V \} \text{ and } \\
\mathcal{V} := \{ g' \in C^0(X, Y) \mid g(V) \subseteq U \}.
\]

For all \((f', g') \in \mathcal{U} \times \mathcal{V}\), \((g' \circ f')(K) \subseteq U\), and the result follows. \(\square\)

We henceforth suppose that \( Y \) and \( Z \) are subsets of normed vector spaces.

Lemma 4.2

For all \( \alpha, \beta \in [0, 1] \), and for all \((f, g) \in C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z)\),
\[
[g \circ f]_{\alpha\beta} \leq [g]_{\beta}[f]_{\alpha}^\beta.
\]

In particular,
\[
\|g \circ f\|_{C^{0,\alpha\beta}} \leq \|g\|_{C^{0,\alpha}} \left( 1 + [f]_{\alpha}^\beta \right).
\]

Remark: It follows that pre-composition by an element of \( C^{0,\alpha}(X, Y) \) defines a bounded linear map from \( C^{0,\beta}(Y, Z) \) to \( C^{0,\alpha\beta}(X, Z) \).
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**Proof:** Indeed,
\[
[g \circ f]_\alpha = \sup_{x \neq y} \frac{\|(g \circ f)(x) - (g \circ f)(y)\|}{d(x, y)^{\alpha \beta}} \\
\leq [g]_\beta \sup_{x \neq y} \frac{d(f(x), f(y))^{\beta}}{d(x, y)^{\alpha \beta}} \\
= [g]_\beta [f]_\alpha^{\beta},
\]
as desired. □

**Lemma 4.3**

For all $\alpha, \beta, \gamma \in [0, 1]$ such that $\gamma < \alpha \beta$, the composition map

\[ C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z) \to C^{0,\gamma}(X, Z); (f, g) \mapsto g \circ f \]

is continuous.

**Proof:** Indeed, consider a sequence $(f_m, g_m)$ converging in $C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z)$ to $(f_\infty, g_\infty)$. By Lemma 4.1,

\[
\lim_{m \to +\infty} \|(g_m \circ f_m) - (g_\infty \circ f_\infty)\|_{C^0} = 0.
\]

By (6), there exists $B > 0$ such that, for all $m$,

\[
[(g_m \circ f_m) - (g_\infty \circ f_\infty)]_{\alpha \beta} \leq [g_m \circ f_m]_{\alpha \beta} + [g_\infty \circ f_\infty]_{\alpha \beta} \leq B.
\]

By (2) with $t := \gamma/\alpha \beta$,

\[
[(g_m \circ f_m) - (g_\infty \circ f_\infty)]_{\gamma} \leq B^t [(g_m \circ f_m) - (g_\infty \circ f_\infty)]_{0}^{(1-t)}
\]

and since this tends to zero as $m$ tends to infinity, the result follows. □

**Lemma 4.4**

For all $\alpha \in [0, 1]$ and for all $\beta \in [0, 1]$, the composition map

\[ C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z) \to C^{0,\alpha \beta}(X, Z); (f, g) \mapsto g \circ f \]

is continuous.

**Proof:** Indeed, consider a sequence $(f_m, g_m)$ converging in $C^{0,\alpha}(X, Y) \times C^{0,\beta}(Y, Z)$ to $(f_\infty, g_\infty)$. For all $m$,

\[
[(g_m \circ f_m) - (g_\infty \circ f_\infty)]_{\alpha \beta} \leq [(g_\infty \circ f_m) - (g_\infty \circ f_\infty)]_{\alpha \beta} + [(g_m - g_\infty) \circ f_m]_{\alpha \beta}.
\]

By (6), the second term on the right hand side satisfies

\[
[(g_m - g_\infty) \circ f_m]_{\alpha \beta} \leq [g_m - g_\infty]_{\beta} [f_m]_{\alpha}^{\beta},
\]

and since this tends to zero as $m$ tends to infinity, the result follows. □
which tends to zero as \( m \) tends to infinity. Consider now the first term on the right hand side. Let \( B > 0 \) be such that, for all \( m \in \mathbb{N} \cup \{\infty\} \),

\[
[f_m]_\alpha \leq B.
\]

Now choose \( \epsilon > 0 \) and choose \( h \in C^\infty(Y,Z) \) such that

\[
[h - g_\infty]_\beta \leq \epsilon/3B^\beta.
\]

Using (6) again, we obtain

\[
\begin{align*}
\left[(g_\infty \circ f_m) - (g_\infty \circ f_\infty)\right]_\alpha &\leq \left[(h - g_\infty) \circ f_m\right]_\alpha \beta \\
&\quad + \left[(h \circ f_\infty) - (h \circ f_\infty)\right]_\alpha \beta \\
&\quad + \left[(h - g_\infty) \circ f_\infty\right]_\alpha \beta \\
&= \left[(h \circ f_m) - (h \circ f_\infty)\right]_\alpha \beta \\
&\quad + [h - g_\infty]_\beta [f_m]_\alpha^\beta \\
&\quad + [h - g_\infty]_\beta [f_\infty]_\alpha^\beta \\
&\leq \left[(h \circ f_m) - (h \circ f_\infty)\right]_\alpha \beta + 2\epsilon/3.
\end{align*}
\]

Since \( h \in C^{0,1}(Y,Z) \), it follows by Lemma 4.3 that, for sufficiently large \( m \),

\[
\left[(h \circ f_m) - (h \circ f_\infty)\right]_\alpha \beta \leq \epsilon/3,
\]

so that

\[
\left[(g_\infty \circ f_m) - (g_\infty \circ f_\infty)\right]_\alpha \beta \leq \epsilon.
\]

Since \( \epsilon \) may be chosen arbitrarily small, the first term on the right hand side also tends to zero as \( m \) tends to infinity, and this completes the proof. \( \Box \)

The case where \( \beta = 1 \) is treated separately. Although it is not strictly necessary for our purposes, we include it for completeness.

**Lemma 4.5**

*If \( Y \) is convex and compact then, for all \( \alpha \in [0,1] \), the composition map*

\[
C^{0,\alpha}(X,Y) \times C^1(Y,Z) \to C^{0,\alpha}(X,Z); (f,g) \mapsto g \circ f
\]

*is continuous.*

**Proof:** Let \( E \) and \( F \) denote the normed vector spaces containing \( Y \) and \( Z \) respectively. Suppose furthermore that \( F \) is complete, so that the integral of continuous curves in \( F \) is well defined. Now let \((f_m, g_m)\) be a sequence converging in \( C^{0,\alpha}(X,Y) \times C^1(Y,Z) \) to \((f_\infty, g_\infty)\). Bearing in mind that \( Y \) is convex, for all \( m \in \mathbb{N} \cup \{\infty\} \), we define \( A_m : X \times X \to \text{Lin}(E,F) \) by

\[
A_m(x,y) := \int_0^1 Dg_m((1-t)f_m(x) + tf_m(y))dt.
\]
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It follows by compactness of $Y$ that the sequence $(A_m)$ converges uniformly to $A_\infty$ as $m$ tends to infinity. Now, for all $m$,

$$
[g \circ f_m - g \circ f_\infty]_\alpha = \sup_{x \neq y} \frac{\|A_m(x, y)(f_m(y) - f_m(x)) - A_\infty(x, y)(f_\infty(y) - f_\infty(x))\|}{d(x, y)\alpha}
$$

$$
\leq \sup_{x \neq y} \frac{\|A_m(x, y)(f_m(y) - f_m(x)) - f_\infty(y) + f_\infty(x))\|}{d(x, y)\alpha}
$$

$$
+ \sup_{x \neq y} \frac{\|(A_m(x, y) - A_\infty(x, y))(f_\infty(y) - f_\infty(x))\|}{d(x, y)\alpha}
$$

and since this tends to zero as $m$ tends to infinity, the result follows. □

Finally, we suppose that $X$ is an open subset of a normed vector space.

**Lemma 4.6**

For all $k \geq 1$, and for all $\alpha \in [0, 1]$, the composition map

$$C^{k,\alpha}(X, Y) \times C^{k,\alpha}(Y, Z) \to C^{k,\alpha}(X, Z); (f, g) \mapsto g \circ f$$

is continuous.

**Proof:** Since $C^{k,1} = C^{k+1}$, the case where $\alpha = 1$ follows by a straightforward argument of elementary calculus. We therefore suppose that $\alpha < 1$, and we prove this result by induction in $k$. Consider a sequence $(f_m, g_m)$ converging to $(f_\infty, g_\infty)$ in $C^{k,\alpha}(X, Y) \times C^{k,\alpha}(Y, Z)$. By Lemma 4.1, the sequence $(g_m \circ f_m)$ converges to $(g_\infty \circ f_\infty)$ in $C^0(X, Z)$. By the chain rule, for all $m \in \mathbb{N} \cup \{\infty\}$,

$$D(g_m \circ f_m) = (Dg_m \circ f_m) Df_m.$$

Denote by $E$ and $F$ the normed vector spaces containing $X$ and $Z$ respectively. If $k > 1$, then it follows by the inductive hypothesis that the sequence $(Dg_m \circ f_m)$ converges to $(Dg_\infty \circ f_\infty)$ in $C^{k-1,\alpha}(X, \text{Lin}(E, F))$. Otherwise, if $k = 1$, then this property follows by Lemma 4.4. In each case, by Lemma 3.3, the sequence $(D(g_m \circ f_m))$ converges to $D(g_\infty \circ f_\infty)$ in $C^{0,\alpha}(X, \text{Lin}(E, F))$, and we conclude that the sequence $(g_m \circ f_m)$ converges to $(g_\infty \circ f_\infty)$ in $C^{k,\alpha}(X, Z)$, as desired. □

Suppose now that $X$ is a smooth, compact, embedded submanifold of some finite-dimensional vector space, and observe that the above results continue to hold in this case. Let $\text{diff}^{k,\alpha}(X)$ denote the space of diffeomorphisms of $X$ which are of type $c^{k,\alpha}$. Setting $Z = Y = X$, Lemma 4.6 immediately yields

**Theorem 4.7**

For all $k \geq 1$, and for all $\alpha \in [0, 1]$, the composition map

$$\text{diff}^{k,\alpha}(X) \times \text{diff}^{k,\alpha}(X) \to \text{diff}^{k,\alpha}(X); (f, g) \mapsto g \circ f$$

is continuous.

We conclude by proving continuity of the inversion map. First, we have
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**Lemma 4.8**

The inversion map

\[
\text{homeo}(X) \rightarrow \text{homeo}(X); \phi \rightarrow \phi^{-1}
\]

is continuous.

**Proof:** Indeed, consider a sequence \((\phi_m)\) converging in \(\text{Homeo}(X)\) to \(\phi_\infty\). Let \(U\) and \(K\) be respectively an open and a compact subset of \(X\) such that \(\phi_\infty^{-1}(K) \subseteq U\). In particular \(\phi_\infty(U^c) \subseteq K^c\) so that, since \(U^c\) is compact and \(K^c\) is open, for sufficiently large \(m\), \(\phi_m(U^c) \subseteq K^c\). It follows that, for sufficiently large \(m\), \(\phi_m^{-1}(K) \subseteq U\) and so \((\phi_m^{-1})\) converges to \(\phi_\infty^{-1}\) in the compact-open topology, as desired. \(\square\)

**Theorem 4.9**

For all \(k \geq 1\) and for all \(\alpha \in [0, 1]\), the inversion map

\[
\text{diff}^{k,\alpha}(X) \rightarrow \text{diff}^{k,\alpha}(X); \phi \mapsto \phi^{-1}
\]

is continuous.

**Proof:** We prove this by induction on \(k\). Consider a sequence \((\phi_m)\) converging in \(\text{diff}^{k,\alpha}(X)\) to \(\phi_\infty\) and for all \(m \in \mathbb{N} \cup \{\infty\}\) denote \(\psi_m := \phi_m^{-1}\). By Lemma 4.8, \((\psi_m)\) converges to \(\psi_\infty\) in the \(C^0\) sense. By the chain rule, for all \(m \in \mathbb{N} \cup \{\infty\}\),

\[
D\psi_m = (D\phi_m)^{-1} \circ \psi_m.
\] (7)

We now claim that \((\psi_m)\) converges to \(\psi_\infty\) in the \(C^{k-1,\alpha}\) sense. Indeed, when \(k > 1\), this follows by the inductive hypothesis. Otherwise, when \(k = 1\), we first observe that (7) implies that \((D\psi_m)\) converges towards \(D\psi_\infty\) in the \(C^0\) sense. It follows that \((\psi_m)\) converges to \(\psi_\infty\) in the \(C^1\) sense, and therefore also in the \(C^{0,\alpha}\) sense, as asserted. In each case, it follows by Lemma 4.6 that \((D\psi_m)\) converges towards \(D\psi_\infty\) in the \(C^{k-1,\alpha}\) sense, and so \((\psi_m)\) converges towards \(\psi_\infty\) in the \(C^{k,\alpha}\) sense, as desired. \(\square\)