Comparison Theorems.

1 - The fundamental theorem of calculus. A function $f : [a,b] \to \mathbb{R}$ is said to be *absolutely continuous* whenever, for every $\epsilon > 0$, there exists $\delta > 0$ such that if

$$a \leq x_1 < x_2 < ... < x_{2n-1} < x_{2n} \leq b$$

is an increasing sequence in $[a,b]$ with

$$\sum_{k=1}^{n} (x_{2k} - x_{2k-1}) < \delta,$$

then

$$\sum_{k=1}^{n} |f(x_{2k}) - f(x_{2k-1})| < \epsilon.$$

Every absolutely continuous function can be written (non-uniquely) as the difference between two non-decreasing, absolutely continuous functions. Furthermore, any given non-decreasing function over $[a,b]$ is absolutely continuous if and only if the measure that it defines over this interval is absolutely continuous with respect to Lebesgue measure. From this is readily follows by the Radon-Nikodym theorem that the absolutely continuous functions over $[a,b]$ are precisely those functions which are integrals of elements of $L^1([a,b])$. More precisely, we have the following version of the fundamental theorem of calculus.

Theorem 1.1

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous then

(a) f is almost everywhere differentiable;

(b) its derivative f' is an element of $L^1([a,b])$; and

(c) the integral of f' differs from f by a constant.

Conversely, if g is an element of $L^1([a,b])$ then

(d) the integral f of g is absolutely continuous; and

(e) the derivative of f is almost everywhere equal to g.

Remark: In particular, all Lipschitz functions are also absolutely continuous. Indeed, they are precisely those functions which are integrals of elements of $L^\infty([a,b])$.

2 - Geodesic charts. Let $\mathbb{R}^d$ denote d-dimensional euclidian space and let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote respectively its canonical metric and norm. For all $\rho > 0$, let $B(\rho)$ denote the ball of radius ρ about the origin in $\mathbb{R}^d$. Let g be a smooth metric over $B(\rho)$ such that

(a) $g(0)_{ij} = \delta_{ij}$; and

(b) for every (euclidean) unit vector x, the curve $\gamma(t) := tx$ is a unit speed geodesic of g.

These conditions are equivalent to specifying that $(B(\rho), g)$ be a geodesic chart of some riemannian manifold.

Let R denote the Riemann curvature tensor of g. Let D and ∇ denote the Levi-Civita covariant derivatives of the euclidean metric and of g respectively. Let $\Gamma := \nabla - D$ denote the relative Christoffel symbol of ∇ with respect to D, that is, for all vector fields X and Y, and for all $x \in B(\rho)$,

$$\Gamma(x)(X,Y) = \big(\nabla_X Y - D_X Y\big)(x).$$

Let ∂_r denote the unit radial vector field in $B(\rho)$. Up to a linear isomorphism of $\mathbb{R}^d$, properties (a) and (b) are equivalent to

(a′) $\Gamma(0) = 0$; and

(b′) $\nabla_{\partial_r}\partial_r = 0$.

Given $r \in]0,\rho[$, let S_r denote the euclidian sphere of radius r about the origin and let A_r denote its Weingarten operator with respect to g. Observe that the restriction of ∂_r to S_r coincides with the outward-pointing unit normal vector field over this sphere with respect to both g and the euclidian metric. In particular, for all r, if A_r denotes the shape operator of S_r, then for every vector X tangent to this sphere,

$$A_r X = \nabla_X \partial_r. \tag{1}$$

Let A denote the endomorphism field defined over $B(\rho) \setminus \{0\}$ such that

(c) for all r, the restriction of A to the tangent bundle of S_r coincides with A_r; and

(d) $A\partial_r = 0$.

Observe that A is at every point symmetric with respect to g.

Lemma 2.1

As x tends to 0, $A(x)$ satisfies

$$A(x)X = \frac{1}{r}\big(X - \langle X, \partial_r\rangle \partial_r\big) + \mathrm{O}(r\|X\|) \tag{2}$$

for every X in $\mathbb{R}^{n+1}$ where here $r = \|x\|$.

Proof: It suffices to consider the case where X is tangent to TS_r. However, by (c), $\Gamma = \mathrm{O}(r)$ so that

$$A(x)X = \nabla_X \partial_r = D_X \partial_r + \mathrm{O}(r\|X\|) = \frac{1}{r}X + \mathrm{O}(r\|X\|),$$

as desired. $\square$

Lemma 2.2

For every bounded vector field X over $B(R)\setminus\{0\}$,

$$\big(\nabla_{\partial_r}A + A^2\big)X = R_{\partial_r X}\partial_r. \tag{3}$$

Proof: Let x be a euclidean unit vector in $\mathbb{R}^d$ and define $\gamma : [0,R[\to B(R)$ by $\gamma(t) = tx$. Let $X : [0,R[\to \mathbb{R}^d$ be a vector field along γ which is parallel with respect to g. Since ∂_r is also parallel and since (3) trivially holds with $X = \partial_r$, we may suppose that X is normal to γ and tangent to S_r for all r. We now have

$$\begin{aligned}\big(\nabla_{\partial_r}A\big)X &= \nabla_{\partial_r}(AX)\\ &= \nabla_{\partial_r}\nabla_X\partial_r\\ &= R_{\partial_r X}\partial_r + \nabla_X\nabla_{\partial_r}\partial_r + \nabla_{[\partial_r,X]}\partial_r\\ &= R_{\partial_r X}\partial_r + \nabla_{[\partial_r,X]}\partial_r.\end{aligned}$$

However, since X is parallel along γ,

$$[\partial_r, X] = \nabla_{\partial_r}X - \nabla_X\partial_r = -\nabla_X\partial_r = -A_rX,$$

so that

$$\big(\nabla_{\partial_r}A\big)X = R_{\partial_r X}\partial_r - A^2X,$$

and the result follows. □

3 - The infinitesimal comparison theorem. For $k \in \mathbb{R}$, define the function c_k by

$$c_k(r) := \begin{cases}\sqrt{k}\cot(\sqrt{k}r) & \text{if } k > 0,\\ 1/r & \text{if } k = 0,\\ \sqrt{|k|}\coth(\sqrt{|k|}r) & \text{if } k < 0,\end{cases} \tag{4}$$

This is the unique solution to the problem

$$\begin{aligned}c' + c^2 &= -k,\\ \operatorname*{Lim}_{r\to 0} rc(r) &= 1.\end{aligned}$$

Theorem 3.1

If the sectional curvature of g is bounded below by k, then, for all $r \in]0,\rho[$,

$$A_r \leq c_k(r)\mathrm{Id}. \tag{5}$$

Likewise, if the sectional curvature of g is bounded above by k, then, for all $r \in]0,\rho[$,

$$A_r \geq c_k(r)\mathrm{Id}. \tag{6}$$

Proof: It suffices to prove (5) since the proof of (6) is identical. Let x be a (euclidean) unit vector in $\mathbb{R}^d$ and define $\gamma : [0,\rho[\rightarrow B(\rho)$ by $\gamma(r) = rx$. Define $\lambda :]0,\rho[\rightarrow \mathbb{R}$ such that, for all r, $\lambda(r)$ is the greatest eigenvalue of $A_r(\gamma(r))$. By (2), as r tends to zero,

$$\lambda(r) = \frac{1}{r} + \mathrm{O}(r),$$

Since A is smooth, λ is locally Lipschitz over $]0,\rho[$. It follows by Theorem 1.1 that λ is almost everywhere differentiable, that its derivative is locally L^1, and that it is equal to the integral of its derivative.

We claim that at every point where λ is differentiable,

$$\lambda'(r) + \lambda(r)^2 \leq -k. \tag{7}$$

Indeed, let r_0 be such a point. Let X be a unit eigenvector of $A_{r_0}(\gamma(r_0))$ with eigenvalue $\lambda(r_0)$. Extend X to a vector field over γ which is parallel with respect to g. Then,

$$\lambda(r_0) = g(\gamma(r_0))(A_r(\gamma(r_0))X, X),$$

whilst, for all $r \in]0,\rho[$,

$$\lambda(r) \geq g(\gamma(r))(A_r(\gamma(r))X, X).$$

Since λ is differentiable at r_0, we have

$$\lambda'(r_0) = \partial_r g(\gamma(r))(A_r(\gamma(r))X, X)|_{r=r_0} = g\big(\big(\nabla_{\partial_r} A\big)X, X\big)|_{r=r_0},$$

and (7) now follows by (3) and the hypotheses on the sectional curvature.

There are many ways to complete the proof from the differential inequality (7) which, we recall, is satisfied for almost all r. The following argument, which only treats the case where $k = -1$, is illustrative of the general idea. Though not the quickest approach, it also provides a nice illustration of the projective nature of Ricatti equations.

Consider first the second order linear equation

$$\phi''(r) - \phi(r) = 0. \tag{8}$$

Its general solution is

$$\phi_{a,b}(r) = a\mathrm{cosh}(R) + b\mathrm{sinh}(r),$$

where a and b are constants. Define $\Phi : \mathbb{RP}^1 \times [0,\rho[\rightarrow \mathbb{RP}^1$ by

$$\Phi([a:b], r) := [\phi_{a,b}(r) : \phi'_{a,b}(r)] = [a\mathrm{cosh}(r) + b\mathrm{sinh}(r) : a\mathrm{sinh}(r) + b\mathrm{cosh}(r)],$$

and for all r denote

$$\Phi_r(\cdot) := \Phi(\cdot, r).$$

Define $\Psi : \mathbb{RP}^1 \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\Psi([a:b]) := \begin{cases} b/a & \text{if } a \neq 0, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

The functions of the form

$$\psi(r) = (\Psi \circ \Phi_r)([a:b]) = \frac{a\mathrm{sinh}(r) + b\mathrm{cosh}(r)}{a\mathrm{cosh}(r) + b\mathrm{sinh}(r)}$$

where $[a:b] \in \mathbb{RP}^1$ are precisely the solutions of the Ricatti equation

$$\phi'(r) + \phi(r)^2 = -1. \tag{9}$$

Consider now the function

$$\tilde{\lambda}(r) := \big(\Psi \circ M \circ \Phi_r^{-1} \circ \Psi^{-1}\big)(\lambda(r))$$

where $M : \mathbb{RP}^1 \to \mathbb{RP}^1$ is the projective rotation

$$M[a:b] = [-b:a].$$

This function is locally Lipschitz and at every point where it is differentiable we have

$$\tilde{\lambda}'(r) \leq 0.$$

Since, in addition,

$$\mathop{\mathrm{Lim}}_{r\rightarrow 0} \tilde{\lambda}(r) = 0,$$

it follows that

$$\lambda(r) \leq \big(\Psi \circ \Phi_r \circ M^{-1} \circ \Psi^{-1}\big)(0) = \mathrm{coth}(r)$$

and the result follows. □

4 - The Rauch comparison theorem. For all $k \in \mathbb{R}$, define the function s_k by

$$s_k(r) := \begin{cases} \sqrt{k}\mathrm{sin}(\sqrt{k}r) & \text{if } k > 0, \\ r & \text{if } k = 0, \\ \sqrt{|k|}\mathrm{sinh}(\sqrt{|k|}r) & \text{if } k < 0, \end{cases} \tag{10}$$

and define the *model metric* g_k over $B(\rho)$ by

$$g_k(r) := dr^2 + s_k(r)^2 d\theta^2. \tag{11}$$

For all k, the functions s_k and c_k are related by

$$\partial_r \mathrm{Log}(s_k) = c_k.$$

Theorem 4.1, Rauch

If the sectional curvature of g is bounded below by k, then

$$g \leq g_k. \tag{12}$$

Likewise, if the sectional curvature of g is bounded above by k, then

$$g \geq g_k. \tag{13}$$

Proof: It suffices to prove (12) as the proof of (13) is identical. It likewise suffices to prove the result for vectors orthogonal to ∂_r. Let X be a tangent vector to S_1 and define $\phi : [0, \rho[\rightarrow]0, \infty[$ by

$$\phi^2(r) = g(\gamma(r))(rX, rX).$$

As r tends to zero, we have

$$\phi^2(r) = r^2\|X\|^2 + \mathrm{O}(r^4\|X\|).$$

Since $[\partial_r, rX] = 0$, and bearing in mind (5),

$$\begin{aligned} 2\phi\phi' &= 2g(\gamma(r))(\nabla_{\partial_r}(rX), rX) \\ &= 2g(\gamma(r))(\nabla_{rX}\partial_r, rX) \\ &= 2g(\gamma(r))(A_r(rX), rX) \\ &\leq 2c_k(r)\phi^2, \end{aligned} \tag{14}$$

so that

$$\partial_r\big(\mathrm{Log}(\phi) - \mathrm{Log}(s_k(r))\big) \leq 0,$$

and the result follows. $\square$

We also establish necessary and sufficient conditions for equality at a single point. For any point $x \in B(\rho)$, let $[0, x]$ denote the straight line segment from 0 to x.

Theorem 4.2

Suppose that one of the two hypotheses of Theorem 4.1 holds. Suppose that the point $x \in B(\rho)$ and the vector $X \perp \partial_r$ satisfy

$$g(x)(X, X) = g_k(x)(X, X).$$

Then, for every point $y \in [0, x]$,

(1) $A(y)X = c_k(\|y\|)X$; and

(2) $g(y)(R_{\partial_r X}X, \partial_r) = kg(y)(X, X)$.

In particular, X and ∂_r are respectively eigenvectors of $R_{\cdot\partial_r}\partial_r$ and $R_{\cdot X}X$.

Proof: Indeed, equality in (14) for all r implies that, for almost all $y \in [0, x]$,

(1) $A(y)X = c_k(\|y\|)X$; and

(2) $g(y)(X,X) = g_k(y)(X,X)$.

By continuity, these relations in fact hold for all $y \in [0,x]$. In particular, that the vector field

$$\tilde{X}(r) := \frac{r}{s_k(r)}X$$

has constant length with respect to g. Consequently,

$$g(y)\big(\nabla_{\partial_r}\tilde{X},\tilde{X}\big) = \frac{1}{2}\partial_r g(y)\big(\tilde{X},\tilde{X}\big) = 0,$$

and since $\tilde{X}$ is an eigenvector of the symmetric matrix $A(y)$,

$$g(y)\big(A(y)\nabla_{\partial_r}\tilde{X},\tilde{X}\big) = g(y)\big(\nabla_{\partial_r}\tilde{X},A(y)\tilde{X}\big) = 0.$$

Thus

$$g(y)\big((\nabla_{\partial_r}A(y))\tilde{X},\tilde{X}\big) = \partial_r g(y)\big(A(y)\tilde{X},\tilde{X}\big) = c_k'(r)g(y)\big(\tilde{X},\tilde{X}\big).$$

Substituting (3) into this relation yields

$$g(y)\big(R_{\partial_r X}\partial_r,X\big) - g(y)\big(A(y)^2X,X\big) = c_k'(r)g(y)\big(X,X\big),$$

so that

$$\begin{aligned} g(y)\big(R_{\partial_r X}\partial_r,X\big) &= \big(c_k(r)^2 + c_k'(r)\big)g(y)\big(X,X\big) \\ &= -kg(y)\big(X,X\big), \end{aligned}$$

and the result follows. □

5 - The Topogonov comparison theorem. Given two points x and y in $B(\rho)$, let $[x,y]_g$ denote the unique g-length minimizing geodesic from x to y in $B(\rho)$ whenever this geodesic exists. In the case where $g = g_k$, denote $[x,y]_k := [x,y]_g$.

Theorem 5.1, Topogonov

Let x and y be points in $B(\rho)$ such that both $[x,y]_g$ and $[x,y]_k$ exist. If the sectional curvature of g is bounded below by k, then

$$l_g([x,y]_g) \leq l_k([x,y]_k), \tag{15}$$

and if the sectional curvature of g is bounded above by k, then

$$l_g([x,y]_g) \geq l_k([x,y]_k). \tag{16}$$

Furthermore, in each case, we have equality if and only if the curves $[0,x]_g$, $[0,y]_g$ and $[x,y]_g$ bound a totally geodesic triangle of constant curvature equal to k.

Proof: It suffices to prove (15) as the proof of (16) is almost identical. By (12),

$$l_g([x,y]_k) \leq l_k([x,y]_k).$$

Since $[x,y]_g$ is the unique g-length minimising geodesic from x to y,

$$l_g([x,y]_g) \leq l_g([x,y]_k) \leq l_k([x,y]_k),$$

as asserted. Suppose now that equality holds. Then, by uniqueness of the length minimiser,

$$\Gamma := [x,y]_g = [x,y]_k.$$

In particular, the triangle formed by the curves $[0,x]_g$, $[0,y]_g$ and $[x,y]_g$ is contained in the euclidian plane P spanned by x and y.

It remains to show that the portion T of P contained inside this triangle is totally geodesic, for then the curvature of this surface is given by Theorem 4.2. To this end, let $\phi :]0,1[\rightarrow \Gamma$ be a constant speed parametrisation and define the parametrisation $\Phi :]0,1[\times]0,1[\rightarrow T$ by

$$\Phi(s,t) := s\gamma(t).$$

Since curves in the s direction are constant speed radial lines,

$$\nabla_{\partial_s}\partial_s = 0.$$

Next, bearing in mind Theorem 4.2 and the fact that $[\partial_s, \partial_t] = 0$,

$$\nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s = a\partial_s + b\partial_t,$$

for suitable functions a and b. By Theorem 4.2 again

$$\begin{aligned}\nabla_{\partial_s}\nabla_{\partial_t}\partial_t &= R_{\partial_s\partial_t}\partial_t + \nabla_{\partial_t}\nabla_{\partial_s}\partial_t\\ &= k\|\partial_t\|^2\partial_s + \nabla_{\partial_t}(a\partial_s + b\partial_t)\\ &= \big(k\|\partial_t\|^2 + a_t + a^2\big)\partial_s + \big(ab + b_t\big)\partial_t + b\nabla_{\partial_t}\partial_t.\end{aligned}$$

Finally, since Γ is a geodesic,

$$\nabla_{\partial_t}\partial_t|_{s=1} = 0,$$

so that

$$\nabla_{\partial_t}\partial_t = f(s)\partial_s + g(s)\partial_t,$$

where (f,g) is the unique solution to the linear system of ODEs

$$\begin{aligned}f' &= bf - ag + k\|\partial_t\|^2 + a_t + a^2,\\ g' &= ab + b_t,\end{aligned}$$

with initial conditions

$$f(1) = g(1) = 0.$$

It follows that for all tangent vector fields X and Y over T, $\nabla_X Y$ is tangent to T, so that T is totally geodesic, and this completes the proof. □