On singular perturbations of the Morse complex.

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Morse Theory

Let M be a compact manifold.

Let $h : M \to \mathbb{R}$ be a smooth function.

h is said to be of **Morse type** whenever every one of its critical points is non-degenerate.

It is said to be of **Morse-Smale type** whenever, in addition, every complete integral curve of its gradient flow is non-degenerate.

Critical points of h are related to topological invariants of M.

This is the basis of Morse theory.

Morse Homology I

Let X_k be the set of critical points of h of Morse index k.

For any $x \in X_{k+1}$ and for any $y \in X_k$, there are only finitely many integral curves of $-\nabla h$ flowing from x to y.

For all k, denote $Z_k := \mathbb{Z}_2[X_k]$.

An operator $\partial : Z_{k+1} \to Z_k$ is determined by counting the integral curves of $-\nabla h$ flowing from points of X_{k+1} to points of X_k.

Remarkably, $\partial^2 = 0$.

Morse Homology II

The complex (Z_*, ∂) has a well-defined homology, MH_*.

This is called the **Morse Homology** of (M, h).

It is (canonically) isomorphic to the singular homology of M.

Information about the number and nature of critical points of h is therefore provided by the singular homology of M.

Morse Homology of the Mean Curvature Flow.

Let Σ be a compact surface.

Let $\mathcal{E}$ be the space of smooth embeddings of Σ into M.

Let $\mathcal{A} : \mathcal{E} \to \mathbb{R}$ be the area functional.

Critical points of $\mathcal{A}$ are minimal surfaces. Complete gradient flows are eternal mean curvature flows.

The Morse Homology of $(\mathcal{M}, \mathcal{A})$ yields information about the number and nature of minimal surfaces in M diffeomorphic to Σ.

Obstacles

This theory is poorly developed and appears to be very hard.

It requires compactness results for families of minimal surfaces of bounded Morse index.

It also requires compactness results for families of eternal mean curvature flows...

...flows which, in general, do not satisfy **any** convexity condition.

Usually, one assumes convexity and proves convergence in **finite** time.

The results required here are complementary to what is currently known about mean curvature flows.

A Tasty Conjecture

Let (M^3, g) be a compact Ricci-positive 3-manifold.

By **[Choi, Schoen, 1985]**, compactness results hold for families of embedded minimal surfaces.

Conjecture [White, 1991]

There exist at least 9 *distinct embedded minimal tori in* (M^3, g).

One uses the $\mathbb{Z}_2$ homology, since the space $\mathcal{M}$ should have the same topology as $\mathbb{RP}^2 \otimes \mathbb{RP}^2$.

A More Tractable Problem

Let (M^{m+1}, g) be a compact Riemannian manifold.

Let Σ^m be the m-sphere.

Let $\mathcal{E}$ be the space of smooth, locally strictly convex Alexandrov embeddedings of Σ into M.

Let $\mathcal{F} : \mathcal{E} \to \mathbb{R}$ be the "Area minus Volume" functional.

Critical points of $\mathcal{F}$ are CMC spheres. Complete gradient flows are eternal forced mean curvature flows.

We claim that for large values of the mean curvature, the Morse complex of the scalar curvature functional approximates the Morse complex of $(\mathcal{M}, \mathcal{F})$.

The Elliptic Component [Ye, 1991]

The scalar curvature function S is of **Morse** type.

Let p be a critical point of S.

Let Σ^m be the sphere of unit radius in T_pM.

For $s \in]0, \infty[$ and $f \in C^1(\Sigma)$, consider the embedding,

$$\begin{aligned} e(s, f) : \Sigma &\to M, \\ x &\mapsto \operatorname{Exp}_p((s(1 + s^2 f(x))x). \end{aligned}$$

Perturbing the Centre

As the scale parameter, s vanishes, different functions dependent on $e(s,f)$ vanish at different rates.

For this reason, it is useful to displace the centre of $e(s,f)$.

Thus, for $s \in]0,\infty[$, $Y \in T_pM$ and $f \in C^1(\Sigma)$, consider the embedding,

$$\begin{aligned} e(s,Y,f) &: \Sigma \to M, \\ & x \mapsto \mathrm{Exp}_{\mathrm{Exp}_p(Y)}(T_{\mathrm{Exp}_p(Y),p}(s(1+s^2f(x))x). \end{aligned}$$

Let $H(s,Y,f)(x)$ be the mean curvature of $e(s,Y,f)$ at x.

Asymptotic Expansions I

Let E be a finite-dimensional vector space.

Let $\phi :]0, \infty] \times E \to \mathbb{R}$ be a smooth function.

Let (ϕ_k) be a sequence of smooth functions where, for all k, $\phi_k : E^k \to \mathbb{R}$.

Let $\xi_x(t) \sim \sum_{k=0}^{\infty} t^k \xi_{k,x}$ be a formal power series in E.

Asymptotic Expansions II

The statement

$$\phi(s,\xi_x) \sim \sum_{k=0}^{\infty} s^k \phi_k(\xi_{0,x},\dots,\xi_{k,x})$$

means that, for all $N \geqslant 0$, there exists a smooth function $R_N : [0,\infty[\times E^{N+1} \to \mathbb{R}$ such that

$$\phi\left(s, \sum_{k=0}^{N} s^k \xi_{x,k}\right) = \sum_{k=0}^{N} s^k \phi_k(\xi_{0,x},\dots,\xi_{k,x}) + s^{N+1} R_N(s,\xi_{x,0},\dots,\xi_{x,N}).$$

Importantly, R_N is **smooth** at $s=0$.

The Asymptotic Expansion of Mean Curvature

Theorem [Ye, 1991]

Denote

$$G(s, Y, f) = \frac{1}{s}\left(H(s, Y, f) - \frac{1}{s}\right).$$

For all $Y \sim \sum_{k=0}^{\infty} s^k Y_k$ and $f_x \sim \sum_{k=0}^{\infty} s^k f_{k,x}$,

$$\begin{aligned} G(s, Y, f) \sim \sum_{k=0}^{\infty} s^k \Bigg[& -\frac{1}{n}(n+\Delta) f_{k,x} \\ & \left(\frac{1}{4}\mathrm{Ric}_{ab;cd} x^a x^b x^c Y^d_{k-2} - \frac{(m+1)}{2(m+3)} S_{;ab} x^a Y^b_{k-2}\right) \\ & + \frac{(m+1)}{2(m+3)} S_{;ab} x^a Y^b_{k-2} - \frac{1}{3}\mathrm{Ric}_{ab;c} x^a x^b Y^c_{k-1} \\ & + H_k(Y_0, \ldots, Y_{k-3}, f_{0,x}, \ldots, f_{k-2,x}) \Bigg]. \end{aligned}$$

Decomposition into Spherical Harmonics

Let $\mathcal{H}_1 \subseteq L^2(\Sigma)$ be the space of first-order spherical harmonics.

Let $\Pi : L^2(\Sigma) \to \mathcal{H}_1$ be the orthogonal projection:

$$\Pi(f)(x) = \frac{(m+1)}{\mathrm{Vol}(\Sigma)} \int_\Sigma f(t,x) x^i \mathrm{dVol} x^j.$$

Let $\Pi^\perp := \mathrm{Id} - \Pi$.

$$\left(\frac{1}{s^2} \Pi(G(s,Y,f)), \Pi^\perp(G(s,Y,f)) \right) = \left(\frac{(m+1)}{2(m+3)} S_{;ab} x^a Y^b, -\frac{1}{n}(n+\Delta) f_x \right) + \ldots$$

This is a **regular** perturbation problem.

The Parabolic Component

The scalar curvature function S is of **Morse-Smale** type.

Let $\gamma : \mathbb{R} \to M$ be a complete orbit of $-\nabla S$ converging towards critical points of S at $\pm\infty$.

Identify $\gamma^* TM$ with $\mathbb{R} \times \mathbb{R}^{m+1}$ by parallel transport.

For $s \in]0, \infty[$ and $f \in C^1(\mathbb{R} \times \Sigma)$, consider the family of embeddings,

$$\begin{aligned} e(s, f) : \mathbb{R} \times \Sigma &\to M, \\ (t, x) &\mapsto \mathrm{Exp}_{\gamma(t)}((s(1 + s^2 f(t, x))x). \end{aligned}$$

The Mean Curvature Flow Operator

It is useful to displace $e(s,f)$ by a vector field $Y : \mathbb{R} \to \mathbb{R}^{m+1}$.

With G defined as before,

$$\Phi(s,Y,f) = G(s,Y,f) - s\langle V(s,Y,f), N(s,Y,f)\rangle,$$

where $N(s,Y,f)(t,\cdot)$ is the outward pointing unit normal vector field over $e(s,Y,f)(t,\cdot)$, and

$$V(s,Y,f)(t,x) = \frac{\partial}{\partial r} e(s,Y,f)(r,x)|_{r=t}.$$

The scale factor in the second term appears because the speed of γ is independent of s.

The Asymptotic Expansion of the MCF Operator

$$
\begin{aligned}
\Phi(s, Y, f) \sim \sum_{k=0}^{\infty} s^k \Bigg[& \left\langle \left(\frac{\partial}{\partial t} + \frac{(m+1)}{2(m+3)} \mathrm{Hess}(S) \right) Y_{k-2,t}, x \right\rangle \\
& + \left(s^4 \frac{\partial}{\partial t} + \frac{1}{m}(m + \overline{\Delta}) \right) f_{k,x,t} \\
& + \left(\frac{1}{4} \mathrm{Ric}_{t,ab;cd} x^a x^b x^c Y^d_{k-2,t} - \frac{(m+1)}{2(m+3)} S_{t,;ab} x^a Y^b_{k-2,t} \right) \\
& - \frac{1}{3} \mathrm{Ric}_{t,ab;c} x^a x^b Y^c_{k-1,t} \\
& + \Phi_k(f_{0,x,t}, \ldots, f_{k-2,x,t}, t^4 \dot{f}_{0,x,t}, \ldots, t^4 \dot{f}_{k-4,x,t}, \\
& \qquad Y_{0,t}, \ldots, Y_{k-3,t}, \dot{Y}_{0,t}, \ldots, \dot{Y}_{k-4,t}) \Bigg],
\end{aligned}
$$

Decomposition into Spherical Harmonics

Define $\Pi : C^1(\mathbb{R} \times \Sigma) \to C^1(\mathbb{R}, \mathcal{H}_1)$ by

$$\Pi(f)(t,x) = \frac{(m+1)}{\mathrm{Vol}(\Sigma)} \int_\Sigma f(t,x) x^i \mathrm{dVol} x^j.$$

denote $\Pi^\perp = \mathrm{Id} - \Pi$.

$$\left(\frac{1}{s^2} \Pi(\Phi(s, Y, f)), \Pi^\perp(\Phi(s, Y, f)) \right) =$$
$$\left(x^a \left(\frac{\partial}{\partial t} + \frac{(m+1)}{2(m+3)} S_{;ab} \right) Y^b, \left(s^4 \frac{\partial}{\partial t} - \frac{1}{n}(n + \Delta) \right) f \right) + \ldots$$

This is a **singular** perturbation problem.

Formal Solutions

Theorem

For all sufficiently small s, there exist canonical sequences $(Y_{s,k})$ and $(f_{s,k})$ such that

$$\Phi\left(s, \sum_{k=0}^{\infty} s^k Y_{s,k}, \sum_{k=0}^{\infty} s^k f_{s,k}\right) \sim 0.$$

The terms in the asymptotic expansions **also depend** on s!

These terms are determined by inverting $s^4 \partial_t - \frac{1}{n}(n + \Delta)$.

We require uniform bounds in the $C^{1,\alpha}$ sense.

We achieve these via **weighted** inhomogeneous Sobolev spaces and a careful argument involving general spherical harmonics.

Exact Solutions

For all N, there exists $C_N > 0$ such that

$$\left\| \Phi\left(s, \sum_{k=0}^{N-1} s^k Y_{s,k}, \sum_{k=0}^{N} s^k f_{s,k}\right)\right\|_{0,\alpha,\mathrm{in}} \leq C_N s^{N+1}$$

Let G_s be the Green's operator of $s^4\partial_t - \frac{1}{n}(n+\Delta)$.

There exists $C > 0$ such that for all s,

$$\|G_s\| \leq Cs^{-4\alpha}.$$

By the inverse function theorem, for sufficiently small s, there exists (Y_s, f_s) such that

$$\Phi(s, Y_s, f_s) = 0.$$

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Thankyou!