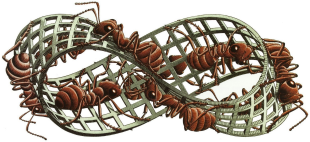
Free-Boundary Minimal Surfaces in Convex 3-Manifolds

Davi Máximo, Ivaldo Paz Nunes, Graham Smith

IMPA, Rio de Janeiro, 30th July 2013



Framework

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with smooth boundary.

Let $S := (S, \partial S)$ be a compact surface with boundary.

Let $e : S \to \Omega$ be an embedding such that $e(\partial S) \subseteq \partial\Omega$.

Definition

We say that e is **free boundary minimal** whenever it is minimal and $e(S)$ meets $\partial\Omega$ orthogonally along ∂S.

Existence Theorem

Theorem, (Máximo, Paz Nunes, Smith, 2013)

If Ω is strictly convex with smooth boundary, then there exists at least 1 free boundary minimal embedded annulus.

The Candidate Space

Davi Máximo, Ivaldo Paz Nunes, Graham Smith

Let $\tilde{\mathcal{E}}$ be the space of smooth embeddings $e : S \to \Omega$ such that $e(\partial S) \subseteq \partial\Omega$.

Let Diff_+ be the group of smooth, orientation-preserving diffeomorphisms $\alpha : S \to S$

Let $\mathcal{E} := \tilde{\mathcal{E}}/\mathrm{Diff}_+$ be the quotient space.

$\mathcal{E}$ is the space of **unparametrised embeddings**.

The Data Space

Let $\mathcal{G}$ be the space of metrics g over Ω such that:

1) g has positive Ricci curvature; and

2) $\partial\Omega$ is strictly convex with respect to g.

$\mathcal{G}$ is the **data space**, which we also call the space of **admissable metrics**

The Solution Space

Let $\mathcal{Z} \subseteq \mathcal{E} \times \mathcal{G}$ be the set of all pairs $([e], g)$ such that e is free boundary minimal with respect to g.

Let $\Pi : \mathcal{Z} \to \mathcal{G}$ be the projection onto the second factor.

Theorem, (Fraser, Li)

Π is a proper mapping.

The Mean Curvature and Angle Functionals

Choose $([e], g) \in \mathcal{E} \times \mathcal{G}$.

Let $\Theta_{e,g} : \partial\Sigma \to \mathbb{R}$ be the function returning the angle that $e(\Sigma)$ makes with $\partial\Omega$ with respect to g at each point.

Let $H_{e,g} : \Sigma \to \mathbb{R}$ be the mean curvature function of $e(\Sigma)$ with respect to g.

$$\mathcal{Z} = \{([e], g) \mid \Theta_{e,g} = 0,\ H_{e,g} = 0\}\,.$$

The Jacobi Operator

For $([e], g) \in \mathcal{Z}$, we define $J = (J_h, J_\theta)$ to be the **Jacobi Operator** of (H, Θ) at $([e], g)$.

J defines a **Fredholm** mapping of index 0 from $C^\infty(\Sigma)$ into $C^\infty(\Sigma) \times C^\infty(\partial\Sigma)$

We say that $([e], g) \in \mathcal{Z}$ is **non-degenerate** whenever J is invertible

The Degree

Theorem

There exists an open, dense subset $\mathcal{G}_0 \subseteq \mathcal{G}$ such that, for all $g \in \mathcal{G}_0$:

1) $\Pi^{-1}(g)$ is finite; and

2) every element $[e]$ of $\Pi^{-1}(g)$ is non-degenerate.

Moreover, there exists a constant $k \in \mathbb{Z}$ such that for all $g \in \mathcal{G}_0$:

$$\mathrm{Deg}(\Pi) := \sum_{[e]\in\Pi^{-1}(g)} (-1)^{\mathrm{Index}([e])} = k.$$

We refer to $\mathrm{Deg}(\Pi)$ as the **mapping degree** of the projection Π.

The Local Degree

Davi Máximo, Ivaldo Paz Nunes, Graham Smith

Choose $g_0 \in \mathcal{G}_0$. Let $\Omega \subseteq \mathcal{E}$ be an open subset such that:

$$\partial\Omega \cap \Pi^{-1}(\{g_0\}) = \emptyset.$$

Proposition

There exists a neighbourhood U of g_0 such that:

$$\partial\Omega \cap \Pi^{-1}(U) = \emptyset.$$

Moreover, there exists $k \in \mathbb{Z}$ such that for all $g \in U \cap \mathcal{G}_0$:

$$\mathrm{Deg}(\Pi|\Omega) := \sum_{[e] \in \Pi^{-1}(g) \cap \Omega} (-1)^{\mathrm{Index}([e])} = k.$$

We refer to $\mathrm{Deg}(\Pi|\Omega)$ as the **local mapping degree** of Π in Ω.

Extremal Annuli

Let Ω be the unit ball in $\mathbb{R}^3$.

Let g_0 be the Euclidean metric.

$\Pi^{-1}(\{g_0\})$ contains the family Z_1 consisting of extremal catenoids.

Z_1 is diffeomorphic to two disjoint copies of $\mathbb{RP}^2$.

Moreover, Z_1 is isolated in $\Pi^{-1}(\{g_0\})$.

Local Degrees

Let $\Omega_1, \Omega_2 \subseteq \mathcal{E}$ be such that:

1) $\Pi^{-1}(g_0) \cap \Omega_1 = Z_1$;

2) $\Pi^{-1}(g_0) \subseteq \Omega_1 \cap \Omega_2$; and

3) $\Omega_1 \cap \Omega_2 = \emptyset$.

$$\mathrm{Deg}(\Pi|\Omega_1) = \chi(Z_1) = 2.$$

$$\mathrm{Deg}(\Pi|\Omega_2) = \chi(Z_1) = 0.$$