Milking the cow: getting the most out of functional norms

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The Hölder seminorm

The *Hölder seminorm* over $\mathbb{R}^d$ is by

$$[f]_\alpha := \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

$[f]_0$ is the *total variation*.

$$[f]_0 \leq 2\|f\|_{C^0}.$$

$[f]_1$ is the *Lipschitz seminorm*.

$$[f]_1 = \|Df\|_{C^0} \;\forall f \in C^1(\mathbb{R}^d).$$

Log convexity I

For all $\alpha, \beta \in [0,1]$, $t \in [0,1]$,

$$[f]_{(1-t)\alpha+t\beta} \leq [f]_{\alpha}^{(1-t)}[f]_{\beta}^{t}.$$

In terms of the logarithms,

$$\log\big([f]_{(1-t)\alpha+t\beta}\big) \leq (1-t)\log\big([f]_{\alpha}\big) + t\log\big([f]_{\beta}\big).$$

By the algebraic-geometric mean inequality,

$$[f]_{(1-t)\alpha+t\beta} \lesssim [f]_{\alpha} + [f]_{\beta}.$$

The Hölder seminorm is dominated by the sum of a term of higher order and a term of lower order.

Log convexity II

Theorem

For all $\alpha, \beta \in [0,1]$, $\alpha + \beta > 0$,

$$\|Df\|_{C^0} \leq C_{\alpha,\beta} [f]_{1-\alpha}^{\frac{\beta}{\alpha+\beta}} [Df]_{\beta}^{\frac{\alpha}{\alpha+\beta}},$$

where

$$C_{\alpha,\beta} = \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}.$$

The same relation holds over $\mathbb{R}^d \setminus K$, for K convex.

$$1 \leq C_{\alpha,\beta} \leq 2 \text{ and}$$
$$\lim_{\beta/\alpha \to 0,\infty} C_{\alpha,\beta} = 1.$$

Log convexity III

For $\alpha \in]0,1[$, define $\phi : \frac{1}{2}\mathbb{N} \to [0,\infty[$ by

$$\phi(x) := \begin{cases} \frac{m^2 \log(2)}{\alpha(1-\alpha)} + \log\big(\|D^m f\|_{C^0}\big) & x = m, \text{ and} \\ \frac{(m+\alpha)^2 \log(2)}{\alpha(1-\alpha)} + \log\big([D^m f]_\alpha\big) & x = m + \frac{1}{2}. \end{cases}$$

Corollary

ϕ is a convex function.

The Hölder norm

The *Hölder norm* over $\mathbb{R}^d$ is

$$\|f\|_{C^{k,\alpha}} := \sum_{m=0}^{k} \|D^m f\|_{C^0} + [D^k f]_\alpha.$$

Corollary

For all (k, α),

$$\|f\|_{C^{k,\alpha}} \lesssim \|f\|_{C^0} + [D^k f]_\alpha.$$

The Hölder norm is dominated by the sum of a term of higher order and a term of lower order.

The Banach space picture

For $T^d := \mathbb{R}^d/\Lambda$ a compact torus, define

$$\Phi : C^{k,\alpha}(T^d) \to C^{0,\alpha}(T^d)/\mathbb{R}; f \mapsto [D^k f].$$

Φ is continuous.

Φ has closed image.

$\mathrm{Ker}(\Phi) \cong \mathbb{R}$.

By the Closed Graph Theorem,

$$\|f\|_{C^{k,\alpha}} \lesssim |f(x_0)| + [D^k f]_\alpha.$$

Sobolev seminorms

The *Sobolev norm* over $\mathbb{R}^d$ is

$$\|f\|_{W^{k,p}} := \sum_{m=0}^{k} \|D^m f\|_{L^p}.$$

Similar interpolation arguments yield

$$\|f\|_{W^{k,p}} \lesssim \|f\|_{L^p} + \|D^k f\|_{L^p}.$$

Scale invariance and differential order

For $\lambda > 0$, define

$$M_\lambda : \mathbb{R}^d \to \mathbb{R}^d; x \mapsto \lambda x.$$

For $f : \mathbb{R}^d \to \mathbb{R}$, define

$$M_\lambda^* f := f \circ M_\lambda = f(\lambda x).$$

Scale invariance is given by

$$\begin{aligned} \|D^k(M_\lambda^* f)\|_{C^0} &= \lambda^k \|D^k f\|_{C^0}, \\ [D^k(M_\lambda^* f)]_\alpha &= \lambda^{k+\alpha}[D^k f]_\alpha, \text{ and} \\ \|D^k(M_\lambda^* f)\|_{L^p} &= \lambda^{k-\frac{n}{p}}[f]_{L^p}. \end{aligned}$$

The exponent is the “order of differentiablity”.

The Sobolev embedding theorem I

For $l - \frac{n}{q} < k - \frac{n}{p}$, $l < k$ and $1 \leq p < q < \infty$,

$$\|D^l f\|_{L^q} \lesssim \|f\|_{L^p} + \|D^k f\|_{L^p}.$$

For $l + \alpha < k - \frac{n}{p}$,

$$[D^l f]_\alpha \lesssim \|f\|_{L^p} + \|D^k f\|_{L^p}.$$

The norm is dominated by the sum of a norm of higher differential order and a norm of lower differential order.

The Sobolev embedding theorem II

Scale invariance yields a little bit more.

Suppose that $l - \frac{n}{q} < k - \frac{n}{p}$, $l < k$ and $1 \leq p < q < \infty$.

For all $\epsilon > 0$, there exists $C > 0$ such that

$$\|D^l f\|_{L^q} \leq C\|f\|_{L^p} + \epsilon\|D^k f\|_{L^p}.$$

Suppose that $l + \alpha < k - \frac{n}{p}$.

For all $\epsilon > 0$, there exists $C > 0$ such that

$$[D^l f]_\alpha \leq C\|f\|_{L^p} + \epsilon\|D^k f\|_{L^p}.$$

The weight of the higher order norm can be chosen arbitrarily small.

Fermi coordinates of hyperbolic space

Fermi coordinates about a complete geodesic in $\mathbb{H}^3$ are

$$g = dr^2 + \sinh^2(r)d\theta^2 + \cosh^2(r)dt^2.$$

In this chart,

(1) the vertical line $\{r = 0\}$ is geodesic;

(2) for all t_0, the horizontal plane $\{t = t_0\}$ is totally geodesic;

(3) for all θ_0, the vertical plane $\{\theta = \theta_0\}$ is totally geodesic; and

(4) for all t_0, θ_0, the radial ray $\{t = t_0, \theta = \theta_0\}$ is geodesic.

Radially symmetric minimal surfaces

For $r_0 \geq 0$, denote

$$A(r_0) := \{r_0 < r < \infty\}.$$

Let Σ be a radially symmetric, minimal graph over the annulus $A(r_0)$. Its profile $u :]r_0, \infty[\to \mathbb{R}$ satisfies

$$u_r = \frac{(F/\pi)}{\cosh(r)\sqrt{\sinh^2(2r) - (F/\pi)^2}}.$$

The *flux* F is

$$F = \pi \sinh(2r_0),$$

where the *neck radius* r_0 is the infimal value such that that Σ extends to a graph over $A(r_0)$.

Diverging parameters

The behaviour that interests us is governed by a small parameter $\epsilon > 0$ and a large parameter $R > 0$ related by

$$\epsilon R^4 \leq \frac{1}{\Lambda}$$
$$\epsilon R^5 \geq \Lambda,$$

where $\Lambda \gg 1$.

As $\Lambda \to +\infty$, $\epsilon \to 0$ and $R \to \infty$ at controlled rates.

An alternative framework would be to suppose that

$$\epsilon = R^{-(4+\gamma)},$$

for some $\gamma \in]0,1[$.

The modified Jacobi operator

The flux is set equal to

$$F = 2\pi\epsilon.$$

A reasonable approximation of the first order variation of mean curvature with respect to vertical perturbations is

$$J_{\epsilon,R} f := \Delta^g f - f - a^i f_i,$$

where Δ^g is the hyperbolic Laplacian, and

$$a^i = \begin{cases} \frac{\epsilon^2 x^i \chi(r)}{r^4}, & r > \epsilon R, \text{ and} \\ \frac{r x^i}{\epsilon^2 R^4}, & r \leq \epsilon R. \end{cases}$$

where χ is a suitable cut-off function.

Observe that a^i is Lipschitz.

Towards uniform invertibility

This operator arises in the study of perturbations of Costa-Hoffman-Meeks (CHM) surfaces.

We require uniform invertibility of $J_{\epsilon,R}$ over Hölder and Sobolev spaces for large values of Λ.

The operators

$$\Delta^g - \mathrm{Id} : C^{2,\alpha}(\mathbb{H}^2) \to C^{0,\alpha}(\mathbb{H}^2), \text{ and}$$
$$\Delta^g - \mathrm{Id} : W^{2,2}(\mathbb{H}^2) \to L^2(\mathbb{H}^2).$$

are invertible. To allow a perturbation argument, the remainder

$$\mathcal{R}f := a^i f_i$$

must *tend to zero* in the operator norms as $\Lambda \to \infty$.

It doesn't. :'-(

The coefficients of $\mathcal{R}$ satisfy

$$\|a\|_{L^2} \lesssim \frac{1}{R^2}, \text{ and}$$
$$\|a\|_{C^{0,\alpha}} \lesssim \frac{1}{(\epsilon R)^\alpha \epsilon R^3}.$$

$\mathcal{R}$ thus tends to 0 in the Sobolev norm...

... but not in the Hölder norm.

Some background

Upon rescaling by a factor of ϵ, functions may be viewed as being defined over the ends of CHM surfaces.

The natural norms over CHM surfaces introduce a multiplicative weight of r with each derivative.

Rescaling this norm by a factor of $\frac{1}{\epsilon}$ would thus give

$$\|Df\|_{C^{0,\alpha}(\mathrm{CHM})} \lesssim (\epsilon R)\|f\|_{C^{2,\alpha}(\mathrm{CHM})}$$

The factor of (ϵR) would balance the singular behaviour of $\mathcal{R}$.

However, the natural norms over $\mathbb{H}^2$ do not have this property.

Imposing this property would break the invertibility of $(\Delta^g - \mathrm{Id})$.

The hybrid norm

Define the *hybrid norm*

$$\|f\|_{k,\alpha} := \|f\|_{C^{k,\alpha}(\mathbb{H}^2)} + \frac{1}{(\epsilon R)}\|f\|_{W^{k,2}(\mathbb{H}^2)},$$

and the *hybrid space*

$$\mathcal{L}_{m,\alpha}(\mathbb{H}^2) := \left\{ f : \mathbb{H}^2 \to \mathbb{R} \mid \|f\|_{m,\alpha} < \infty \right\}.$$

Since $\|f\|_{W^{k,2}}$ has differential order $(k-1)$, we expect, more or less,

$$\|f\|_{C^{k-1}(\mathbb{H}^2)} \lesssim (\epsilon R)\|f\|_{k,\alpha}.$$

The perturbation step

The operator

$$\Delta^g - \mathsf{Id} : \mathcal{L}_{2,\alpha}(\mathbb{H}^2) \to \mathcal{L}_{0,\alpha}(\mathbb{H}^2)$$

is uniformly invertible as $\Lambda \to \infty$.

Theorem

For sufficiently small α, the operator norm of

$$\mathcal{R} : \mathcal{L}_{2,\alpha}(\mathbb{H}^2) \to \mathcal{L}_{0,\alpha}(\mathbb{H}^2)$$

tends to zero as $\Lambda \to \infty$.

The trick

Lemma

For all f,

$$\|f\|_{C^{1,\alpha}(\mathbb{H}^2)} \lesssim (\epsilon R)^{1-2\alpha} \|f\|_{2,\alpha}.$$

Proof By the Sobolev embedding theorem,

$$\|f\|_{C^{0,1-\alpha}(\mathbb{H}^2)} \lesssim \|f\|_{W^{2,2}(\mathbb{H}^2)} \lesssim (\epsilon R) \|f\|_{2,\alpha}.$$

By log-convexity,

$$\|f\|_{C^{1,\alpha}(\mathbb{H}^2)} \lesssim (\epsilon R)^{\frac{1}{1+2\alpha}} \|f\|_{2,\alpha} \lesssim (\epsilon R)^{1-2\alpha} \|f\|_{2,\alpha}.$$

Uniform invertibility

For all f

$$\|\mathcal{R}f\|_{C^{0,\alpha}} \lesssim \|a\|_{C^{0,\alpha}}\|f\|_{C^{1,\alpha}} \lesssim \frac{1}{(\epsilon R)^{3\alpha}R^2}\|f\|_{2,\alpha}, \text{ and}$$

$$\frac{1}{\epsilon R}\|\mathcal{R}f\|_{L^2(\mathbb{H}^2)} \lesssim \frac{1}{\epsilon R}\|a\|_{L^2(\mathbb{H}^2)}\|f\|_{C^{1,\alpha}} \lesssim \frac{1}{R^2}\|f\|_{2,\alpha}.$$

It follows that

$$\|\mathcal{R}f\|_{0,\alpha} \lesssim \frac{1}{(\epsilon R)^{3\alpha}R^2}\|f\|_{2,\alpha}.$$

For α sufficiently small, this tends to zero as Λ tends to infinity.

The main result

Theorem

For sufficiently small α, the operator

$$J_{\epsilon,R} : \mathcal{L}_{2,\alpha}(\mathbb{H}^2) \to \mathcal{L}_{0,\alpha}(\mathbb{H}^2)$$

is uniformly invertible as Λ tends to infinity.

Thankyou!