Constant scalar curvature time functions over GHMC Minkowski spacetimes

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Minkowski spacetimes

We define $(d + 1)$-dimensional **Minkowski space** to be $\mathbb{R}^{d+1}$ furnished with the semi-riemannian metric

$$ds^2 := dx_1^2 + ... + dx_d^2 - dx_{d+1}^2.$$

A $(d+1)$-dimensional **Minkowski spacetime** is a semi-riemannian manifold X that is everywhere locally isometric to $\mathbb{R}^{d,1}$.

We will be interested in Minkowski spacetimes that are **GHMC**.

This stands for **Globally Hyperbolic**, **Maximal** and **Cauchy compact** (GHMC)...

Causality

A piecewise smooth, embedded curve $c : [0,1] \to X$ is said to be **causal** whenever every tangent vector is either timelike or null, that is

$$\|\partial_t c\|^2 \leq 0.$$

The spacetime X is said to be **causal** whenever it contains no closed causal curves.

Grossly speaking, you cannot travel into the past by moving into the future.

Cauchy hypersurfaces and global hyperbolicity

A smooth, embedded, spacelike hypersurface Σ in a causal spacetime X is said to be **Cauchy** whenever it intersects **every** inextensible causal curve exactly once.

A causal spacetime X is said to be **globally hyperbolic** whenever it has a Cauchy hypersurface Σ.

In this case,

$$X \cong \Sigma \times \mathbb{R}.$$

A globally hyperbolic spacetime is said to be **Cauchy compact** whenever its Cauchy hypersurface is compact.

Maximality

A globally hyperbolic spacetime X is said to be **maximal** whenever there exists no isometric embedding

$$e : X \to \hat{X}$$

into another globally hyperbolic spacetime which sends Cauchy hypersurfaces to Cauchy hypersurfaces.

The group theoretic perspective - I

A subgroup $\Gamma \subseteq \mathrm{SO}(d,1)$ is said to be **Kleinian** whenever it is **discrete**, **torsion free** and **cocompact**.

Recall that $\mathrm{SO}(d,1)$ is also the group of isometries $\mathbb{H}^d$.

In particular, Γ is Kleinian if and only if $\mathbb{H}^d/\Gamma$ is a compact manifold.

A map $\tau : \Gamma \to \mathbb{R}^{d,1}$ is said to be a **cocyle** whenever

$$\tau(\alpha\beta) = \tau(\alpha) + \alpha\tau(\beta).$$

By abuse of notation, we denote

$$\Gamma \ltimes \tau = \left\{(\alpha, \tau(\alpha)) \mid \alpha \in \Gamma\right\}.$$

This is a discrete subgroup of $\mathrm{SO}(d,1) \ltimes \mathbb{R}^{d,1}$.

The group theoretic perspective - II

Given $\Gamma \ltimes \tau$, there exists a unique, future complete convex subset $K \subseteq \mathbb{R}^{d,1}$ such that

1 : $\Gamma \ltimes \tau$ acts properly discontinuously on K; and

2 : K is maximal amongst all convex subsets which satisfy (1).

The quotient $X := K/\Gamma \ltimes \tau$ is a GHMC Minkowski spacetime.

Barbot, Bonsante and Mess show that **all** GHMC Minkowski spacetimes arise in this manner.

That is GHMC Minkowski spacetimes are parametrised by pairs (Γ, τ) consisting of Kleinian groups and cocycles.

Time functions

It is well known that in special relativity there is no such thing as absolute time.

However, in certain cases of general relativity, the global geometry of the spacetime imposes valid notions of absolute time.

The **cosmological time** in the GHMC spacetime is the (timelike) distance from the initial singularity.

The **York time** is the unique time function with isochrones of constant mean curvature (c.f. Andersson, Barbot, Béguin & Zeghib).

Other time functions can be defined using other notions of curvature, but...

The elliptic defect

...not all curvature functions yield smooth time functions!

For example, Barbot, Béguin & Zeghib and Schlenker construct smooth time functions over $(2+1)$-dimensional spacetimes with isochrones of constant **extrinsic curvature**.

However, Bonsante & Fillastre show that there are $(3+1)$-dimensional spacetimes which **do not** admit smooth time functions with isochrones of constant **extrinsic curvature**.

The reason is that in higher dimensions extrinsic curvature is not **uniformly elliptic**.

General curvature functions

Let Λ be a homogeneous cone in $\mathrm{Symm}(\mathbb{R}^d)$ invariant under the action of $\mathrm{O}(d)$.

A **curvature function** is a function $K : \Lambda \to \mathbb{R}$ such that

1 : K is invariant under the action of $\mathrm{O}(d)$;

2 : K is homogeneous of order 1; and

3 : $K(\mathrm{Id}) = 1$.

We say that a curvature function is **elliptic** whenever, in addition

4 : K is concave; and

5 : $DK(A)$ is positive definite for all A.

Ellipticity allows us to study curvature functions analytically.

Special lagrangian curvature

Let Λ be the cone of positive definite matrices.

Define first $\Theta : \Lambda \times]0, \infty[\to \mathbb{R}$ by

$$\Theta(A, t) := \operatorname{tr}\left(\arctan\left(\frac{1}{t}A\right)\right) = \operatorname{im}\left(\log\left(\det\left(\operatorname{Id} + \frac{i}{t}A\right)\right)\right).$$

For all A and for all $\theta \in]0, d\pi/2[$, there exists a unique $R_\theta(A)$ such that

$$\Theta(A, R_\theta(A)) = \theta.$$

We call R_θ the **special Lagrangian curvature** with parameter θ.

This curvature function **is** uniformly elliptic.

Special cases

When $d = 2$ and $\theta = \pi/2$, R_θ is **extrinsic curvature**.

When $d = 3$ and $\theta = \pi/2$, R_θ is **scalar curvature**.

Theorem, S. (2017)

For every $(3+1)$-dimensional GHMC Minkowski spacetime X there exists a unique smooth time function $T :]0, \infty[$ with isochrones of constant scalar curvature.

Real Calabi-Yau structures - I

A **real Calabi-Yau structure** over a real vector space E is a triplet (ω, J, R) where

1 : ω is a symplectic form;

2 : $J^2 = -\mathrm{Id}$;

3 : $R^2 = \mathrm{Id}$;

4 : $\omega(J\cdot, J\cdot) = \omega$;

5 : $\omega(R\cdot, R\cdot) = -\omega$; and

6 : $\{J, R\} := RJ + JR = 0$.

Real Calabi-Yau structures - II

The bilinear form

$$g := \omega(\cdot, J\cdot)$$

is symmetric and non-degenerate.

The **signature** of (Ω, J, R) is defined to be the signature $(2p, 2q)$ of this bilinear form.

The isomorphism group of (Ω, J, R) is $\mathrm{O}(p, q)$, acting diagonally.

Any two real Calabi-Yau structures with the same signature are isomorphic.

Real Calabi-Yau structures - III

The bilinear form

$$m := \omega(\cdot, R\cdot)$$

is symmetric, non-degenerate and has signature (d, d).

Let $\mathcal{R}$ and $\mathcal{I}$ be the $+1$ and -1 eigenspaces of R.

$\mathcal{R}$ and $\mathcal{I}$ are complementary, d-dimensional Lagrangian subspaces.

There is (up to sign) a unique J-complex d-form Ω such that

$$\Omega|_{\mathcal{R}} = \mathrm{dVol}_g,$$

where dVol_g is the volume form of the metric g.

Positive special Lagrangian submanifolds

Let X be a manifold. Let $D \subseteq TX$ be a $2d$-dimensional distribution carrying a positive-definite, real Calabi-Yau structure (ω, J, R) on each fibre.

Let Σ be a d-dimensional manifold and let $e : \Sigma \to X$ be an immersion.

The immersion e is said to be **Lagrangian** whenever $e_* T\Sigma$ is contained in D and

$$e^*\omega = 0.$$

It is said to be **θ-special Lagrangian** whenever

$$e^*\mathrm{Re}(e^{i\theta}\Omega) = 0.$$

It is said to be **non-negative** whenever

$$e^* m \geq 0.$$

The compactness result

Theorem, Smith (2004)

Let (Σ_m, p_m) be a sequence of pointed, non-negative, special Lagrangian submanifolds which are complete with respect to the metric g.

If there exists a compact subset K of X such that $p_m \in K$ for all m, then there exists a complete, pointed, non-negative, special Lagrangian submanifold $(\Sigma_\infty, p_\infty)$ towards which (Σ_m, p_m) subconverges in the C^∞-Cheeger-Gromov sense.

The case of Minkowski space

Let $\mathrm{U}^+\mathbb{R}^{d,1}$ denote the bundle of future oriented, unit, timelike vectors over $\mathbb{R}^{d,1}$.

$\mathrm{U}^+\mathbb{R}^{d,1}$ embeds into the Cartesian product $\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}$ by

$$\mathrm{U}^+\mathbb{R}^{d,1} = \left\{(x,y) \mid \|y\|^2 = -1,\ y_{d+1} > 0\right\}.$$

$\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}$ carries a natural real Calabi-Yau structure.

Furthermore, $\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}$ carries a natural Liouville form λ.

The restriction of λ to $\mathrm{U}^+\mathbb{R}^{d,1}$ defines a contact structure. In fact,

$$W_{(x,y)} := \mathrm{Ker}(\lambda) = \{(U,V) \mid \langle U,y\rangle = \langle V,y\rangle = 0\}.$$

The real Calabi-Yau structure of $\mathbb{R}^{d,1} \times \mathbb{R}^{d,1}$ restricts to a real Calabi-Yau structure over each fibre of W.

Spacelike hypersurfaces in Minkowski space

Let Σ be a embedded, spacelike hypersurface in $\mathbb{R}^{d,1}$.

Let $N : \Sigma \to \mathrm{U}^{+}\mathbb{R}^{d,1}$ be the future-oriented, unit, normal vector field over Σ.

N defines an embedding into $\mathrm{U}^{+}\mathbb{R}^{d,1}$.

1 : N is always Lagrangian;

2 : N is non-negative if and only if Σ is locally convex; and

3 : N is θ-special Lagrangian if and only if Σ has constant θ-special Lagrangian curvature equal to $1/3$.

Degenerate limits - I

We say that a non-negative, special Lagrangian submanifold of $\mathrm{U}^+\mathbb{R}^{d,1}$ is **degenerate** whenever it is not the lift of some embedded hypersurfaces.

Given a complete, totally geodesic submanifold X of $\mathbb{H}^d$, define

$$\hat{X} := \{(x, y) \mid y \in X,\ \langle x, y\rangle = 0,\ \langle x, U\rangle = 0\ \forall\ U \in T_y X\}\,.$$

Theorem, S. (2017)

For all complete, totally geodesic X in $\mathbb{H}^d$, $\hat{X}$ is a degenerate, non-negative, special Lagrangian submanifold of $\mathrm{U}^+\mathbb{R}^{d,1}$.

Furthermore every degenerate, non-negative, special Lagrangian submanifold of $\mathrm{U}^+\mathbb{R}^{d,1}$ coincides with $\hat{X}$ for some X.

Degenerate limits - II

In particular, every such submanifold of $\mathrm{U}^{+}\mathbb{R}^{d,1}$ projects onto a complete, spacelike, affine subspace of $\mathbb{R}^{d,1}$.

Theorem, S. (2017)

Let (Σ_m, p_m) be a sequence of complete, pointed, embedded, locally strictly convex, spacelike hypersurfaces in $\mathbb{R}^{d,1}$ of constant $\pi/2$-special Lagrangian curvature equal to $1/3$.

If there exists a compact subset K of $\mathbb{R}^{d,1}$ such that $p_m \in K$ for all m then, either,

1 : there exists a pointed, embedded, locally strictly convex, spacelike hypersurface $(\Sigma_\infty, p_\infty)$ towards which (Σ_m, p_m) subconverges in the C^∞-Cheeger-Gromov sense; or

2 : there exists a complete, pointed, spacelike geodesic $(\Gamma_\infty, p_\infty)$ towards which (Σ_m, p_m) subconverges in the Hausdorff sense.

¡Gracias!