On translating solitons for the mean curvature flow that are of finite genus.

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Let $A_{ij} : \mathbb{R} \to \text{Symm}(\mathbb{R}^n)$ be smooth functions such that

1. $A_{ij}(t)$ tends to $A_i$ (resp. $A_j$) as $t$ tends to $-\infty$ (resp. $+\infty$);
2. $A^{(k)}_{ij}(t)$ tends to 0 as $|t|$ tends to $\infty$.

Define $D : H^1(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ by

$$(D_{ij}f)(t) = (\partial_t f)(t) + A_{ij}(t)f(t).$$
Theorem

If \( A_{ij}(t) =: A \) is constant, and if \( A \) is invertible, then \( D_{ij} \) defines an invertible linear map from \( H^1(\mathbb{R}, \mathbb{R}^n) \) into \( L^2(\mathbb{R}, \mathbb{R}^n) \).

Theorem

If \( A_i \) and \( A_j \) are invertible, then \( D_{ij} \) is Fredholm. Furthermore

\[
\text{Ind}(D_{ij}) = \text{Index}(A_i) - \text{Index}(A_j),
\]

where \( \text{Index}(A_i) \) and \( \text{Index}(A_j) \) are the Morse Indices of \( A_i \) and \( A_j \) respectively.

Theorem

For generic \( A_{ij} \), \( \text{Coker}(D_{ij}) \) is maximal.
Define $\xi : \mathbb{R} \to [0, 1]$ such that

$$\xi(t) = \begin{cases} 1 & \text{if } t \leq 0, \\ 0 & \text{if } t \geq 1. \end{cases}$$

For $R > 0$, define $D_{02,R}^R : H^1(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ by

$$(D_{02}^R f)(t) = (\partial_t f)(t) + (\xi(t)A_{01}(t + R) + (1 - \xi(t)A_{12}(t - R))f(t).$$

**Theorem**

For sufficiently large $R$, if $\xi(t)f(t + R) \in \text{Ker}(D_{01})^\perp$, if $(1 - \xi(t))f(t - R) \in \text{Ker}(D_{12})^\perp$, and if $f \in \text{Ker}(D^R)$, then $f = 0$. 
Step 1: For all $L, \epsilon > 0$ and for sufficiently large $R$,

$$\|f|_{[-L,L]}\|_{H^1} \leq \epsilon.$$ 

Indeed, define

$$\eta^R(t) = \xi(4(x - 1)/R)\xi(4(1 - x)/R).$$

Then, denoting $A := A_1$,

$$(\partial_t - A)(\eta^R f) = D_{02}^R(\eta^R f) + \text{Err}^R(\eta^R f)$$
$$= [D_{02}^R, \eta^R]f + \eta^R D_{02}^R f + \text{Err}^R(\eta^R f)$$
$$= (\partial_t \eta^R)f + \text{Err}^R(\eta^R f).$$

This is small for large $R$, and the assertion follows by invertibility of $\partial_t - A$. 
**Step II:** For all $\epsilon > 0$ and for sufficiently large $R$,

$$\|\xi f\|_{H^1} \leq \epsilon.$$ 

Indeed

$$D_{01}(\xi f) = D_{02}^R(\xi f) + Err^R(\xi f)$$

$$= [D_{02}^R, \xi]f + \xi D_{02}^R f + Err^R(\xi f)$$

$$= (\partial_t \xi)f + Err^R(\xi f).$$

This is small for large $R$, and the assertion follows since $\xi f \in \text{Ker}(D_{01})^\perp$. 
Mean Curvature Flow

Let $M$ be a surface. Let $i_t : M \to \mathbb{R}^3$ be a smooth family of complete, smooth immersions of $M$ into $\mathbb{R}^3$.

Let $N_t : M \to S^2$ be the unit normal vector field over $i_t$.

Let $H_t : M \to \mathbb{R}$ be the mean curvature of $M$.

We say that $i_t$ is a **mean curvature flow** whenever it satisfies

$$\langle \partial_t i_t, N_t \rangle + H_t = 0.$$  

This is a degenerate-parabolic partial differential equation.
Mean Curvature Solitons

Let $A_t$ be a group of isometries of $\mathbb{R}^3$.
Let $i : M \rightarrow \mathbb{R}^3$ be a complete, smooth immersion.
We say that $i$ is a soliton for the mean curvature flow whenever the family $i_t := A_t \circ i$ is a mean curvature flow.
When $A_t$ is a group of translations (usually in the z-direction), we say that $i$ is a translating soliton.
Translating solitons are solutions of
\[ \langle e_z, N_t \rangle + H_t = 0, \]
where $e_z$ here denotes the unit vector in the $z$ direction.
This is a degenerate-elliptic partial differential equation.
Invariant Examples I - Translation Invariance

Simple examples are constructed by supposing invariance with respect to a continuous group of isometries of \( \mathbb{R}^3 \).

This reduces the dimension of the problem from 2 to 1.

First suppose horizontal translation invariance.

This reduces the problem to mean curvature solitons in \( \mathbb{R}^2 \). Up to translation, there are two examples.

(1) The vertical line in \( \mathbb{R}^2 \). This yields the vertical plane in \( \mathbb{R}^3 \).

(2) The graph of the function

\[
g : [ - \pi/2, \pi/2 ] \to \mathbb{R}, \quad t \mapsto \log(\sec(t)).
\]

This is known as the Grim Reaper Curve. It yields the Grim plane in \( \mathbb{R}^3 \).
Now suppose rotation invariance about the $z$-axis.

There are two types of solutions: \textbf{simply connected} solutions and \textbf{doubly connected} solutions.

The simply connected solution, $S_0$, is called the \textbf{Grim paraboloid}.

$S_0$ is the graph of a function $g : \mathbb{R}^2 \rightarrow \mathbb{R}$.

For large $x$,

$$g(x) \sim C + \frac{1}{2} \|x\|^2 - \log(\|x\|) + \ldots$$

Up to vertical translation, $S_0$ is unique.
The doubly connected solution, $S_r$, is called the **Grim catenoid**.

$S_r$ is the union of the graphs of two functions $g_{u,r}, g_{l,r} : A(r, \infty) \rightarrow \mathbb{R}$.

For large $x$,

$$g_{u,r}(x), g_{l,r}(x) \sim C + \frac{1}{2} \|x\|^2 - \log(\|x\|) + \ldots$$

Up to vertical translation, this family is 1-dimensional, parametrised by the inner-radius, $r$.

As $r$ tends to 0, $S_r$ converges to 2 copies of $S_0$. 
In 2013, Nguyen constructed solutions that are invariant under the action of a discrete group of translations.

She calls these solutions **translating tridents**.

They are obtained by desingularising the union of a **Grim plane** and a **vertical plane** along a **singly-periodic Sherk surface**.

In particular, like the singly-periodic Sherk surface, they have infinite genus.
Invariant Examples - Discrete Rotation Invariance

In 2014, Martín, Savas-Halilaj and Smoczyk ask for the existence of solutions of non-trivial finite topology.

We achieve this by singularising the union of a Grim catenoid and a Grim paraboloid about the Costa-Hoffman-Meeks surface, $C_g$, of genus $g$. 
$C_g$ is a complete, embedded minimal surface of genus $g$ with 3 horizontal ends.

The upper and lower ends are catenoidal, that is, they are asymptotic to the graphs of

$$f_{\pm}(x) = \pm A \pm B \log(\|x\|) + \ldots$$

The middle end is planar.

For all $k \in \{0, \ldots, g\}$, $C_g$ is invariant by reflection in the plane spanned by the vectors

$$\cos(k\pi/(g + 1))e_x + \sin(k\pi/(g + 1))e_y \text{ and } e_z.$$

We refer to the isometry group spanned by these reflections as the group of **horizontal symmetries** of $C_g$. 

Theorem A

Let $g$ be a positive integer and fix $\eta > 0$. For all sufficiently large $\Delta$, and for all $\epsilon, R > 0$ such that

$$\epsilon R^{4+\eta} < \frac{1}{\Delta}, \quad \epsilon R^{5-\eta} > \Delta,$$

there exists a complete, embedded, translating soliton, $M_{g,\epsilon,R}$, with genus $g$ and 3 ends. Furthermore

1. $M_{g,\epsilon,R}$ is preserved by the horizontal symmetries of $C_g$;
2. $M_{g,\epsilon,R} \setminus B_{\epsilon R}(0)$ consists of 3 disjoint Grim ends each of which converges towards the Grim paraboloid as $\Delta$ tends to infinity; and
3. upon rescaling by $1/\epsilon$, $M_{g,\epsilon,R} \cap B_{\epsilon R}(0)$ converges towards $C_g$ as $\Delta$ tends to infinity.
Transition Regions and Cut-Off Functions

For $R > 0$, define

\[ B(R) := \{(x, y, z) \mid \|x, y\| \leq R\}. \]

\[ A(R, 2R) := \{(x, y, z) \mid R \leq \|x, y\| \leq 2R\}. \]

Define $\chi : [0, \infty] \rightarrow [0, 1]$ such that

\[ \chi(t) = \begin{cases} 
1 & \text{if } t \leq 1, \\
0 & \text{if } t \geq 2.
\end{cases} \]

For $R > 0$, define $\chi_R : \mathbb{R}^3 \rightarrow [0, 1]$ by

\[ \chi_R(x, y, z) = \chi \left( \frac{\sqrt{x^2 + y^2}}{R} \right). \]

We call $\chi_R$ the \textbf{cut-off function} over the \textbf{transition region} $A(R, 2R)$.
We pretend that $C_g$ is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$.

(It is outside a compact set.)

We pretend that $J_C$ is invertible, with Green's operator $G_C$.

(It's almost invertible.)

We pretend that $G$ is the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$.

(It is outside a compact set.)

We pretend that $J_G$ is invertible, with Green's operator $G_G$.

(It is.)
Building the Approximate Soliton

Given $\epsilon > 0$, we define the rescaled Grim end

$$g_\epsilon(x, y) := g(\epsilon x, \epsilon y).$$

Given $R > 0$, we define the approximate soliton

$$h(x, y) := h_{\epsilon, R}(x, y) = \chi_R(x, y)f(x, y) + (1 - \chi_R)(x, y)g_\epsilon(x, y).$$

Let $H$ be the graph of $h$.

$H$ solves the translating soliton equation over $A(2R, \infty)$.

The error over $B(2R)$ is bounded by $\epsilon$.

We consider $H$ as an **approximate soliton**.
Building the Green’s operator

We construct the translating soliton by perturbing $H$.

This is a simple application of Schauder’s fixed point theorem.

However, we require a Green’s operator for $J_H$.

Furthermore, the Green’s operator is not unique: we need to choose the right one.

We construct a suitable Green’s operator out of $G_C$ and $G_G$.

We do this via a “ping-pong” argument.
For $\phi \in C^\infty_0(B(0, 2R))$, we define
\[ \|\phi\|_{k,C} \]
to be the $C^k$-norm of $\phi$ with respect to the intrinsic metric of $C$.

For $\psi \in C^\infty_0(A(R, \infty))$, we define
\[ \|\phi\|_{k,G} \]
to be the $C^k$-norm of $\phi$ with respect to the intrinsic metric of $G$.

Care should be taken, as $G$ has been **rescaled** by a factor of $1/\epsilon$!
For $\phi$ supported in $B(0, 2R)$, we define

$$A\phi := J_H \chi_{R^4} G_C \phi - \phi.$$ 

$A\phi$ measures the obstruction to $\chi_{R^4} G_C \phi$ being the $J_H$ inverse of $\phi$.

We rewrite $A$ as

$$A\phi = \chi_{R^4} (J_H - J_C) G_C \phi - [J_G, \chi_{R^4}] G_C \phi.$$ 

This is supported in $A(R, \infty)$. Furthermore

$$\|A\phi\|_{0, G} \leq \frac{1}{R^{6+\delta}} \|\phi\|_{0, C}.$$
The “Ping-Pong” Argument - Part II

For $\psi$ supported in $A(R, \infty)$, we define

$$B\phi := J_H(1 - \chi_{R/2})G_G\psi - \psi.$$  

$B\phi$ measures the obstruction to $(1 - \chi_{R/2})G_G\psi$ being the $J_H$ inverse of $\psi$.

We rewrite $B$ as

$$B\phi = \chi_{R/4}(J_H - J_C)G_C\phi + [J_G, \chi_{R/4}]G_C\phi.$$  

This is supported in $B(2R)$. Furthermore

$$\|B\psi\|_{0,c} \leq \frac{R^2}{\epsilon R} \|\psi\|_{0,c}.$$
The “Ping-Pong” Argument - Part III

We define

\[ Q_C := \sum_{m=0}^{\infty} (BA)^m, \quad Q_G := \sum_{m=0}^{\infty} (AB)^m \]

If \( \phi \) is supported in \( B(0, 2R) \), we define

\[ G_{H,C}\phi := \chi_{R^4} G_C Q_C \phi - (1 - \chi_{R/4}) G_G A Q_C \phi. \]

This yields

\[ J_H G_{H,C}\phi = J_H \chi_{R^4} G_C Q_C \phi - J_H (1 - \chi_{R/4}) G_G A Q_C \phi, \]

\[ = A Q_C \phi + Q_C \phi - B A Q_C \phi - A Q_C \phi, \]

\[ = \phi. \]

We define \( G_{H,G} \) in a similar manner.
Let $D$ denote the differentiation operator of $\mathbb{R}^2$ and define

$$D_C := RD, \quad D_G := \frac{1}{\epsilon}D$$

$D_C$ is the natural differentiation operator over the Costa-Hoffman-Meeks surface.

$D_G$ is the natural differentiation operator over the rescaled Grimm end.

In particular

$$\|\phi\|_{2,C} = \|\phi\|_{L^\infty} + \|D_C\phi\|_{L^\infty} + \|D_C^2\phi\|_{L^\infty}$$

$$\|\phi\|_{2,G} = \|\phi\|_{L^\infty} + \|D_G\phi\|_{L^\infty} + \|D_G^2\phi\|_{L^\infty}$$
For $\phi$ supported in $A(R/4, 2R)$,

$$\|\phi\|_{L^\infty} = \|\phi\|_{L^\infty},$$

$$\|D_G\phi\|_{L^\infty} = (\epsilon R)^{-1}\|D_C\phi\|_{L^\infty},$$

$$\|D_G^2\phi\|_{L^\infty} = (\epsilon R)^{-2}\|D_C^2\phi\|_{L^\infty}.$$  

In particular

$$\|\phi\|_{2,G} \leq (\epsilon R)^{-2}\|\phi\|_{2,C}.$$  

This constitutes a loss of information concerning the lower order derivatives of $\phi$!
To recover this lost information, we actually use the norm

\[ \| \phi \|_{k,G} := \| \phi \|_{C^k(G)} + \frac{1}{(\epsilon R)} \| \phi \|_{H^k(G)}, \]

where \( \| \cdot \|_{C^k(G)} \) is the rescaled \( C^k \) norm over \( G \) and \( H^k(G) \) is the rescaled Sobolev norm over \( G \).

In particular,

\[ \| \phi \|_{C^1(G)} \leq \| \phi \|_{H^2(G)} \leq (\epsilon R) \| \phi \|_{2,G}. \]

Well... not quite...
Thankyou!