On translating solitons for the mean curvature flow that are of finite genus.

Graham Smith

Universidade Federal do Rio de Janeiro, Rio de Janeiro

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Mean Curvature Flow

Let M be a surface. Let $i_t : M \to \mathbb{R}^3$ be a smooth family of complete, smooth immersions of M into $\mathbb{R}^3$.

Let $N_t : M \to S^2$ be the unit normal vector field over i_t.

Let $H_t : M \to \mathbb{R}$ be the mean curvature of M.

We say that i_t is a **mean curvature flow** whenever it satisfies

$$\langle \partial_t i_t, N_t \rangle + H_t = 0.$$

This is a degenerate-parabolic partial differential equation.

Mean Curvature Solitons

Let A_t be a group of isometries of $\mathbb{R}^3$.

Let $i : M \to \mathbb{R}^3$ be a complete, smooth immersion.

We say that i is a **soliton** for the mean curvature flow whenever the family $i_t := A_t \circ i$ is a mean curvature flow.

When A_t is a group of translations (usually in the z-direction), we say that i is a **translating soliton**.

Translating solitons are solutions of

$$\langle e_z, N_t \rangle + H_t = 0,$$

where e_z here denotes the unit vector in the z direction.

This is a degenerate-elliptic partial differential equation.

Invariant Examples I - Translation Invariance

Simple examples are constructed by supposing invariance with respect to a continuous group of isometries of $\mathbb{R}^3$.

This reduces the dimension of the problem from 2 to 1.

First suppose horizontal translation invariance.

This reduces the problem to mean curvature solitons in $\mathbb{R}^2$. Up to translation, there are two examples.

(1) The vertical line in $\mathbb{R}^2$. This yields the vertical plane in $\mathbb{R}^3$.

(2) The graph of the function

$$g :]-\pi/2, \pi/2[\to \mathbb{R}, \ t \mapsto \log(\sec(t)).$$

This is known as the **Grim Reaper Curve**. It yields the **Grim plane** in $\mathbb{R}^3$.

The Grim Reaper - a Word of Culture

I have no idea why this curve is called the **Grim Reaper** curve.

Basically, it meant bad news for someone.

Invariant Examples II - Rotation Invariance I

Now suppose rotation invariance about the z-axis.

There are two types of solutions: **simply connected** solutions and **doubly connected** solutions.

The simply connected solution, S_0, is called the **Grim paraboloid**.

S_0 is the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$.

For large x,

$$g(x) \sim C + \frac{1}{2}\|x\|^2 - \log(\|x\|) + ...$$

Up to vertical translation, S_0 is unique.

Invariant Examples - Rotation Invariance II

The doubly connected solution, S_r, is called the **Grim catenoid**.

S_r is the union of the graphs of two functions $g_{u,r}, g_{l,r} : A(r,\infty) \to \mathbb{R}$.

For large x,

$$g_{u,r}(x), g_{l,r}(x) \sim C + \frac{1}{2}\|x\|^2 - \log(\|x\|) + ...$$

Up to vertical translation, this family is 1-dimensional, parametrised by the inner-radius, r.

As r tends to 0, S_r converges to 2 copies of S_0.

Invariant Examples - Discrete Translation Invariance

In 2013, Nguyen constructed solutions that are invariant under the action of a discrete group of translations.

She calls these solutions **translating tridents**.

They are obtained by desingularising the union of a **Grim plane** and a **vertical plane** along a **singly-periodic Sherk surface**.

In particular, like the singly-periodic Sherk surface, they have infinite genus.

Invariant Examples - Discrete Rotation Invariance

In 2014, Martín, Savas-Halilaj and Smoczyk ask for the existence of solutions of non-trivial finite topology.

We achieve this by singularising the union of a **Grim catenoid** and a **Grim paraboloid** about the **Costa-Hoffman-Meeks surface**, C_g, of genus g.

Revision - Costa-Hoffman-Meeks Surfaces

C_g is a complete, embedded minimal surface of genus g with 3 horizontal ends.

The upper and lower ends are catenoidal, that is, they are asymptotic to the graphs of

$$f_{\pm}(x) = \pm A \pm B\log(\|x\|) + ...$$

The middle end is planar.

For all $k \in \{0, ..., g\}$, C_g is invariant by reflection in the plane spanned by the vectors

$$\cos(k\pi/(g+1))e_x + \sin(k\pi/(g+1))e_y \text{ and } e_z.$$

We refer to the isometry group spanned by these reflections as the group of **horizontal symmetries** of C_g.

Solitons of Non-Trivial, Finite Genus

Theorem A

Let g be a positive integer and fix $\eta > 0$. For all sufficiently large Δ, and for all $\epsilon, R > 0$ such that

$$\epsilon R^{4+\eta} < \frac{1}{\Delta},\ \epsilon R^{5-\eta} > \Delta,$$

there exists a complete, embedded, translating soliton, $M_{g,\epsilon,R}$, with genus g and 3 ends. Furthermore

(1) $M_{g,\epsilon,R}$ is preserved by the horizontal symmetries of C_g;

(2) $M_{g,\epsilon,R} \setminus B_{\epsilon R}(0)$ consists of 3 disjoint Grim ends each of which converges towards the Grim paraboloid as Δ tends to infinity; and

(3) upon rescaling by $1/\epsilon$, $M_{g,\epsilon,R} \cap B_{\epsilon R}(0)$ converges towards C_g as Δ tends to infinity.

Transition Regions and Cut-Off Functions

For $R > 0$, define

$$B(R) := \{(x, y, z) \mid \|x, y\| \leq R\}.$$
$$A(R, 2R) := \{(x, y, z) \mid R \leq \|x, y\| \leq 2R\}.$$

Define $\chi : [0, \infty[\to [0, 1]$ such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

For $R > 0$, define $\chi_R : \mathbb{R}^3 \to [0, 1]$ by

$$\chi_R(x, y, z) = \chi\left(\frac{\sqrt{x^2 + y^2}}{R}\right).$$

We call χ_R the **cut-off function** over the **transition region** $A(R, 2R)$.

Lies, Damned Lies, and Mathematics

We pretend that C_g is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$.

(It is outside a compact set.)

We pretend that J_C is invertible, with Green's operator G_C.

(It's almost invertible.)

We pretend that G is the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$.

(It is outside a compact set.)

We pretend that J_G is invertible, with Green's operator G_G.

(It is.)

Building the Approximate Soliton

Given $\epsilon > 0$, we define the rescaled Grim end

$$g_\epsilon(x, y) := g(\epsilon x, \epsilon y).$$

Given $R > 0$, we define the approximate soliton

$$h(x, y) := h_{\epsilon,R}(x, y) = \chi_R(x, y) f(x, y) + (1 - \chi_R)(x, y) g_\epsilon(x, y).$$

Let H be the graph of h.

H solves the translating soliton equation over $A(2R, \infty)$.

The error over $B(2R)$ is bounded by ϵ.

We consider H as an **approximate soliton**.

Building the Green's operator

We construct the translating soliton by perturbing H.

This is a simple application of Schauder's fixed point theorem.

However, we require a Green's operator for J_H.

Furthermore, the Green's operator is not unique: we need to choose the right one.

We construct a suitable Green's operator out of G_C and G_G.

We do this via a "ping-pong" argument.

Semi-Norms

For $\phi \in C_0^\infty(B(0, 2R))$, we define

$$\|\phi\|_{k,C}$$

to be the C^k-norm of ϕ with respect to the intrinsic metric of C.

For $\psi \in C_0^\infty(A(R, \infty))$, we define

$$\|\phi\|_{k,G}$$

to be the C^k-norm of ϕ with respect to the instrinsic metric of G.

Care should be taken, as G has been **rescaled** by a factor of $1/\epsilon$!

The "Ping-Pong" Argument - Part I

For ϕ supported in $B(0, 2R)$, we define

$$A\phi := J_H \chi_{R^4} G_C \phi - \phi.$$

$A\phi$ measures the obstruction to $\chi_{R^4} G_C \phi$ being the J_H inverse of ϕ.

We rewrite A as

$$A\phi = \chi_{R^4}(J_H - J_C) G_C \phi - [J_G, \chi_{R^4}] G_C \phi.$$

This is supported in $A(R, \infty)$. Furthermore

$$\|A\phi\|_{0,G} \leq \frac{1}{R^{6+\delta}} \|\phi\|_{0,C}.$$

The "Ping-Pong" Argument - Part II

For ψ supported in $A(R, \infty)$, we define

$$B\phi := J_H(1 - \chi_{R/2})G_G\psi - \psi.$$

$B\phi$ measures the obstruction to $(1 - \chi_{R/2})G_G\psi$ being the J_H inverse of ψ.

We rewrite B as

$$B\phi = \chi_{R/4}(J_H - J_C)G_C\phi + [J_G, \chi_{R/4}]G_C\phi.$$

This is supported in $B(2R)$. Furthermore

$$\|B\psi\|_{0,C} \leq \frac{R^2}{\epsilon R}\|\psi\|_{0,C}.$$

The "Ping-Pong" Argument - Part III

We define

$$Q_C := \sum_{m=0}^{\infty} (BA)^m, \; Q_G := \sum_{m=0}^{\infty} (AB)^m$$

If ϕ is supported in $B(0, 2R)$, we define

$$G_{H,C}\phi := \chi_{R^4} G_C Q_C \phi - (1 - \chi_{R/4}) G_G A Q_C \phi.$$

This yields

$$\begin{aligned} J_H G_{H,C}\phi &= J_H \chi_{R^4} G_C Q_C \phi - J_H (1 - \chi_{R/4}) G_C A Q_C \phi, \\ &= A Q_C \phi + Q_C \phi - B A Q_C \phi - A Q_C \phi, \\ &= \phi. \end{aligned}$$

We define $G_{H,G}$ in a similar manner.

Information Loss I - Differentiation Operators

Let D denote the differentiation operator of $\mathbb{R}^2$ and define

$$D_C := RD, \; D_G := \frac{1}{\epsilon} D$$

D_C is the natural differentiation operator over the Costa-Hoffman-Meeks surface.

D_G is the natural differentiation operator over the rescaled Grimm end.

In particular

$$\|\phi\|_{2,C} = \|\phi\|_{L^\infty} + \|D_C \phi\|_{L^\infty} + \|D_C^2 \phi\|_{L^\infty}$$
$$\|\phi\|_{2,G} = \|\phi\|_{L^\infty} + \|D_G \phi\|_{L^\infty} + \|D_G^2 \phi\|_{L^\infty}$$

Information Loss II - Changing Norms

For ϕ supported in $A(R/4, 2R)$,

$$\|\phi\|_{L^\infty} = \|\phi\|_{L^\infty},$$
$$\|D_G\phi\|_{L^\infty} = (\epsilon R)^{-1}\|D_C\phi\|_{L^\infty},$$
$$\|D_G^2\phi\|_{L^\infty} = (\epsilon R)^{-2}\|D_C^2\phi\|_{L^\infty}.$$

In particular

$$\|\phi\|_{2,G} \leq (\epsilon R)^{-2}\|\phi\|_{2,C}.$$

This constitutes a loss of information concerning the lower order derivatives of ϕ!

Information Recovery - The Hybrid Norm

To recover this lost information, we actually use the norm

$$\|\phi\|_{k,G} := \|\phi\|_{C^k(G)} + \frac{1}{(\epsilon R)}\|\phi\|_{H^k(G)},$$

where $\|\cdot\|_{C^k(G)}$ is the rescaled C^k norm over G and $H^k(G)$ is the rescaled Sobolev norm over G.

In particular,

$$\|\phi\|_{C^1(G)} \leq \|\phi\|_{H^2(G)} \leq (\epsilon R)\|\phi\|_{2,G}.$$

Well... not quite...

Thankyou!