Rényi’s Parking Problem Revisited
Probability in Dynamics at IM-UFRJ

Matthew Clay & Nándor Simányi

mclay6@gatech.edu  Georgia Institute of Technology
simanyi@uab.edu  University of Alabama at Birmingham

26 May 2014
Outline

1 Background and Motivation
   • Rényi’s Parking Problem
   • Discrete Models

2 The Discrete Model
   • Model Description
   • Research Questions
   • The Recursion

3 Numerical Results
   • Numerical Approach
   • Numerical Results
   • Filling Density of Blocks
Rényi’s Parking Problem

Model (1958)
- Consider an interval $I$ of length $x \gg 1$.
- Sequentially and randomly pack disjoint unit intervals in $I$ as long as there is space in $I$.
- Each new interval is chosen uniformly from the available space.
- $M(x)$ is the expected value of the measure of the covered part.

Rényi’s Parking Constant

$$m = \lim_{x \to \infty} \frac{M(x)}{x} = \int_0^\infty \exp \left[ -2 \int_0^x \frac{1 - e^{-y}}{y} dy \right] dx = 0.747597...$$

$$M(x) = mx + m - 1 + O(x^{-n})$$
Some Previous Discrete Models

- Rényi studied a continuous process, but discrete processes have been investigated as well.
- Page (1959) investigated the sequential packings of non-overlapping neighboring pairs of integer points.
- Such discrete models have direct applications in physics, e.g., the sequential absorption of molecules.

Model of Gargano et al. (2005)

- Considered packings with disjoint blocks of $k + 1$ consecutive integer lattice points.
- Asymmetric filling process: each block center is distanced at least $k + 2$ from at least one of its two neighbors.
Model Description

Consider the discrete lattice interval \( \{1, 2, \ldots, n + k - 1\} \) (\( n \gg 1 \)).

Sequentially pack the interval with disjoint blocks of \( k + 1 \) integers until the space does not permit additional intervals.

The packing process is symmetric, hence more natural.

Example: \( k = 2 \)

We then have gaps of \( k \leq r \leq 2k \) lattice points between the block centers.
Research Questions

1. In similar fashion to Rényi’s continuous recursion, can we obtain a recursion to determine the filling by the discrete process?
2. What is the distribution of gap lengths $r$?
3. How do the results for the discrete model compare to those from Rényi’s continuous model?
Recursion for Gap Length Expectation

Definition

For any \( r (k \leq r \leq 2k) \), let \( a_n^{(r)} \) be the expected number of \( r \)-gaps.

We begin with the lattice

A car chooses the \( i + k \) slot, where \( 1 \leq i \leq n - k - 1 \).
Recursion for Gap Length Expectation

The process repeats for each new lattice.

The probability of occupying the available \( n - k - 1 \) lattice points are equal, hence we obtain the following formula:

\[
a_n^{(r)} = \frac{1}{n - k - 1} \sum_{i=1}^{n-k-1} \left[ a_i^{(r)} + a_{n-k-i}^{(r)} \right]
\]
Recursion for Gap Length Expectation

Hence, for \( n \geq k + 2 \) we have that

\[
a_n^{(r)} = \frac{2}{n - k - 1} \sum_{i=1}^{n-k-1} a_i^{(r)}
\]

\[
a_n^{(r)} = \begin{cases} 
1 & \text{if } n = r - k + 1 \\
0 & \text{if } 1 \leq n \leq k + 1, n \neq r - k + 1 
\end{cases}
\]

We have a linear recursion with an unbounded step.
Obtaining a \( k \)-step Recursion

**Definition**

\[
\begin{align*}
    s_n^{(r)} &= \sum_{i=1}^{n} a_i^{(r)} \\
    t_n^{(r)} &= \frac{s_n^{(r)}}{n(n+2k+1)}
\end{align*}
\]

Let \( u_n^{(r)} = t_n^{(r)} - t_{n-1}^{(r)} \), which yields the following linear \( k \)-step recursion.

\[
u_n^{(r)} = \frac{-2(n+k)}{n(n+2k+1)} \cdot \sum_{i=1}^{k} u_{n-i}^{(r)}
\]

\[
u_n^{(r)} = \begin{cases} 
    0 & \text{if } 2 \leq n \leq r - k \\
    \frac{1}{(r-k+1)(r+k+2)} & \text{if } n = r - k + 1 \text{ and } r \geq k + 1 \\
    \frac{1}{n(n+2k+1)} - \frac{1}{(n-1)(n+2k)} & \text{if } r - k + 2 \leq n \leq k + 1
\end{cases}
\]
Limits

The limiting densities

\[ D(k, r) = (r + 1) \lim_{n \to \infty} \frac{a_n^{(r)}}{n} = 2(r + 1)t^{(r)} \]

exist for all \( r \) \((k \leq r \leq 2k)\), and

\[ \sum_{r=k}^{2k} D(k, r) = 1 \]

The factor \((r + 1)\) accounts for the occupied lattice sites.

Example: \( k = 2 \)
Limits

Of particular interest is the limiting cumulative distribution function

\[ F(t) = \lim_{k \to \infty} \sum_{r=k}^{[(1+t)k]} D(k, r) \]

which indicates how the \( r \) gaps are distributed, and the corresponding limiting density function

\[ F'(t) = \lim_{k \to \infty} kD(k, [(1 + t)k]) \]

for \( 0 \leq t \leq 1 \).
Numerical Methods

Driving Factors
- Must use large $k$, and therefore large $n$, to accurately estimate $F(t)$ and $F'(t)$.
- Precision is important when estimating limits accurately.

Program Design
- Fortran module with quadruple precision to calculate each $D(k, r)$.
- Driving program is a Python script utilizing MPI to parallelize the calculation of each $D(k, r)$.
- Communication is only required at program completion, since each $D(k, r)$ calculation is independent.
Figure: Plot of the distribution function $\sum_{s=k}^{\lfloor (1+t)k \rfloor} D(k, s)$ for $k = 2^{20}$. 
Figure: Plot of the density function $kD(k, [(1 + t)k])$ for $k = 2^{20}$. The maximum value is at $t = 0$ and is marked with the symbol.
Figure: Plot of the growth of $kD(k, k)$ as $k$ is increased. The values of $k$ used were $2^n$, where $3 \leq n \leq 30$. 
Figure: Plot of the growth of $kD(k, 2k)$ as $k$ is increased. The values of $k$ used were $2^n$, where $3 \leq n \leq 30$. 
The numerical results suggest that

- Small gaps are favored, but not aggressively so.
- $F'(0) \approx kD(k, k)$ grows at a logarithmic rate with $k$.
- $F'(1) \approx kD(k, 2k)$ converges to a number 0.63047... in a monotone increasing fashion as $k$ is increased.
- $D(k, r)$ is decreasing in $r$ (not shown).
Filling Density of Integer Blocks

Definition

\[ D(k) = \sum_{r=k}^{2k} \frac{k+1}{r+1} D(k, r) \]

- Gives the limiting filling density of cars getting a parking slot.
- Reminiscent of the filling density investigated by Rényi.

Clearly

\[ \frac{k+1}{2k+1} \leq D(k) \leq 1 \]

Example of Inefficient Packing: \( k = 2 \)
Recovering Rényi’s Parking Constant

Figure: Plot depicting the difference between the calculated values of $D(k)$ and Rényi’s constant $m$ (to machine precision) versus $k$. The values of $k$ used were $2^n$, where $3 \leq n \leq 20$. 
Thanks!