On a few statistical properties of sequential (non-autonomous) dynamical systems

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I will give a description of three topics:

- **Almost sure invariance principle for sequential and non-stationary dynamical systems**, joint with N Haydn, M Nicol and A Torok.
- **Extreme value theory for sequential dynamical systems**, joint with Ana and Jorge Freitas.
- **Loss of memory for non-uniformly expanding maps**, joint with R Aimino, H Hu, M Nicol and A Torok.
Survey

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We will consider families $\mathcal{F}$ of non-invertible maps $T_\beta$ defined on compact subsets $M$ of $\mathbb{R}^n$ or on the torus $\mathbb{T}^n$ and non-singular with respect to the Lebesgue or the Haar measure $m$: $T_\beta m \ll m$. Let us now fix one of these families $\mathcal{F}$ and take a countable sequence of maps $\{T_k\}_{k \geq 1}$ in it: this sequence defines a *sequential dynamical system*. A *sequential orbit* will be defined by the concatenation

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$$T_n \circ \cdots \circ T_1, \ n \geq 1 \quad (1)$$

We will denote with $P_\beta$ the Perron-Frobenius (transfer) operator associated to $T_\beta$ and defined by the duality relation

$$\int_M P_\beta f \ g \ dm = \int_M f \ g \circ T_\beta \ dm, \ f \in L^1_m, \ g \in L^\infty_m$$
Loss of memory
Let us consider

$$\left| \int \psi(x) \phi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1}(x) dm - \int \psi(x) dm \int \phi \circ T_{\beta_n} \circ \cdots \circ T_{\beta_1}(x) dm \right| \leq ||\phi||_{\infty} ||P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(1(\int \psi dm))||_1$$

provided $\phi$ is essentially bounded and $1(\int \psi dm)$ remains in the functional space where the convergence of the operator takes place.
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\[
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||\phi||_{\infty} ||P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(1(\int \psi dm))||_1
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Conze and Raugy called the decorrelation described above *decorrelation* for the *sequential dynamical system* \(T_{\beta_n} \circ \cdots \circ T_{\beta_1}\).
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$$\|\phi\|_\infty \|P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(1(\int \psi dm))\|_1$$

provided $\phi$ is essentially bounded and $1(\int \psi dm)$ remains in the functional space where the convergence of the operator takes place.

Conze and Raugy called the decorrelation described above *decorrelation* for the *sequential dynamical system* $T_{\beta_n} \circ \cdots \circ T_{\beta_1}$.

Estimates on the rate of decorrelation (and the function space in which decay occurs) are a key ingredient in the Conze-Raugy theory to establish central limit theorems for the sums $\sum_{k=0}^{n-1} \phi(T_{\beta_k} \circ \cdots \circ T_{\beta_1} x)$, after centering and normalisation. The question could be formulated in this way: does the ratio

$$\frac{\sum_{k=0}^{n-1} [\phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1}(x) - \int \phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1} dm]}{\| \sum_{k=0}^{n-1} \phi \circ T_{\beta_k} \circ \cdots \circ T_{\beta_1} \|_2}$$

converge in distribution to the normal law $\mathcal{N}(0, 1)$?
Some history
Some history

Sequential
Some history

**Sequential**

- Berend-Bergelson (1984)
- Bakhtin (1995) (hyperbolic maps)
- Polterovich-Rudnick (2004) (cat maps and quasi-morphisms of the modular group)
- Berger, Bunimovich, Hill (2005) ("Bendford law")
- Conze-Raugi (2007), (one of the main contributions in the field)
- Nándori, Szasz, Varjú (2006) (CLT for time dependent dynamical systems)
- Haydn, Nicol, V., Zhang (2014), (CLT for the shrinking target problem)
- Aimino-Roussaeu (2014) (Concentration inequalities for sequential)
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Loss of memory
- Ott, Stendlund, Young (2009): 1-D piecewise expanding maps (coupling)
- Stenlund, Young, and Zhang (2012): Sinai billiards with moving scatterers (coupling)
- Gupta, Ott, Torok (2013): multidimensional piecewise expanding maps (Hilbert projective metric).
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**Property (Dec):**
Given the family $\mathcal{F}$ there exist constants $A > 0, B > 0, 0 < \rho < 1$, such that for any $n$ and any sequence of operators $P_n, \ldots, P_1$ in $\mathcal{F}$ and any $f \in \mathcal{V}$ we have

$$\|P_n \circ \cdots \circ P_1 f\|_\alpha \leq A \rho^n \|f\|_\alpha + B \|f\|_m \quad (2)$$
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$$||P_n \circ \cdots \circ P_1 f||_\alpha \leq A\rho^n ||f||_\alpha + B||f||_m$$  \hspace{1cm} (2)

**Property (Min):**
There exists $\delta > 0$ such that for any sequence $P_n, \ldots, P_1$ in $\mathcal{F}$ and $1 \in \mathcal{V}$ we have

$$P_n \circ \cdots \circ P_1 1(x) \geq \delta, \ \forall x \in M, \ \forall n \geq 0.$$  \hspace{1cm} (3)
There are two main assumptions

**Property (Dec):**
Given the family \( \mathcal{F} \) there exist constants \( A > 0, B > 0, 0 < \rho < 1 \), such that for any \( n \) and any sequence of operators \( P_n, \cdots, P_1 \) in \( \mathcal{F} \) and any \( f \in V \) we have

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\| P_n \circ \cdots \circ P_1 f \|_\alpha \leq A \rho^n \| f \|_\alpha + B \| f \|_m
\]  

\[(2)\]

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There exists \( \delta > 0 \) such that for any sequence \( P_n, \cdots, P_1 \) in \( \mathcal{F} \) and \( 1 \in V \) we have

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P_n \circ \cdots \circ P_1 1(x) \geq \delta, \quad \forall x \in M, \quad \forall n \geq 0.
\]  

\[(3)\]

We will accompany \( L^1_m \) with a Banach space \( V \) of functions over \( M \) with norm \( \| \cdot \|_\alpha \), such that \( \| \phi \|_\infty \leq C \| \phi \|_\alpha \).
Let $\mathcal{V}$ be a Banach space with norm $\|\cdot\|_\alpha$ such that $\|\phi\|_\infty \leq C \|\phi\|_\alpha$. If $(\phi_n)$ is a sequence in $\mathcal{V}$ define $\sigma^2_n = E\left(\sum_{i=1}^n \tilde{\phi}_n(T_n \cdots T_1)\right)^2$ where $\tilde{\phi} = \phi - m(\phi)$. We write $E[\phi]$ for the expectation of $\phi$ with respect to Lebesgue measure.
Let $\mathcal{V}$ be a Banach space with norm $\|\cdot\|_\alpha$ such that $\|\phi\|_\infty \leq C\|\phi\|_\alpha$. If $(\phi_n)$ is a sequence in $\mathcal{V}$ define $\sigma_n^2 = E(\sum_{i=1}^{n} \tilde{\phi}_n(T_n \cdots T_1))^2$ where $\tilde{\phi} = \phi - m(\phi)$. We write $E[\phi]$ for the expectation of $\phi$ with respect to Lebesgue measure.

**Theorem**

Let $(\phi_n)$ be a sequence in $\mathcal{V}$ such that sup$_n \|\phi_n\|_\alpha < \infty$ and hence sup$_n E|\phi_n|^2 < \infty$. Assume (Dec) and (Min) and $\sigma_n \geq n^{1/4+\delta}$ for some $0 < \delta < \frac{1}{4}$. Then $(\phi_n)$ satisfies the ASIP i.e. enlarging our probability space if necessary it is possible to find a sequence $(Z_k)_{k \geq 1}$ of independent centered Gaussian variables $Z_k$ such that for any $\gamma < \delta$

$$\sup_{1 \leq k \leq n} |\sum_{i=1}^{k} \tilde{\phi}_i - \sum_{i=1}^{k} Z_i| = o(\sigma_n^{1-\gamma}) \quad m - a.s.$$ 

Furthermore $\sum_{j=1}^{n} E[Z_j^2] = \sigma_n^2 + O(\sigma_n)$. 

**SV** Sequential Dynamical Systems
Our result is based on an important result of Cuny and Merlevéde (2014)

**Theorem (Cuny and Merlevéde)**

Let \((X_n)\) be sequence of square integrable random variables adapted to a non-increasing filtration \((\mathcal{G}_n)_{n \in \mathbb{N}}\). Assume that \(E(X_n|\mathcal{G}_{n+1}) = 0\) a.s., that \(\sigma_n^2 := \sum_{k=1}^{n} E(X_k^2) \to \infty\) and that \(\sup_n E(X_n^2) < \infty\). Let \((a_n)_{n \in \mathbb{N}}\) be a non-decreasing sequence of positive numbers such that \((a_n/\sigma_n^2)_{n \in \mathbb{N}}\) is non-increasing and \((a_n/\sigma_n)_{n \in \mathbb{N}}\) is non-decreasing. Assume that

\[
\begin{align*}
(A) & \quad \sum_{k=1}^{n} (E(X_k^2|\mathcal{G}_{k+1}) - E(X_k^2)) = o(a_n) \quad P - a.s. \\
(B) & \quad \sum_{n \geq 1} a_n^{-v} E(|X_n|^{2v}) < \infty \quad \text{for some } 1 \leq v \leq 2
\end{align*}
\]

Then enlarging our probability space if necessary it is possible to find a sequence \((Z_k)_{k \geq 1}\) of independent centered Gaussian variables with \(E(Z_k^2) = E(X_k^2)\) such that

\[
\sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Z_i \right| = o((a_n(| \log(\sigma_n^2/a_n)| + \log \log a_n))^{1/2}) \quad P - a.s.
\]
ASIP III - sketch of the proof
ASIP III-sketch of the proof

Having set $T_n := T_n \circ \cdots \circ T_1(x)$, $n \geq 1$ and $P_n := P_n \circ \cdots \circ P_1$, $n \geq 1$ and defined $B_n := T_1^{-1} \cdots T_n^{-1}B$, the σ-algebra associated to the $n$-fold pull back of the Borel σ-algebra $B$, one can show that
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\[
\mathbb{E}(f | B_k) = \left( \frac{P_k f}{P_k 1} \right) \circ T_k
\]

(4)

\[
\mathbb{E}(T_l f | B_k) = \left( \frac{P_k \cdots P_{l+1}(f P_{l+1})}{P_k 1} \right) \circ T_k, \quad 0 \leq l \leq k \leq n
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(5)
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E(T_l f | B_k) = \left( \frac{P_k \cdots P_{l+1}(f P_l 1)}{P_k 1} \right) \circ T_k, \ 0 \leq l \leq k \leq n \quad (5)
\]

We put \( P_n = P_n P_{n-1} \cdots P_1 \) and the operators \( Q_n \phi = \frac{P_n(\phi P_{n-1})}{P_n 1} \). Define \( h_n \) by \( h_n = Q_n \tilde{\phi}_{n-1} + Q_n Q_{n-1} \tilde{\phi}_{n-2} + \cdots + Q_n Q_{n-1} \cdots Q_1 \tilde{\phi}_0 \). Writing \( \psi_n = \tilde{\phi}_n + h_n - T_{n+1} h_{n+1} \) and

\[
U_n = T_1 \cdots T_n \psi_n,
\]

we see, as proven by Conze and Raugi, that \( (U_n) \) is a sequence of reversed martingale differences for the filtration \((B_n)\).
ASIP III-sketch of the proof, continued
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Given the sequence of reversed martingale differences $U_n = T_1 \cdots T_n \psi_n$, one can easily prove since our observables are basically in $L^\infty$, that 
\[
\lim_{n \to \infty} \sum_i E[U_i^2] = \sigma_n^2 + O(\sigma_n).
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Thus we may use $\sigma_n = \sum_{j=1}^n E[U_i^2]$ as our variance.
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We know that $E[U_k^2 | B_{k+1}] = T_1 \cdots T_k T_{k+1} \left( \frac{P_{k+1}(\psi_k^2 \bar{P}_k 1)}{\bar{P}_{k+1} 1} \right)$ and
\[
\int \left[ \sum_{k=1}^n E(U_k^2 | B_{k+1}) - E(U_k^2) \right]^2 dm \leq C \sum_{k=1}^n E(U_k^2)
\]
for some constant $C > 0$. This implies by the Gal-Koksma theorem that
\[
\sum_{k=1}^n E(U_k^2 | B_{k+1}) - E(U_k^2) = o(\sigma_n^{1+\eta}) = o(a_n)
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m a.s. for any \( \eta > 0 \).

In Theorem 2.3 of Cuny and Merlevede, we will take \( a_n \) to be \( \sigma_n^{2-\epsilon} \), for some \( \epsilon > 0 \) sufficiently small (\( \epsilon < 2\delta \) will do) that \( a_n^2 > n^{1/2+\delta'} \) where \( \delta' > 0 \). Thus \( U_n \) satisfies the ASIP with error term \( o(\sigma_n^{1-\gamma}) \) for any \( \gamma < \delta \).
Infinite Variance I
Conze and Raugi show sequential systems formed by taking maps in a small neighborhood of a given $\beta$-transformation, by which we mean for all $T_{\beta'}$ where $\beta' \in (\beta - \delta, \beta + \delta)$ for sufficiently small $\delta > 0$ the conditions $(Dec)$ and $(Min)$ are satisfied and if $\phi$ is not a coboundary for $T_{\beta}$ then the variance of $\phi$ grows as $\sqrt{n}$. 
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Nandori, Szasz and Varju consider sequential systems of form $T_a(x) = (ax)(mod1)$, the theorem applies here to show that the ASIP holds in all the examples in which they establish the CLT (if the variance grows at the stated rate).
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We now introduce new class of maps for which the conditions $(\text{Dec})$ and $(\text{Min})$ are satisfied, but in order to guarantee the unboundedness of the variance when $\phi$ is not a coboundary, we need to introduce new assumptions.
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We now introduce new class of maps for which the conditions $(\text{Dec})$ and $(\text{Min})$ are satisfied, but in order to guarantee the unboundedness of the variance when $\phi$ is not a coboundary, we need to introduce new assumptions.

First of all all the maps in $\mathcal{F}$ will be close to a given map $T_0$. Call $P_0$ the transfer operator associated to $T_0$. Conze and Raugi introduced the following distance between two operators $P_k, P_j$:

$$d(P_k, P_j) = \sup_{\{f \in BV, \|f\|_{BV} \leq 1\}} \|P_k f - P_j f\|_1.$$  

$$d(P_r \circ \cdots \circ P_1, P_0^r) \leq M \sum_{i=1}^r d(P_j, P_0), \quad (DS)$$
Infinite Variance II
Property (Exa): exactness The operator $P_0$ has a spectral gap, for all $f \in \mathcal{V}$ of zero (Lebesgue) mean and $n \geq 1$:

$$||P^n_0||_{BV} \leq C_1 \gamma_0^n ||f||_{BV}$$

Properties (DS) and (Exa) allow to get that there exists a constant $C_2$ such that for all integers $p \leq n$ and all functions $\phi$ of bounded variation we have

$$||P_n \circ \cdots \circ P_1 \phi - P^n_0 \phi||_1 \leq C_2 ||\phi||_{BV} \left( \sum_{k=1}^{p} d(P_{n-k+1}, P_0) + (1 - \gamma_0)^{-1} \gamma_0^p \right)$$

Sequential Dynamical Systems
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$$

We now suppose that the maps and the operators are parametrized by a sequence of numbers $\varepsilon_k$, $k \in \mathbb{N}$ such that $\varepsilon_k$ converge to a given $\varepsilon_*$ when $k \to \infty$ and

**Property (clos)- closeness:**

$$
\|P_{\varepsilon_k} f - P_{\varepsilon_j} f\|_1 \leq C_3 \|f\|_{BV} |\varepsilon_k - \varepsilon_j|, \ \forall \{\varepsilon_k\}_{k \in \mathbb{N}} \cup \{\varepsilon_*\}.
$$

for $f \in \mathcal{V}$ and $C_3 > 0$ a positive constant independent of $f$ and of the sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$, and where $P_{\varepsilon_*} = P_0$. 
\textbf{Infinite Variance III}

\textbf{Property (con)-convergence:}

\[ |\varepsilon_k - \varepsilon_*| \leq \frac{C_4}{n^{\kappa}}, \kappa > 0 \]

With this last assumption, we get a polynomial decay of the type $O(n^{-\kappa})$ for $P_n \circ \cdots \circ P_1 \phi$ to $h \int \phi \, dm$ being $h$ the density of the absolutely continuous mixing measure of the map $T$. This convergence is necessary to establish the growing of the variance $\sigma_n^2$. In particular all the above conditions plus the following one
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- **Property (pos)-positivity**

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With this last assumption, we get a polynomial decay of the type \( O(n^{-\kappa}) \) for \( P_n \circ \cdots \circ P_1 \phi \) to \( h \int \phi \, dm \) being \( h \) the density of the absolutely continuous mixing measure of the map \( T \). This convergence is necessary to establish the growing of the variance \( \sigma_n^2 \). In particular all the above conditions plus the following one

Property (pos)-positivity

The density \( h \) is uniformly bounded from below away from 0, namely \( \exists h_m > 0 \), such that \( ||h||_{\infty} > h_m \).

imply that \( n^{-1} \sigma_n^2 \) converge, whenever \( \phi \) is not a cobraundary, to \( \sigma^2 \) given by

\[ \sigma^2 = \int \hat{P} [G \phi - \hat{P}G \phi]^2(x) h(x) \, dx \]

where \( \hat{P} \phi = \frac{P(h\phi)}{h} \) is the normalized transfer operator of \( T \) and

\( G \phi = \sum_{k \geq 0} \frac{P^k(h\phi)}{h} \).
An example: covering maps

Let us put

\[ A_{\varepsilon_1, \ldots, \varepsilon_n}^{k_1, \ldots, k_n} = T_{k_1, \varepsilon_1}^{-1} \circ \cdots \circ T_{k_n-1, \varepsilon_{n-1}}^{-1} A_{k_n, \varepsilon_n} \cap \cdots \cap T_{k_1, \varepsilon_1}^{-1} A_{k_2, \varepsilon_2} \cap A_{k_1 \varepsilon_1} \]

Since we have supposed that \( \inf_{T \in \mathcal{F}, i=1, \ldots, q, x \in A_i, \varepsilon} |DT_{\varepsilon}(x)| \geq \beta_0 > 2 \), it follows that the previous intervals have all length bounded by \( \beta_0^n \) independently of the concatenation we have chosen. We are now ready to strengthen the assumptions on our maps by requiring the following condition:

**Property Covering**

There exist \( n_0 \) and \( N(n_0) \) such that:

(i) The partition into sets \( A_{\varepsilon_1, \ldots, \varepsilon_{n_0}}^{k_1, \ldots, k_{n_0}} \) has diameter less than \( \frac{1}{2au} \).

(ii) For any sequence \( (\varepsilon_1, \ldots, \varepsilon_{N(n_0)}) \) and \( k_1, \ldots, k_{n_0} \) we have

\[ T_{\varepsilon_{N(n_0)}}^{-1} \circ \cdots \circ T_{\varepsilon_{n_0}+1}^{-1} A_{\varepsilon_1, \ldots, \varepsilon_{n_0}}^{k_1, \ldots, k_{n_0}} = M \]
We now consider a fixed expanding map acting on the unit interval (for example a $\beta$-transformation, smooth expanding map, the Gauss map, Rychlik maps...) whose transfer operator is quasi compact in the bounded variation norm so that we have exponential decay of correlations in the bounded variation norm. Suppose $\phi_j = 1_{A_j}$ are indicator functions of a sequence of nested intervals $A_j$, where $\mu$ is the unique invariant measure for the map $T$.

**Theorem**

Suppose $(T, X, \mu)$ is a dynamical system with exponential decay in the BV norm versus $L^1(\mu)$. Suppose $\phi_j = 1_{A_j}$ are indicator functions of a sequence of nested sets $A_j$ such that $\sup_n \|\phi_n\|_{BV} < \infty$ and $\frac{c_2}{n^{\gamma_2}} \leq \mu(A_n) \leq \frac{c_1}{n^{\gamma_1}}$ where $0 < \gamma_1 \leq \gamma_2 < 1$. If $2\gamma_2 - \gamma_1 < 1$ then $(\phi_n \circ T^n)$ satisfies the ASIP i.e. enlarging our probability space if necessary it is possible to find a sequence $(Z_k)_{k \geq 1}$ of independent centered Gaussian variables $Z_k$ such that for $\gamma > 0$ such that $2\gamma < 1 - (2\gamma_2 - \gamma_1)$

$$
\sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \phi_i \circ T^i - \sum_{i=1}^{k} Z_i \right| = o(\sigma^{1-\gamma}) \mu - a.s.
$$

Furthermore $\sum_{i=1}^{n} E[Z_i^2] = \sigma_n^2 + O(\sigma_n)$. 

**ASIP for the shrinking target problem I**
ASIP for the shrinking target problem II
We see already proved (Haydn, Nicol, V, Zhang), that for sufficiently large $n$, $\sigma_n^2 \geq E_n \geq Cn^{1-\gamma}$ for some constant $C > 0$. We follow again the proof of Conze-Raugi by taking $T_k = T$ for all $k, m$ as the invariant measure $\mu$ and $f_n = 1_{A_n}$. Note that conditions (Dec) and (Min) are satisfied automatically under the assumption that we have exponential decay of correlations in BV norm and the transfer operator $P$ is defined with respect to the invariant measure $\mu$, so that $P1 = 1$. 
ASIP for the shrinking target problem II

We see already proved (Haydn, Nicol, V, Zhang), that for sufficiently large $n$, $\sigma_n^2 \geq E_n \geq Cn^{1-\gamma_2}$ for some constant $C > 0$. We follow again the proof of Conze-Raugi by taking $T_k = T$ for all $k, m$ as the invariant measure $\mu$ and $f_n = 1_{A_n}$. Note that conditions (Dec) and (Min) are satisfied automatically under the assumption that we have exponential decay of correlations in BV norm and the transfer operator $P$ is defined with respect to the invariant measure $\mu$, so that $P1 = 1$.

We write $P^n$ for the $n$-fold composition of the linear operator $P$. Let $\tilde{\phi}_i = \phi_i - \mu(\phi_i)$. Define $h_n$ by $h_n = P\tilde{\phi}_{n-1} + P^2\tilde{\phi}_{n-2} + \cdots + P^n\tilde{\phi}_0$. Writing

$$\psi_n = \tilde{\phi}_n + h_n - h_{n+1} \circ T$$

and

$$U_n = \psi_n \circ T^n$$

as before $(U_n)$ is a sequence of reversed martingale differences for the filtration $(\mathcal{B}_n)$. Note that $\mu(|U_n|^4) \leq \frac{C}{n^{\gamma_1}}$ where $C$ is a uniform constant, as $h_n$ is uniformly bounded in $L^\infty(\mu)$. 
ASIP for the shrinking target problem III
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The proof that $\sum_{i=1}^{n} E[U_i^2] = \sigma_n^2 + O(\sigma_n)$ is exactly the same as in the case of sequential expanding maps.
ASIP for the shrinking target problem III

- The proof that $\sum_{i=1}^{n} E[U_i^2] = \sigma_n^2 + O(\sigma_n)$ is exactly the same as in the case of sequential expanding maps.
- We take $a_n$ to be $\sigma_n^{2(1-\epsilon)}$, for some small $\epsilon > 0$ so that $a_n \geq n^{1-\gamma_2-\epsilon'}$ for any small $\epsilon' > 0$. Condition (A) of Cuny Merlevede holds exactly as before.
ASIP for the shrinking target problem III

The proof that \( \sum_{i=1}^{n} E[U_i^2] = \sigma_n^2 + O(\sigma_n) \) is exactly the same as in the case of sequential expanding maps.

We take \( a_n = \sigma_n^{2(1-\epsilon)} \), for some small \( \epsilon > 0 \) so that \( a_n \geq n^{1-\gamma_2-\epsilon'} \) for any small \( \epsilon' > 0 \). Condition (A) of Cuny Merlevede holds exactly as before.

We will show Condition (B) of Cuny Merlevede holds under our assumption that \( \frac{C_2}{n^{\gamma_2}} \leq \mu(A_n) \leq \frac{C_1}{n^{\gamma_1}} \) where \( 0 < \gamma_1 \leq \gamma_2 < 1 \). We take \( v = 2 \) in Condition (B) and estimate

\[
\sum_{n \geq 1} a_n^{-2} \mu(|U_n|^4) \leq \sum_{n} Cn^{-2(1-\gamma_2-\epsilon')-\gamma_1}
\]

which converges under our assumption \( 2\gamma_2 - \gamma_1 < 1 \) by taking \( \epsilon' \) so that \( 2\epsilon' < 1 - (2\gamma_2 - \gamma_1) \).
The proof that \( \sum_{i=1}^{n} E[U_i^2] = \sigma_n^2 + O(\sigma_n) \) is exactly the same as in the case of sequential expanding maps.

We take \( a_n \) to be \( \sigma_n^{2(1-\epsilon)} \), for some small \( \epsilon > 0 \) so that \( a_n \geq n^{1-\gamma_2-\epsilon'} \) for any small \( \epsilon' > 0 \). Condition (A) of Cuny Merlevede holds exactly as before.

We will show Condition (B) of Cuny Merlevede holds under our assumption that \( \frac{C_2}{n^{\gamma_2}} \leq \mu(A_n) \leq \frac{C_1}{n^{\gamma_1}} \) where \( 0 < \gamma_1 \leq \gamma_2 < 1 \). We take \( \nu = 2 \) in Condition (B) and estimate

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which converges under our assumption \( 2\gamma_2 - \gamma_1 < 1 \) by taking \( \epsilon' \) so that \( 2\epsilon' < 1 - (2\gamma_2 - \gamma_1) \).

Thus \( U_n \) satisfies the ASIP with error term \( o(\sigma_n^{1-\gamma}) \) if \( 2\gamma < 1 - (2\gamma_2 - \gamma_1) \).

Finally

\[
\sum_{j=1}^{n} U_j = \left[ \sum_{j=1}^{n} \tilde{\phi}_j(T^j) \right] + h_1(T_1) - h_n(T^n)
\]

As \( |h_n| \) is uniformly bounded we see that \( (\phi_j(T^j)) \) satisfies the ASIP with error term \( o(\sigma_n^{1-\gamma}) \) for the same \( \gamma > 0 \).
EVT: the spectral approach I
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Let us first consider a deterministic dynamical system \((X, T, \mu)\) and consider the observable \(\phi(x) = -\log d(x, z)\), where \(d(\cdot, \cdot)\) denote a distance and \(z\) is a given point in \(X\). We put \(M_n = \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}\). We also set \(U_n = B(z, e^{-u_n})\), where the sequence \(u_n\) will be chosen in such a way that \(u_n \to \infty\), when \(n \to \infty\), (and therefore the ball \(U_n\) shrinks on the center \(z\)), and \(n\mu(U_n)\) converges to a fixed prescribed value \(\tau\).
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It easy to see that

\[
\mu(M_n \leq u_n) = \mu(U_n^c \cap \cdots \cap T^{-(n-1)}U_n^c) = \int [h1_{U_n^c}(x) \cdots 1_{U_n^c}(T^{n-1}(x))] dm = \int \mathcal{P}_n g
\]

where \(h\) is the density of \(\mu\) w.r.t. the Lebesgue measure \(m\) (we will suppose that since we are using the corresponding duality with the transfer operator), and

\[
\mathcal{P} g := P(g1_{U_n^c})
\]

where \(P\) is the PF (Perron-Frobenius) operator associated to \(T\) and \(g\) is an observable living in a good functional space (we will take later on \(g\) in BV).
EVT II
EVT II

We will suppose that \( \mathcal{P} \) is a small perturbation of \( P \) in the sense that we could find eigenfunction \( \psi_n \) and eigenmeasure \( \nu_n \) such that

\[
\mathcal{P}\psi_n = \lambda_n \psi_n; \quad \nu_n \mathcal{P} = \lambda_n \nu_n; \quad \lambda_n^{-1} \mathcal{P} = \psi_n \otimes \nu_n + Q_n
\]

and integrating w.r.t. the density \( h \) we have

\[
\mathcal{P}_n^g = \lambda_n^n \psi_n \int g d\nu_n + \lambda_n^n Q_n g
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and integrating w.r.t. the density $h$ we have

$$\int \mathcal{P}_n^g h dm = \lambda_n \int \psi_n dm \int h d\nu_n + \lambda_n \int Q_n^g h dm$$

with $\int \psi_n dm = 1, \forall n$, $\int h d\nu_n \to \int h dm$, when $n \to \infty$, $Q_n$ is an operator whose $L^\infty$ norm goes exponentially fast to zero and finally (Keller-Liverani)

$$\lim_{n \to \infty} \frac{1 - \lambda_n}{\mu(U_n)} = \theta > 0.$$
We will suppose that $\mathcal{P}$ is a small perturbation of $\mathcal{P}$ in the sense that we could find eigenfunction $\psi_n$ and eigenmeasure $\nu_n$ such that

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$$\mathcal{P}^n g = \lambda^n_n \psi_n \int gd\nu_n + \lambda^n_n Q_n g$$

and integrating w.r.t. the density $h$ we have

$$\int \mathcal{P}^n h \, dm = \lambda^n_n \int \psi_n \, dm \int h \, d\nu_n + \lambda^n_n \int Q^n_n h \, dm$$

with $\int \psi_n \, dm = 1$, $\forall n$, $\int h \, d\nu_n \to \int h \, dm$, when $n \to \infty$, $Q_n$ is an operator whose $L^\infty$ norm goes exponentially fast to zero and finally (Keller-Liverani)

$$\lim_{n \to \infty} \frac{1 - \lambda_n}{\mu(U_n)} = \theta > 0.$$

Let us note that with these prescriptions and remembering that on $\mu(U_n)$ we easily have:

$$\mu(M_n \leq u_n) \to e^{-\theta \tau} \quad n \to \infty, \; \text{Gumbel’s law.}$$
EVT III
We now consider sequential dynamical systems and we denote with $T_n \circ \cdots \circ T_1$ the concatenation of $T_j$ maps chosen in a given set. We call $P_j$ the PF operator associated to $T_j$. We consider the Lebesgue measure $m$ as the probability measure which rules out the distribution of the maxima. These are now defined in the natural way (the quantity $u_n$ verifies the same assumptions as above):

$$m(M_n \leq u_n) = \int [1_{U_n^c}(x) \cdots 1_{U_n^c}(T_{n-1} \circ \cdots \circ T_1(x))]dm = \int \hat{P}_n 1 dm$$

where

$$\hat{P}_{ng} := \hat{P}_{n,n} \circ \cdots \circ \hat{P}_{1,n} g; \quad \hat{P}_{l,n} g := P_l(g1_{U_n^c})$$
We now consider sequential dynamical systems and we denote with $T_n \circ \cdots \circ T_1$ the concatenation of $T_j$ maps chosen in a given set. We call $P_j$ the PF operator associated to $T_j$. We consider the Lebesgue measure $m$ as the probability measure which rules out the distribution of the maxima. These are now defined in the natural way (the quantity $u_n$ verifies the same assumptions as above):

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where

$$\hat{P}_{ng} := \hat{P}_{n,n} \circ \cdots \circ \hat{P}_{1,ng}; \quad \hat{P}_{l,ng} := P_l(g1_{U_n^c})$$

$$m(M_n \leq u_n) = \int \hat{P}_n1 dm = \int [\hat{P}_n \circ \cdots \circ \hat{P}_1 - \tilde{P}^n1 + \tilde{P}^n1] dm$$

where $\tilde{P}g = P(g1_{U_n^c})$ is defined in terms of the PF operator of a given map $T$ which satisfies the spectral properties quoted above.
EVT IV
EVT IV

If we can show that

$$\int |\hat{P}_n \circ \cdots \circ \hat{P}_1 1 - \hat{P}_1^n 1| dm (\emptyset)$$

goes to zero when $n$ goes to infinity, then

$$m(M_n \leq u_n) \rightarrow e^{-\theta \tau}, \quad n \rightarrow \infty$$
If we can show that

$$\int |\hat{P}_n \circ \cdots \circ \hat{P}_1 1 - \tilde{P}^n 1| dm \ (\Theta)$$

goes to zero when $n$ goes to infinity, then

$$m(M_n \leq u_n) \to e^{-\theta \tau}, \ n \to \infty$$

In order to prove the decay of $(\Theta)$, we should consider very particular dynamical systems, namely a class of $\beta$-transformations considered in the paper by Conze and Raugi or the other maps previously investigated. Let us call $T(x) = \beta x \mod 1$, the original unperturbed $\beta$-transformation. We will take the other transformations of the same kind $x \to \beta_k x \mod 1$, with $\beta_k \geq 1 + a$, $\forall k \geq 1$, and $a$ a given positive number.
If we can show that
\[ \int |\hat{P}_n \circ \cdots \circ \hat{P}_1 1 - \tilde{P}^n 1| dm (\Theta) \]
\[ \text{goes to zero when } n \text{ goes to infinity, then} \]
\[ m(M_n \leq u_n) \to e^{-\theta \tau}, \quad n \to \infty \]

In order to prove the decay of \((\Theta)\), we should consider very particular dynamical systems, namely a class of \(\beta\)-transformations considered in the paper by Conze and Raugi or the other maps previously investigated. Let us call \(T(x) = \beta x \mod 1\), the original unperturbed \(\beta\)-transformation. We will take the other transformations of the same kind \(x \to \beta_k x \mod 1\), with \(\beta_k \geq 1 + a, \forall k \geq 1\), and \(a\) a given positive number.

Our next step is to compute the distance \(d(\hat{P}_j, \tilde{P})\) for \(\beta\) transformations. We remind that \(\hat{P}_j f = P_j (f1_{U_{n_j}})\) and \(\tilde{P} f = P(f1_{U_{n_j}})\). Since \(P\) is nothing but a particular \(P_j\) we will compute the distance between two PF operators \(\hat{P}_1, \hat{P}_2\) associated respectively to the \(\beta\) transformations \(\beta_1\) and \(\beta_2\).
EVT V
One gets that there is a constant $C$ such that for any $\beta_1, \beta_2$:

$$||\hat{P}_2 f - \hat{P}_1 f||_1 \leq C ||f||_{BV} |\beta_2 - \beta_1|$$

which immediately implies

$$d(\hat{P}_1, \hat{P}_2) \leq C |\beta_2 - \beta_1|, \ (D)$$
One gets that there is a constant \( C \) such that for any \( \beta_1, \beta_2 \):
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which immediately implies
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\]

The assumptions made allow to get that there exists a constant \( C_2 \) such that for all integers \( p \leq n \) and all functions \( \phi \) of bounded variation we have
\[
||\hat{P}_n \circ \cdots \circ \hat{P}_1 \phi - \tilde{P}^n \phi||_1 \leq C_2 ||\phi||_{BV}(\sum_{k=1}^{p} d(\hat{P}_{n-k+1}, \tilde{P}) + (1 - \gamma_0)^{-1} \gamma_0^p)
\]
One gets that there is a constant $C$ such that for any $\beta_1, \beta_2$:

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The assumptions made allow to get that there exists a constant $C_2$ such that for all integers $p \leq n$ and all functions $\phi$ of bounded variation we have

$$||\hat{P}_n \circ \cdots \circ \hat{P}_1 \phi - \tilde{P}^n \phi||_1 \leq C_2 ||\phi||_{BV} \left( \sum_{k=1}^{p} d(\hat{P}_{n-k+1}, \tilde{P}) + (1 - \gamma_0)^{-1} \gamma_0^p \right)$$

Then

$$||\hat{P}_n \circ \cdots \circ \hat{P}_1 \phi - \tilde{P}^n \phi||_1 \leq C_2 ||\phi||_{BV} \left( \sum_{k=1}^{p} |\beta_{n-k+1} - \beta| + (1 - \gamma_0)^{-1} \gamma_0^p \right)$$

In particular, whenever $|\beta_n - \beta| \leq \frac{1}{n^\zeta}$, with $\zeta > 0$, then we have, for a constant $\hat{C}$ independent from $n$ and from $\phi \in BV$:

$$||\hat{P}_n \circ \cdots \circ \hat{P}_1 \phi - \tilde{P}^n \phi||_1 \leq \frac{\hat{C} \log n}{n^\zeta} ||\phi||_{BV}.$$
Loss of memory
Loss of memory

Are there examples with polynomial loss of memory?

\[ ||P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\phi)||_1 \]
Loss of memory

- Are there examples with polynomial loss of memory?
  \[ ||P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\phi)||_1 \]

- The obvious candidate is a (sort of) Pomeau-Manneville map of this kind
  \[ T_\alpha(x) = x + \frac{3^\alpha}{2^{1+\alpha}} x^{1+\alpha}, \quad 0 \leq x \leq 2/3 \]
  \[ T_\alpha(x) = 3x - 2, \quad 2/3 \leq x \leq 1 \]

- The perturbation: it will be defined by considering maps \( T_\beta(x) \) like above with \( 0 < \beta* \leq \beta \leq \alpha \) in \( 0 \leq x \leq 2/3 \), and \( T_\beta = T_\alpha \) on \( 2/3 \leq x \leq 1 \).

- The reference measure will be Lebesgue (\( m \)) and we will move functions as convenable densities of the ”unperturbed” acim.
The result

We define the cone of functions

\[ C_1 := \{ f \in C^0([0,1]; f \geq 0; f \text{ decreasing}; X^{\alpha+1}f \text{ increasing}, f(x) \leq ax^\alpha \int f dm \} \]

where \( X(x) = x \) is the identity function.

**Theorem**

Suppose \( \psi, \phi \) are in \( C_2 \) for some \( a \) with equal expectation \( \int \phi dm = \int \psi dm \).

Then for any \( 0 < \beta^* \leq \alpha < 1 \) and for any sequence \( T_{\beta_1}, \ldots, T_{\beta_n}, n > 1 \) of maps of Pomeau-Manneville type with \( \beta^* \leq \beta_k \leq \alpha, \ k \in [1,n] \), we have

\[
\int |P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\phi) - P_{\beta_n} \circ \cdots \circ P_{\beta_1}(\psi)| dm \leq C_\alpha (||\phi||_{C_1} + ||\psi||_{C_1}) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}},
\]

where the constant \( C_\alpha \) depends only on the map \( T_\alpha \), and \( || \cdot ||_{C_1} \) denotes the \( L^1 \) norm.

A similar rate of decay holds for \( C^1 \) observables \( \phi \) and \( \psi \) on \( S^1 \); in this case the rate of decay has an upper bound given by

\[
C_\alpha \ F(||\phi||_{C^1} + ||\psi||_{C^1}) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}
\]

where the function \( F : \mathbb{R} \to \mathbb{R} \) is affine.