Numerical scheme with uniform exponential decay for the critical case of KdV equation with damping.

Felipe Linares ∗  Ademir Pazoto †
Mauricio Sepúlveda ‡  Octavio Vera Villagrán §

Abstract

We describe a conservative numerical scheme with the property of uniform exponential decay for the critical case of the Generalized Korteweg-de Vries equation ($p = 4$), with damping.

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1 Introduction

We consider the following generalized Korteweg-de Vries (GKdV) equation

$$
u_t + u_{xxx} + u^4 u_x + u + a(x) = 0, \quad (0, L) \times (0, +\infty), \tag{1.1}$$

$$u(0, t) = u(L, t) = 0, \quad t \in (0, +\infty), \tag{1.2}$$

$$u_t(L, t) = 0, \quad t \in (0, +\infty), \tag{1.3}$$

$$u(x, 0) = u_0(x), \quad x \in (0, L) \tag{1.4}$$

and the additional condition on $a(x)$:

$$a \in L^\infty(0, L) \text{ and } a(x) \geq a_0 > 0 \text{ a. e. in } \Omega \text{ where } \Omega \text{ is a nonempty open subset of } (0, L).$$

We establish a series of estimates for the continuous problem (1.1)-(1.4) that will be used for the numerical scheme in the next section. We start by stating the following existence result due to Faminskii [2]

Theorem 1.1 (See [2], Theorem 1) . Let $u_0 \in L^2(0, L)$ and $T > 0$ be given. Then, there exists a $T^* \in (0, T]$ such that the problem (1.1)-(1.4) admits a unique solution $u \in C([0, T^*] : L^2(0, L)) \cap L^2([0, T^*] : H^1_0(0, L)).$

Proposition 1.1 Let $u$ be the solution of problem (1.1)-(1.4) obtained in Theorem 1.1.

If $||u_0||_{L^2(0, L)} < \sqrt{\frac{3}{2}},$ then

$$||u||_{L^2(0, T; H^1_0(0, L))} \leq c \frac{||u_0||_{L^2(0, L)}}{1 - \frac{4}{9} ||u_0||_{L^2(0, L)}}, \tag{1.5}$$

where $c = c(T, L).$ Furthermore,

$$u_t \in L^{6/5}(0, T : H^{-2}(0, L)). \tag{1.6}$$
The following result was proven in [3], and corresponds to a result of stabilization and exponential decay for the continuous case of the solution of the equation (1.1)-(1.4)

**Theorem 1.2 (See [3], Theorem 3.3).** Let $u$ be the solution of problem (1.1)-(1.4) given by Theorem 1.1 and let $\Omega$ and $a = a(x)$ be as in introduction. Then, for any $0 < R < \sqrt{3/2}$ and $T > 0$, there exist positive constants $c = c(R, T)$ and $\mu = \mu(R)$ such that

$$E(t) \leq c \||u_0||_{L^2(0, L)}^2 e^{-\mu t}$$ (1.7)

holds for all $t > 0$ and $u_0$ satisfying $||u_0||_{L^2(0, L)} \leq R$.

For the discrete case, it is possible to find several numerical scheme with stabilization properties. The important here is to give a result of stabilization an exponential decay uniform respect to $\delta t$ and $\delta x$ (see for instance [7]). This is the result that we will prove in section 5.

We start first with a description of the scheme, the wellposedness, some estimate results and the convergence of the scheme.

## 2 Description of the numerical scheme

We consider finite differences based on the unconditionally stable schemes described in [1, 4]. We note by $u^n$ the approximate value of $u(j \delta x, n \delta t)$, solutions of the nonlinear problem (1.1)-(1.4) where $\delta x$ is the space-step, and $\delta t$ is the time-step, for $j = 0, \ldots, J$, and $n = 0, \ldots, N$. Let the discrete space :

$$X_J = \{ u = (u_0, u_1, \ldots, u_J) \in \mathbb{R}^{J+1} | \text{ with } u_0 = 0 \text{ and } u_J = u_{J-1} = 0 \}$$

and $(D^+u)_j = \frac{u_{j+1} - u_j}{\delta x}$, $(D^-u)_j = \frac{u_{j} - u_{j-1}}{\delta x}$, for $j = 1, \ldots, J - 1$, and $D = \frac{1}{2}(D^+ + D^-)$ the classical difference operators.

We consider a completely implicit numerical scheme for the approximation of the nonlinear problem (1.1)-(1.4) which reads as following :

$$\begin{align*}
\frac{u^{n+1}_j - u^n_j}{\delta t} + (Au^{n+1})_j + F(u^{n+1})_j + a_3 u^{n+1} = 0, & \quad j = 1, \ldots, J - 1, \\
u^n_0 = u^n_J = u^n_{J-1} = 0, \\
u^0 = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) dx, & \quad j = 1, \ldots, J - 1,
\end{align*}$$ (2.1-2.3)

where $x_{j+\frac{1}{2}} = j + \frac{1}{2} \delta x$ and $x_j = j \delta x$. The matrix $A \in \mathbb{R}^{(J-1) \times (J-1)}$ is an approximation of the dispersive term $u_{xxx}$ and the linear convective term $u_x$, for which, we add an artificial numerical viscosity of 4th order multiply by $\delta x^\theta$

$$A = \delta x^\theta D^- D^+ D^- + D^- D^+ D^+ + D,$$ (2.4)

with $0 < \theta \leq 1$. The different choices of the discrete operator $D^+$ and $D^-$ in the definition of $A$, is do it in order to have a positive defined matrix (see for instance [1, 4]). We will need to impose an additional boundary condition for the numerical viscosity term in this numerical scheme, which we specify more latter.

The approximation of the damping is given by $a_5 = (a_j)_{j=1}^{J-1} \in \mathbb{R}^{J-1}$, which is an approximation of $a(x)$ defined for each component by $a_j = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} a(x) dx$.

The nonlinearity $u^4 u_x$ of the equation (1.1), is approached by a nonlinear function $F(u^n)$, where $F : \mathbb{R}^{J-1} \rightarrow \mathbb{R}^{J-1}$ is given by the following expression

$$F(u)_j = u^3_j(Du)_j - \frac{5}{2} u^3_j (Du^2)_j + \frac{10}{3} a^3_j (Du^3)_j - \frac{5}{2} a_j (Du^4)_j + (Du^5)_j$$ (2.5)

for all $j = 1, \ldots, J - 1$. 

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Notation 2.1 We note the following internal product in $X_J$

$$(z,w) = \sum_{j=1}^{J-1} \delta x z_j w_j, \quad \text{and} \quad (z,w)_x = (z, xw) = \sum_{j=1}^{J-1} \delta x^2 z_j w_j$$

for all $z,w \in \mathbb{R}^{J+1}$, and the norms: $|z|_2 = \sqrt{(z,z)}$, $|z|_x = \sqrt{(z,z)_x}$, for all $z \in \mathbb{R}^{J+1}$. Additionally, we note the $p-$norms in $X_J$ defined by:

$$|z|_p = \left( \sum_{j=1}^{J-1} \delta x |z_j|^p \right)^{1/p}, \quad \text{and} \quad |z|_\infty = \max_{j=1,\ldots,J-1} |z_j|,$$

for all $z,w \in \mathbb{R}^{J+1}$.

First, we suppose that we can solve (2.1)-(2.3), and we give some estimate results for the solution of the numerical scheme. Thus, we will proof the existence and uniqueness of the solution of (2.1)-(2.3) more later. We have

Proposition 2.1 Let $(u^n)_{n \in \mathbb{N}}$ a sequence in $\bar{X}_J$ built by the numerical scheme (2.1)-(2.3), with $A$ and $F(u^n)$ defined by (2.4)-(2.5), and let be $T = n \delta t$. If $|u^n|_2 \leq \sqrt{\frac{3}{2}}$, then, there exist a constant $C > 0$ independent of $\delta t$ and $\delta x$, such that

$$\|Q_3 u^n\|_{L^\infty(0,T; L^2(0,L))} \leq |u^0|_2$$

$$\|P_3 u^n\|_{L^2(0,T; H^1_0(0,L))} \leq \left( T + L + \delta x + 2 \delta x^\theta \right) |u^0|_2$$

$$\|Q_3 D_+ D_- u^n\|_{L^2((0,T) \times (0,L))} \leq \frac{\delta x^{-\theta/2}}{\sqrt{2}} |u^0|_2$$

$$\frac{\partial}{\partial t} \left| P_3 u^n \right|_{L^{18/5}(0,T; H^{-3/2}(0,L))} \leq C.$$ (2.7)

Proposition 2.2 Let $u^0$ in $\bar{X}_J$, for $\delta t, \delta x > 0$ fixed. Then, there exists a unique solution $u_\delta = (u^n)_{n \in \mathbb{N}}$, with $u^n \in \bar{X}_J$, for the numerical scheme (2.1)-(2.3).

Theorem 2.1 Let $u_\delta = (u^n)_{n \in \mathbb{N}}$ a sequence in $X_J$ built by the numerical scheme (2.1)-(2.3), with $A$ and $F(u^n)$ defined by (2.4)-(2.5) If $|u^0|_2 \leq \sqrt{\frac{3}{2}}$, then

$$Q_3 u_\delta \rightarrow u, \quad \text{in} \quad L^4(0,T; L^4(0,L)) \text{ - strong}$$

as $\delta t, \delta x \rightarrow 0$, where $u$ is the solution of (1.1)-(1.4) in the sense of Theorem 1.1 and Proposition 1.2.

3 Uniform Numerical Exponential Decay

We consider the energy of the system (2.1)-(2.3) defined by

$$E_\delta(t) = |u^n|_2^2, \quad \text{for all} \quad n \delta t \leq t \leq (n+1)\delta t, \quad n \in \mathbb{N}.$$ (2.10)

We have the following result of uniform exponential decay of the energy for the solution of the numerical scheme

Theorem 3.1 Let $u_\delta = (u^n)_{n \in \mathbb{N}}$ the sequence in $\bar{X}_J$ built by the numerical scheme (2.1)-(2.3) and let $\Omega$ and $a = a(x)$ be as in introduction. Then, for any $0 < R < \sqrt{3/2}$ and $T > 0$, there exist positive constants $c = c(R,T)$ and $\mu = \mu(R)$, but both independent of $\delta t$ and $\delta x$ such that

$$E_\delta(t) \leq c |u^0|_2^2 e^{-\mu t}$$ (3.1)

holds for all $t > 0$ and $u_0$ satisfying $|u^0|_2 \leq R$. 

3
References


