Phase-field systems with nonlinear coupling and dynamic boundary conditions

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VIII Workshop
on Partial Differential Equations
Rio de Janeiro, August 25-28, 2009

Joint work with C.G.Gal, M.Grasselli, A.Miranville
Phase-field system

A well-known system of partial differential equations which describes the behavior of a two-phase material in presence of temperature variations, and neglecting mechanical stresses, is the so-called Caginalp phase-field system

\[
\begin{align*}
\delta \psi_t - D \Delta \psi + f(\psi) - \lambda'(\psi)\theta &= 0, \quad \Omega \times (0, \infty) \\
(\varepsilon \theta + \lambda(\psi))_t - k \Delta \theta &= 0, \quad \Omega \times (0, \infty)
\end{align*}
\]

- \( \Omega \subset \mathbb{R}^3 \), bounded domain with smooth boundary \( \Gamma \)
- \( \psi \) order parameter (or phase-field)
- \( \theta \) (relative) temperature
- \( \delta, D, \varepsilon, k \) positive coefficients
- \( \lambda \) function related to the latent heat
- \( f = F', F \) nonconvex potential (e.g., double well potential)
A naive derivation of the system

- **bulk free energy functional**

\[
E_\Omega(\psi, \theta) = \int_\Omega \left[ \frac{D}{2} |\nabla \psi|^2 + F(\psi) - \lambda(\psi)\theta - \frac{\varepsilon}{2} \theta^2 \right] \, dx
\]

- \( \partial_\theta E_\Omega(\psi, \theta), \partial_\psi E_\Omega(\psi, \theta) \): Frechét derivatives of \( E_\Omega(\psi, \theta) \)

- equation for the temperature

\[
(-\partial_\theta E_\Omega(\psi, \theta))_t + \nabla \cdot \mathbf{q} = 0, \quad \mathbf{q} = -k \nabla \theta
\]

- equation for the order parameter

\[
\psi_t = -\partial_\psi E_\Omega(\psi, \theta)
\]
Motivation for studying the case $\lambda$ nonlinear

- Assuming $\lambda$ linear is satisfactory for solid-liquid phase transitions

- However, when one deals, for instance, with phase transitions in ferromagnetic materials, where $\psi$ represents the fraction of lattice sites at which the spins are pointing “up”, then a quadratic $\lambda$ is a more appropriate choice (see M. Brokate & J. Sprekels (1996))
Standard boundary conditions for $\psi$ and $\theta$

- $\psi$: Neumann boundary condition
  \[ \partial_n \psi = 0 \]

- $n$: outward normal to $\Gamma$

- $\theta$: Dirichlet, Neumann, Robin boundary conditions
  \[ b \partial_n \theta + c \theta = 0 \]

- (D): $b = 0, c > 0$
- (N): $b > 0, c = 0$
- (R): $b > 0, c > 0$
In order to account for possible interaction with the boundary $\Gamma$, we can consider dynamic boundary conditions for $\psi$

- surface free energy functional

$$E_\Gamma(\psi) = \int_\Gamma \left[ \frac{\alpha}{2} |\nabla_\Gamma \psi|^2 + \frac{\beta}{2} \psi^2 + G(\psi) \right] dS$$

- $\nabla_\Gamma$ tangential gradient operator
- $\alpha, \beta > 0$, $G$ nonconvex boundary potential

- $\partial_\psi E_\Gamma(\psi)$: Frechét derivative of $E_\Gamma(\psi)$
- $\psi_t = -\partial_\psi E_\Gamma(\psi, \theta) - \partial_n \psi$

$$\psi_t - \alpha \Delta_\Gamma \psi + \partial_n \psi + \beta \psi + g(\psi) = 0$$

- $\Delta_\Gamma$ Laplace-Beltrami operator, $g = G'$
Without loss of generality we take $D = k = \delta = \varepsilon = 1$

- evolution equations in $\Omega \times (0, \infty)$

\[
\begin{align*}
\psi_t - \Delta \psi + f(\psi) - \lambda'(\psi)\theta &= 0 \\
(\theta + \lambda(\psi))_t - \Delta \theta &= 0
\end{align*}
\]

- boundary conditions on $\Gamma \times (0, \infty)$

\[
\begin{align*}
\psi_t - \alpha \Delta_{\Gamma} \psi + \partial_n \psi + \beta \psi + g(\psi) &= 0 \\
b \partial_n \theta + c \theta &= 0
\end{align*}
\]

- initial conditions on $\Omega$

\[
\theta(0) = \theta_0, \quad \psi(0) = \psi_0
\]
Main goals

- Well posedness
- Existence of the global attractor
- Existence of an exponential attractor
- Convergence of solutions to single equilibria
Definitions of global attractor and exponential attractor

- Dynamical system: \((\mathcal{X}, S(t))\)
- \(\mathcal{X}\): complete metric sp. \(S(t) : \mathcal{X} \to \mathcal{X}\) semigroup of op.
- Hausdorff semidistance: \(\text{dist}_\mathcal{X}(\mathcal{W}, \mathcal{Z}) = \sup_{w \in \mathcal{W}} \inf_{z \in \mathcal{Z}} d(w, z)\)

The **global attractor** \(A \subset \mathcal{X}\) is a compact set in \(\mathcal{X}\):
- \(A\) is fully invariant \((S(t)A = A, \ \forall t \geq 0)\)
- \(A\) is an attracting set w.r.t. H-semidistance:
  \[
  \forall \text{bdd } B \Rightarrow \lim_{t \to \infty} \text{dist}_\mathcal{X}(S(t)B, A) = 0
  \]

An **exponential attractor** \(E \subset \mathcal{X}\) is a compact set in \(\mathcal{X}\):
- \(E\) is invariant \((S(t)E \subseteq E, \ \forall t \geq 0)\)
- \(E\) has finite fractal dimension
- \(E\) is an exponential attracting set w.r.t. H-semidistance:
  \[
  \forall \text{bdd } B \Rightarrow \exists C_B > 0, \omega > 0 \text{ s.t.} \\
  \text{dist}_\mathcal{X}(S(t)B, E) \leq C_B e^{-\omega t}, \ \forall t \geq 0
  \]
- well-posedness

- longtime behavior of solutions
- existence and smoothness of global attractors
- existence of exponential attractors

Z. Zhang (2005)
- asymptotic behavior of single solutions
Literature:

\( \lambda \) nonlinear + standard b.c. + smooth potentials


- well-posedness, global and exponential attractors


- global and exponential attractors, asymptotic behavior of single solutions (nonhomogenous b.c.)
Literature:
\[ \lambda \text{ nonlinear} + \text{ standard b.c.} + \text{ singular potentials} \]


- well-posedness and asymptotic behavior
Literature:

λ linear + dynamic b.c. + smooth potentials

- $F$ with polynomial controlled growth of degree 6, $G \equiv 0$
- well-posedness
- convergence to single equilibria via Łojasiewicz-Simon inequality (when $F$ is also real analytic)

S.Gatti & A.Miranville (2006)
- construction of a s-continuous dissipative semigroup
- $\exists$ global attractor $A_\varepsilon$ upper semicontinuous at $\varepsilon = 0$
- $\exists$ exponential attractors $\mathcal{E}_\varepsilon$
Literature:
λ linear + dynamic b.c. + smooth potentials


- ∃ family of exponential attractors \{E_\varepsilon\} stable as \varepsilon \downarrow 0 when \partial_n \theta = 0


- F and G smooth potentials (more general than S.Gatti & A.Miranville)
- (possibly) dynamic boundary condition for \theta (a, b, c \geq 0)

\[
a \theta_t + b \partial_n \theta + c \theta = 0
\]
- construction of a dissipative semigroup (larger phase spaces w.r.t. S.Gatti & A.Miranville)
- ∃ global attractor, ∃ exponential attractors
Literature:
\( \lambda \) linear + dynamic b.c. + singular potential

**L. Cherfils & A. Miranville** (2007)
- \( F \) singular potential defined on \((-1, 1)\)
- \( G \) smooth potential (sign restrictions)
- construction of a s-continuous dissipative semigroup
- \( \exists \) global attractor of finite fractal dimension
- convergence to single equilibria via Ł-S method
  (when \( F \) is real analytic and \( G \equiv 0 \))

- \( F \) (strongly) singular potential defined on \((-1, 1)\)
- \( G \) smooth potential (sign restrictions are removed)
- separation property and existence of global solutions
- existence of global and exponential attractors
Notations

- $\| \cdot \|_p$ norm on $L^p(\Omega)$
- $\| \cdot \|_{p,\Gamma}$ norm on $L^p(\Gamma)$
- $\langle \cdot, \cdot \rangle_2$ usual scalar product inducing the norm on $L^2(\Omega)$ (even for vector-valued functions)
- $\langle \cdot, \cdot \rangle_{2,\Gamma}$ usual scalar product inducing the norm on $L^2(\Gamma)$ (even for vector-valued functions)
- $\| \cdot \|_{H^s(\Omega)}$ norm on $H^s(\Omega)$, for any $s > 0$
- $\| \cdot \|_{H^s(\Gamma)}$ norm on $H^s(\Gamma)$, for any $s > 0$
In order to account different cases of boundary conditions, we introduce the linear operators

$$A_K = -\Delta : D(A_K) \to L^2(\Omega)$$

- $D(A_K) = H_0^1(\Omega) \cap H^2(\Omega)$, if $K = D$
- $D(A_K) = \{ \theta \in H^2(\Omega) : b\partial_n \theta + c\theta = 0 \}$, if $K = N, R$

$D, N, R$ stand for Dirichlet, Neumann, or Robin bdry conds

- $A_K$ generates an analytic semigroup $e^{-A_K t}$ on $L^2(\Omega)$
- $A_K$ is nonnegative and self-adjoint on $L^2(\Omega)$
The functional spaces $\mathbb{Z}_K^1$

- $\mathbb{Z}_D^1 = H_0^1(\Omega)$
- $\mathbb{Z}_K^1 = H^1(\Omega)$, if $K \in \{N, R\}$

\[
\| \theta \|_{\mathbb{Z}_K^1}^2 = \begin{cases} 
\| \nabla \theta \|_2^2, & \text{if } K = D, \\
\| \nabla \theta \|_2^2 + \frac{c}{b} \| \theta \|_{2,\Gamma}^2, & \text{if } K = R, \\
\| \nabla \theta \|_2^2 + \langle \theta \rangle_{\Omega}^2, & \text{if } K = N,
\end{cases}
\]

where we have set

\[
\langle v \rangle_{\Omega} := |\Omega|^{-1} \int_{\Omega} v(x) \, dx
\]

the norm in $\mathbb{Z}_K^1$ is equivalent to the standard $H^1$-norm
The function spaces $\mathbb{V}_s$

- $\mathbb{V}_s = C^s(\Omega)^1, \quad s > 0$
- $\|\psi\|_{\mathbb{V}_s} = \left(\|\psi\|_{H^s(\Omega)}^2 + \|\psi\|_{H^s(\Gamma)}^2\right)^{1/2}$
- $\mathbb{V}_s = H^s(\Omega) \oplus H^s(\Gamma)$
- $\mathbb{V}_0 = L^2(\Omega) \oplus L^2(\Gamma)$
- $\mathbb{V}_s$ is compactly embedded in $\mathbb{V}_{s-1}, \quad \forall s \geq 1$

- $H^1(\Omega) \hookrightarrow L^6(\Omega)$
- $H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma)$
- $H^1(\Gamma) \hookrightarrow L^s(\Gamma), \quad \forall s \geq 1$
In the case $K = N$ we define the enthalpy

$$I_N (\psi(t), \theta(t)) := \langle \lambda (\psi(t)) + \theta(t) \rangle_\Omega$$

and this quantity is conserved in time for any given solution
The weak formulation

Problem $P^w_K$

$\forall (\psi_0, \theta_0) \in \mathcal{V}_1 \times L^2(\Omega)$ find $(\psi, \theta) \in C([0, +\infty); \mathcal{V}_1 \times L^2(\Omega))$:

- $\psi_t \in L^2([0, +\infty); \mathcal{V}_0)$, $\nabla \theta \in L^2([0, +\infty); (L^2(\Omega))^3)$

- $\langle \psi_t, u \rangle_2 + \langle \nabla \psi, \nabla u \rangle_2 + \langle f(\psi) - \lambda' \psi \theta, u \rangle_2 + \langle \psi_t, u \rangle_{2,\Gamma}$
  $+$ $\alpha \langle \nabla \Gamma \psi, \nabla \Gamma u \rangle_{2,\Gamma} + \langle \beta \psi + g(\psi), u \rangle_{2,\Gamma} = 0,$
  $\forall u \in \mathcal{V}_1, \text{ a.e. in } (0, \infty)$

- $\langle (\theta + \lambda(\psi))_t, v \rangle_2 + \langle \nabla \theta, \nabla v \rangle_2 + d \langle \theta, v \rangle_{2,\Gamma} = 0,$
  $\forall v \in \mathcal{Z}_K^1, \text{ a.e. in } (0, \infty)$

- $\theta(0) = \theta_0$, $\psi(0) = \psi_0$

and, if $K = N$,

- $I_N(\psi(t), \theta(t)) = I_N(\psi_0, \theta_0), \quad \forall t \geq 0$

Here $d = \frac{c}{b}$ if $K = R$, $d = 0$ otherwise
The strong formulation

Problem $P^s_K$

$\forall (\psi_0, \theta_0) \in V_2 \times Z^1_K$ find $(\psi, \theta) \in C ([0, +\infty); V_2 \times Z^1_K)$ :

- $(\psi_t, \theta_t) \in L^2 ([0, +\infty); V_1 \times L^2 (\Omega))$
- $A_K \theta \in L^2 ([0, +\infty) \times \Omega)$
- $\psi_t - \Delta \psi + f(\chi) - \lambda'(\psi) \theta = 0, \text{ a.e. in } \Omega \times (0, +\infty)$
- $(\theta + \lambda(\psi))_t - \Delta \theta = 0, \text{ a.e. in } \Omega \times (0, +\infty)$
- $\psi_t - \alpha \Delta \Gamma \psi + \partial_n \psi + \beta \psi + g(\psi) = 0, \text{ a.e. in } \Gamma \times (0, +\infty)$
- $b \partial_n \theta + c \theta = 0, \text{ a.e. in } \Gamma \times (0, +\infty)$

$\theta(0) = \theta_0, \quad \psi(0) = \psi_0$

and, if $K = N$,$$
\text{l}_N (\psi(t), \theta(t)) = \text{l}_N (\psi_0, \theta_0), \quad \forall t \geq 0$$
Assumptions on the nonlinearities $f$, $g$ and $\lambda$

(H1) $f, g \in C^1(\mathbb{R})$ satisfy
\[
\lim_{|y|\to+\infty} \inf f' (y) > 0, \quad \lim_{|y|\to+\infty} \inf g' (y) > 0
\]

(H2) \( \exists \ c_f, c_g > 0, \ q \in [1, +\infty) : \)
\[
|f' (y)| \leq c_f \left(1 + |y|^2\right), \quad |g' (y)| \leq c_g \left(1 + |y|^q\right), \quad \forall y \in \mathbb{R}
\]

(H3) $\lambda \in C^2(\mathbb{R}) : \ |\lambda'' (y)| \leq c_\lambda, \quad \forall y \in \mathbb{R},$ where $c_\lambda > 0$

(H4) \( \exists \ a > 0 \text{ and } \gamma \in C^2(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) : \)
\[
\lambda (y) = \gamma (y) - ay^2, \quad \forall y \in \mathbb{R}
\]

(H5) \( \exists \ \eta_1 > 0 \text{ and } \eta_2 \geq 0 : \)
\[
f (y) y \geq \eta_1 |y|^4 - \eta_2, \quad \forall y \in \mathbb{R}
\]
Remarks

Remark

*The growth restriction (H2) is only needed to analyze $P^w_K$. *

Remark

*Assumption (H4) is only needed to handle the case $K = N$. Note that (H4) is justified from a physical viewpoint (see, e.g., M. Brokate & J. Sprekels (1996)).*
The global existence result for $P^w_K$

Let $f, g, \lambda$ satisfy assumptions (H1), (H2) and (H3).

Then, for each $K \in \{D, N, R\}$, problem $P^w_K$ admits a global weak solution.
Sketch of the proof

Proof.

- suitable Faedo-Galerkin approximation scheme based on the characterization of a suitable self-adjoint positive operator $\mathcal{B}$ on $\mathcal{V}_0$ satisfying Wentzell b.c. (cf. C.G.Gal, G.Goldstein, J.Goldstein, S.Romanelli & M.Warma (2009))
- system of nonlinear ODEs for the approximating solutions
- a priori estimates for the approximating solutions
- global existence (in time) + strong convergence
- passage to the limit in the nonlinear terms
- convergence of the approximating solutions to the solution of $P^w_K$
The global existence result for $P^s_K$

**Theorem**

Let $f$, $g$, $\lambda$ satisfy (H1) and (H3).

Then, for each $K \in \{D, N, R\}$, problem $P^s_K$ admits a global strong solution.

**Proof.**

- global existence result for $P^w_K$
- higher order estimates
Lemma

Let $f$, $g$, $\lambda$ satisfy assumptions (H1), (H2) and (H3). Let $(\psi_{wi}, \theta_{wi})$ be two global solutions to $P^w_K$ corresponding to the initial data $(\psi_{0i}, \theta_{0i}) \in V_1 \times L^2(\Omega)$, $i = 1, 2$.

Then, for any $t \geq 0$, the following estimate holds:

$$
\| (\psi_{w1} - \psi_{w2}) (t) \|_{V_1}^2 + \| (\theta_{w1} - \theta_{w2}) (t) \|_2^2
+ \int_0^t \left[ \| (\psi_{w1} - \psi_{w2})_t (s) \|_{V_0}^2 + \| (\theta_{w1} - \theta_{w2}) (s) \|_{Z_1^K}^2 \right] ds
\leq C_w e^{L_w t} \left( \| \psi_{01} - \psi_{02} \|_{V_1}^2 + \| \theta_{01} - \theta_{02} \|_2^2 \right)
$$

Here $C_w$ and $L_w$ are positive constants depending on the norms of the initial data in $V_1 \times L^2(\Omega)$, on $\Omega$ and on the parameters of the problem, but are both independent of time.
Lemma

Let $f$, $g$, $\lambda$ satisfy (H1) and (H3).
Let $(\psi_{si}, \theta_{si})$ two global solutions to $P_s^K$ corresponding to the initial data $(\psi_{0i}, \theta_{0i}) \in V_2 \times Z_1^K$, $i = 1, 2$.

Then, for any $t \geq 0$, the following estimate holds:

$$
\|(\psi_{s1} - \psi_{s2})(t)\|_V^2 + \|(\theta_{s1} - \theta_{s2})(t)\|_Z^2 \\
+ \int_0^t \left[ \|(\psi_{s1} - \psi_{s2})_t(s)\|_V^2 + \|(\theta_{s1} - \theta_{s2})(s)\|_Z^1 \right] ds \\
\leq C_s e^{L_s t} \left( \|\psi_{01} - \psi_{02}\|_V^2 + \|\theta_{01} - \theta_{02}\|_Z^2 \right)
$$

Here $C_s$ and $L_s$ are positive constants depending on the norms of the initial data in $V_2 \times Z_1^K$, on $\Omega$ and on the parameters of the problem, but are both independent of time.
Corollary

Let $f$, $g$, $\lambda$ satisfy assumptions (H1), (H2) and (H3).

Then, for each $K \in \{D, N, R\}$, we can define a strongly continuous semigroup

$$S^K_w(t) : \mathbb{V}_1 \times L^2(\Omega) \to \mathbb{V}_1 \times L^2(\Omega)$$

by setting, for all $t \geq 0$,

$$S^K_w(t)(\psi_0, \theta_0) = (\psi_w(t), \theta_w(t))$$

where $(\psi_w, \theta_w)$ is the unique solution to $P^K_w$. 

The dynamical system generated by $P^s_K$

Corollary

Let $f$, $g$, $\lambda$ satisfy $(H1)$ and $(H3)$.

Then, for each $K \in \{D, N, R\}$, we can define a semigroup

$$S^s_K(t) : \mathbb{V}_2 \times \mathbb{Z}_K^1 \to \mathbb{V}_2 \times \mathbb{Z}_K^1$$

by setting, for all $t \geq 0$,

$$S^s_K(t) (\psi_0, \theta_0) = (\psi_s(t), \theta_s(t))$$

where $(\psi_s, \theta_s)$ is the unique solution to $P^s_K$. 
The phase spaces for $S^w_K$ and $S^s_K$

Due to the enthalpy conservation, in the case $K = N$ we define the complete metric spaces w.r.t. the metrics induced by the norms where the constraint $M \geq 0$ is fixed

\[
\left( V_1 \times L^2(\Omega) \right)^M := \left\{ (u, v) \in V_1 \times L^2(\Omega) : |I_N(u, v)| \leq M \right\}
\]

\[
\left( V_2 \times \mathbb{Z}_N^1 \right)^M := \left\{ (u, v) \in V_2 \times \mathbb{Z}_N^1 : |I_N(u, v)| \leq M \right\}
\]

- Phase-space for $S^w_K(t)$

\[
\mathbb{Y}_{0,K} = \begin{cases} 
V_1 \times L^2(\Omega), & \text{if } K \in \{D, R\} \\
\left( V_1 \times L^2(\Omega) \right)^M, & \text{if } K = N
\end{cases}
\]

- Phase-space for $S^s_K(t)$

\[
\mathbb{Y}_{1,K} = \begin{cases} 
V_2 \times \mathbb{Z}_K^1, & \text{if } K \in \{D, R\} \\
\left( V_2 \times \mathbb{Z}_N^1 \right)^M, & \text{if } K = N
\end{cases}
\]
Lemma

Let $f$, $g$ satisfy assumptions (H1) and (H5). Let $\lambda$ satisfies either (H3), if $K \in \{D, R\}$, or (H4), if $K = N$.

Then, $\forall (\psi_0, \theta_0) \in \mathbb{Y}_{0,K}$, the following estimate holds

$$\|(\psi(t), \theta(t))\|_{\mathbb{Y}_{0,K}}^2 + \int_{t}^{t+1} \left( \|\psi_t(s)\|_{\mathbb{V}_0}^2 + \|\theta(s)\|_{\mathbb{Z}_K^1}^2 + \|\psi(s)\|_{L^4(\Omega)}^4 \right) ds$$

$$\leq C_K \left( \|(\psi_0, \theta_0)\|_{\mathbb{Y}_{0,K}}^2 + \langle F(\psi_0), 1 \rangle_2 + \langle G(\psi_0), 1 \rangle_{2,\Gamma} + 1 \right) e^{-\rho t}$$

$$+ C^*_K, \quad \forall t \geq 0$$

where $F(y) = \int_0^y f(r) dr$, $G(y) = \int_0^y g(r) dr$, $\forall y \in \mathbb{R}$.

Here $\rho$, $C_K$, $C^*_K$ are independent of $t$ and of the initial data.
Existence of a compact absorbing set

Lemma

Let $f, g, \lambda$ satisfy (H1), (H2), (H3) and (H5). If $K = N$, assume also $\lambda$ fulfilling (H4). Then, $\forall R_0 > 0$, $\exists$ a positive nondecreasing monotone function $Q$ and $t_0 = t_0(R_0) > 0$:

$$\| S^w_K (t) (\psi_0, \theta_0) \|_{V^2 \times H^2(\Omega)} \leq Q(R_0), \quad \forall t \geq t_0$$

$\forall (\psi_0, \theta_0) \in B(R_0) \subset Y_{0,K}$, where $B(R_0)$ is a ball of radius $R_0$.

Lemma

Let $f, g, \lambda$ satisfy (H1), (H3) and (H5). If $K = N$, assume also $\lambda$ fulfilling (H4). Then, $\forall R_1 > 0$, $\exists$ a positive nondecreasing monotone function $Q$ and $t_1 = t_1(R_1) > 0$:

$$\| S^s_K (t) (\psi_0, \theta_0) \|_{V^3 \times H^3(\Omega)} \leq Q(R_1), \quad \forall t \geq t_1$$

$\forall (\psi_0, \theta_0) \in B(R_1) \subset Y_{1,K}$, where $B(R_1)$ is a ball of radius $R_1$. 
Existence of the global attractor

**Theorem**

Let $f$, $g$, $\lambda$ satisfy assumptions (H1), (H2), (H3) and (H5). If $K = N$, assume also $\lambda$ fulfilling (H4).

Then, $S^w_K(t)$ possesses the connected global attractor $A^w_K \subset \mathbb{Y}_0,K$, which is bounded in $\mathbb{V}_2 \times H^2(\Omega)$.

**Theorem**

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Then, $S^s_K(t)$ possesses the connected global attractor $A^s_K \subset \mathbb{Y}_1,K$, which is bounded in $\mathbb{V}_3 \times H^3(\Omega)$. 
Existence of exponential attractors for $S^w_K(t)$

**Theorem**

Let $f, g \in C^2(\mathbb{R})$ and $\lambda \in C^3(\mathbb{R})$ satisfy (H1), (H2), (H3) and (H5). If $K = N$, assume also $\lambda$ fulfilling (H4). Then, $S^w_K(t)$ has an exponential attractor $\mathcal{M}^w_K$, bounded in $\mathcal{V}_2 \times H^2(\Omega)$, namely, (I) $\mathcal{M}^w_K$ is compact and positively invariant w.r.t $S^w_K(t)$, i.e.,

$$S^w_K(t)(\mathcal{M}^w_K) \subset \mathcal{M}^w_K, \quad \forall \ t \geq 0.$$

(II) The fractal dimension of $\mathcal{M}^w_K$ w.r.t. $\mathcal{Y}_{0,K}$-metric is finite.

(III) There exist a positive nondecreasing monotone function $Q_w$ and a constant $\rho_w > 0$ such that

$$\text{dist}_{\mathcal{Y}_{0,K}}(S^w_K(t)B, \mathcal{M}^w_K) \leq Q_w(\|B\|_{\mathcal{Y}_{0,K}})e^{-\rho_w t}, \quad \forall \ t \geq 0,$$

where $B$ is any bounded set of initial data in $\mathcal{Y}_{0,K}$. Here $\text{dist}_{\mathcal{Y}_{0,K}}$ denotes the non-symmetric Hausdorff distance in $\mathcal{Y}_{0,K}$ and $\|B\|_{\mathcal{Y}_{0,K}}$ stands for the size of $B$ in $\mathcal{Y}_{0,K}$. 
Existence of exponential attractors for $S^s_K(t)$

**Theorem**

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$$S^s_K(t)(\mathcal{M}^s_K) \subset \mathcal{M}^s_K, \quad \forall \ t \geq 0.$$

(II) The fractal dimension of $\mathcal{M}^s_K$ w.r.t. $\mathbb{V}_1,K$-metric is finite.

(III) There exist a positive nondecreasing monotone function $Q_s$ and a constant $\rho_s > 0$ such that

$$\text{dist}_{\mathbb{V}_1,K}(S^s_K(t)B, \mathcal{M}^s_K) \leq Q_s(\|B\|_{\mathbb{V}_1,K})e^{-\rho_s t}, \quad \forall \ t \geq 0,$$

where $B$ is any bounded set of initial data in $\mathbb{V}_1,K$. Here $\text{dist}_{\mathbb{V}_1,K}$ denotes the non-symmetric Hausdorff distance in $\mathbb{V}_1,K$ and $\|B\|_{\mathbb{V}_1,K}$ stands for the size of $B$ in $\mathbb{V}_1,K$. 

Remark

Thanks to the existence of exponential attractors for $S^K_w(t)$ and $S^K_s(t)$ we deduce that the global attractors $A^K_w$ and $A^K_s$ have finite fractal dimension.
Proposition

Let \( f, g, \lambda \) satisfy (H1) and (H3). Then \( S^s_K(t) \) has a (strict) Lyapunov functional defined by the free energy, namely,

\[
\mathcal{L}_K(\psi_0, \theta_0) := \frac{1}{2} \left[ \| \nabla \psi_0 \|_2^2 + \alpha \| \nabla \Gamma \psi_0 \|_{2, \Gamma}^2 + \beta \| \psi_0 \|_{2, \Gamma}^2 + \| \theta_0 \|_2^2 \right]
+ \int_\Omega F(\psi_0(x)) \, dx + \int_\Gamma G(\psi_0(S)) \, dS, \quad K \in \{D, N, R\}
\]

where \( F(y) = \int_0^y f(r) \, dr \), \( G(y) = \int_0^y g(r) \, dr \), \( \forall y \in \mathbb{R} \).

In particular, for all \( t > 0 \), we have

\[
\frac{d}{dt} \mathcal{L}_K(S^s_K(t)(\psi_0, \theta_0)) = -\| \psi_t(t) \|_2^2 - \| \psi_t(t) \|_{2, \Gamma}^2 - \| \nabla \theta(t) \|_2^2 - d \| \theta |_{\Gamma}(t) \|_{2, \Gamma}^2
\]
Equilibrium points (strong solutions)

$(\psi_\infty, \theta_\infty) \in Y_{1,K}$ is an equilibrium for $P^s_K$ if and only if it is a solution to the boundary value problem

\[- \Delta \psi_\infty + f(\psi_\infty) - \lambda'(\psi_\infty)\theta_\infty = 0, \quad \text{in } \Omega\]
\[- \alpha \Delta_{\Gamma} \psi_\infty + \partial_n \psi_\infty + \beta \psi_\infty + g(\psi_\infty) = 0, \quad \text{on } \Gamma\]
\[- \Delta \theta_\infty = 0, \quad \text{in } \Omega,\]
\[b \partial_n \theta_\infty + c \theta_\infty = 0, \quad \text{on } \Gamma\]

- if $K \in \{D, R\}$, then $\theta_\infty \equiv 0$
- if $K = N$, then $\theta_\infty = l_N(\psi_0, \theta_0) - \langle \lambda(\psi_\infty) \rangle_\Omega$

If $K = N$, then $\psi_\infty$ is solution of a nonlocal boundary value problem ($\Rightarrow$ a special version of the Łojasiewicz-Simon inequality is needed)
$K = N$: convergence to equilibrium (strong solutions)

**Theorem**

Let $f$, $g$, $\lambda$ satisfy assumptions (H1), (H3), (H4) and (H5). Suppose, in addition, that the nonlinearities $F$, $G$, $\lambda$ are analytic.

Then, $\forall (\psi_0, \theta_0) \in \mathbb{Y}_{1,N}$, the solution $(\psi(t), \theta(t)) = S^s_N(t)(\psi_0, \theta_0)$ converges to a single equilibrium $(\psi_\infty, \theta_\infty)$ in the topology of $\mathbb{V}_2 \times \mathbb{Z}_N^1$, that is,

$$\lim_{t \to +\infty} \left( \|\psi(t) - \psi_\infty\|_{\mathbb{V}_2^2} + \|\theta(t) - \theta_\infty\|_{\mathbb{Z}_N^1} \right) = 0$$

Moreover, $\exists C > 0$ and $\xi \in (0, 1/2)$ depending on $(\psi_\infty, \theta_\infty)$:

$$\|\psi(t) - \psi_\infty\|_{\mathbb{V}_2^2} + \|\theta(t) - \theta_\infty\|_{\mathbb{Z}_N^1} + \|\psi_t(t)\|_{\mathbb{V}_2^0} \leq C(1 + t)^{-\xi/(1-2\xi)}$$

for all $t \geq 0$. 
<table>
<thead>
<tr>
<th>Remark</th>
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<tbody>
<tr>
<td>$A^s_K$ coincides with the unstable manifold of the set of equilibria.</td>
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<td>Since we have gradient systems with a set of equilibria bounded in the phase-space, we could avoid to prove the existence of a bounded absorbing set and we could directly show the existence of the global attractor.</td>
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<tr>
<td>However, the existence of a uniform dissipative estimate can be easily adapted to some nonautonomous systems.</td>
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</tbody>
</table>
Further issue: coupled dynamic boundary conditions

- surface free energy functional

\[
E_\Gamma(\psi, \theta) = \int_\Gamma \left[ \frac{\alpha}{2} |\nabla \Gamma \psi|^2 + \frac{\beta}{2} \psi^2 + G(\psi) - \ell(\psi)\theta - \frac{a}{2} \theta^2 \right] dS
\]

- \( \partial_\psi E_\Gamma(\psi, \theta) \), \( \partial_\theta E_\Gamma(\psi, \theta) \): Frechét derivatives of \( E_\Gamma(\psi, \theta) \)
- \( \psi_t = -\partial_\psi E_\Gamma(\psi, \theta) - \partial_n \psi \)
- \(- (\partial_\theta E_\Gamma(\psi, \theta))_t = -b\partial_n \theta - c\theta \)

\[
\begin{cases}
\psi_t - \alpha \Delta \Gamma \psi + \partial_n \psi + \beta \psi + g(\psi) - \ell'(\psi)\theta = 0 \\
(a\theta + \ell(\psi))_t + b\partial_n \theta + c\theta = 0
\end{cases}
\]

- \( a > 0 \), \( g = G' \)
- \( \ell \in C^2(\mathbb{R}) : |\ell''(y)| \leq c_\ell, \quad \forall y \in \mathbb{R} \), where \( c_\ell > 0 \)
Open issue: singular potentials

- bdry coupling with singular $F$ (and, possibly, $G$) of the form

$$F(s) = \gamma_1 [(1 + s) \ln(1 + s) + (1 - s) \ln(1 - s)] - \gamma_2 s^2$$
Open issue: memory effects

\[
\begin{align*}
\psi_t + \int_0^\infty h_1(s)(-\Delta \psi + f(\psi) - \lambda'(\psi)\theta)(t - s)ds &= 0 \\
(\theta + \lambda(\psi))_t - \int_0^\infty h_2(s)\Delta \theta(t - s)ds &= 0 \\
\psi_t + \int_0^\infty h_3(s)(-\Delta \Gamma \psi + \psi + g(\psi) + \partial_n \psi)(t - s)ds &= 0 \\
b \partial_n \theta + c \theta &= 0
\end{align*}
\]

\[
\theta(s) = \tilde{\theta}_0(-s), \quad \psi(s) = \tilde{\psi}_0(-s) \quad \text{in} \ \Omega, \ s \geq 0
\]

- $h_1, h_2, h_3 \geq 0$ smooth exp. decreasing relaxation kernels
Thanks for your attention!