On Zakharov-Kuznetsov Equation

VIII Workshop on Partial Differential Equations

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In this talk we will consider the initial value problem associated to the nonlinear equation
\[
\begin{aligned}
&\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \Delta u = 0, \\
&u(x, y, 0) = u_0(x, y)
\end{aligned}
\] 
(1)
called the modified Zakharov-Kuznetsov equation, where \( u \) is a real function defined in \( \mathbb{R}^2 \times \mathbb{R} \).
Outline

• Model
• Motivation and Previous Results
• Main Results
• Ingredients
• Ideas of the Proofs
• Final Remarks

Joint work with Ademir Pastor (IMPA), Jean-Claude Saut (Paris-Sud)
The equation under consideration is a 2D generalization of the Zakharov-Kuznetsov equation, that is,

\[ u_t + u \partial_x u + \partial_x \Delta u = 0, \]  

(2)

This equation was first derived by Zakharov and Kuznetsov (1974) in three-dimensional form to describe nonlinear ion–acoustic waves in a magnetized plasma. A variety of physical phenomena, are governed by this type of equation; for example, the long waves on a thin liquid film, the Rosby waves in rotating atmosphere, and the isolated vortex of the drift waves in three-dimensional plasma.
Even though the Zakharov-Kuznetsov equation seems a natural generalization of the Korteweg-de Vries equation,

$$\partial_t v + v \partial_x v + \partial_x^3 v = 0$$

The ZK equation is derived from the Euler-Poisson system for nonlinear ion-acoustic waves in a magnetized plasma.

$$\begin{cases}
    n_t + \text{div}(nv) = 0 \\
    v_t + (v \cdot \nabla)v + \nabla \varphi + a e_x \times v = 0 \\
    \Delta \varphi - e^{\varphi} + n = 0
\end{cases}$$

where

- $n =$ ion density
- $v =$ ion velocity
- $\varphi =$ electrostatic potential
- $a \geq 0$ measures the applied magnetic field
Critical character of the modified ZK equation.

- Local well-posedness

If we consider the IVP associated to the generalized Zakharov-Kuznetsov equation, i.e.,
\[ u_t + u^p \partial_x u + \partial_x \Delta u = 0, \quad u(x, y, 0) = u_0(x, y). \]

We can see that if \( u \) is a solution with data \( u_0 \), then \( u_\lambda(x, y, t) = \lambda u(\lambda x, \lambda y, \lambda^3 t) \) is also a solution with data \( u_\lambda(x, y, 0) = \lambda u_0(\lambda x, \lambda y) \). In particular, we have that
\[
\| u_\lambda(0) \|_{\dot{H}^s(\mathbb{R}^2)} = \lambda^{s-1 + \frac{2}{p}} \| u_0 \|_{\dot{H}^s(\mathbb{R}^2)},
\]
This means that derivatives of the solutions remain invariant only if
\[
s = 1 - \frac{2}{p}
\]
This scaling argument suggests local well-posedness for \( s \geq 1 - \frac{2}{p} \). In case \( p = 2 \), we have \( L^2(\mathbb{R}^2) \) as the possible larger space where local well-posedness can be obtained.
Global well-posedness

We note that the modified Zakharov-Kuznetsov equation has two conserved quantities, namely,

\[ I_1(u(t)) = \int_{\mathbb{R}^2} u^2(t) \, dx \, dy = \int_{\mathbb{R}^2} u_0^2 \, dx \, dy, \]

\[ I_2(u(t)) = \int_{\mathbb{R}^2} (u_x^2 + u_y^2 - \frac{1}{6} u^4)(t) \, dx \, dy = \int_{\mathbb{R}^2} (u_0^2_x + u_0^2_y - \frac{1}{6} u_0^4) \, dx \, dy. \]

One can establish a \( H^1(\mathbb{R}^2) \) an a priori estimate combining \( I_1 \) and \( I_2 \). Indeed,

\[ \|u(t)\|_{H^1}^2 = \|u(t)\|_{L^2}^2 + \|\partial_x u(t)\|_{L^2}^2 + \|\partial_y u(t)\|_{L^2}^2 \]

\[ = \|u_0\|_{L^2}^2 + I_2(u(0)) + \frac{1}{6} \int u^4(t) \, dx \, dy \]

Using Gagliardo-Nirenberg interpolation estimate we see that the last term is bounded by

\[ c \|u(t)\|_{L^2}^2 \left( \|\partial_x u\|_{L^2}^2 + \|\partial_y u\|_{L^2}^2 \right) \]

\[ = c \|u_0\|_{L^2}^2 \left( \|\partial_x u\|_{L^2}^2 + \|\partial_y u\|_{L^2}^2 \right). \]
Thus to obtain an *a priori* estimate we require $c\|u_0\|_{L^2}^2 < 1$. In fact,

$$\|u(t)\|_{H^1} \leq \|u_0\|_{L^2} + (1 - c\|u_0\|_{L^2}^2)^{-1} I_2(u_0).$$

One can be more precise regarding the size of the $L^2$-norm of the data.

Observe that a similar analysis can be done for the generalized ZK equation. In particular, solutions of the generalized ZK equation satisfy two conserved quantities as above and a priori estimate in $H^1$ can be established for data in $H^1$ with $H^1$-norm small and $p \geq 3$.

It is an open problem to show global well-posedness for the modified ZK equation (and generalized ZK equation) for any data. Numerical evidence suggests blow-up of solutions in finite time.
– Stability / Instability of solitary wave solutions

The existence of solitary wave solutions of the form \( \varphi(x, y) = \varphi(r), \ r = \sqrt{x^2 + y^2} \) for the generalized ZK equation was established by de Bouard.

• \( p = 1 \) stable

• \( p \geq 3 \) unstable
Previous Results

- Faminskii (1995) Local and Global well-posedness for ZK equation for data in $H^1(\mathbb{R}^2)$

- Biagioni-L (2003) Local and Global well-posedness for modified ZK equation for data in $H^1(\mathbb{R}^2)$

- L-Saut (2008) Local well-posedness for ZK equation in 3D for data $H^{1+}(\mathbb{R}^3)$

The notion of well-posedness we use is the one given by Kato, that is, existence, uniqueness, persistence property and continuous dependence upon the data.
Main Results

**Theorem 1.** For any \( u_0 \in H^s(\mathbb{R}^2), \ s > 3/4, \) there exist \( T = T(\|u_0\|_{H^s}) > 0 \) and a unique solution of the IVP associated to the modified ZK equation, defined in the interval \([0, T]\), such that

\[
\begin{align*}
  u & \in C([0, T]; H^s(\mathbb{R}^2)), \quad (3) \\
  \|D_x^s u_x\|_{L_x^\infty L_y^2} + \|D_y^s u_x\|_{L_x^\infty L_y^2} & < \infty, \quad (4) \\
  \|u\|_{L_T^3 L_x^\infty} + \|u_x\|_{L_T^{9/4} L_x^\infty} & < \infty, \quad (5) \\
  \|u\|_{L_x^2 L_y^\infty} & < \infty. \quad (6)
\end{align*}
\]

Moreover, for any \( T' \in (0, T) \) there exists a neighborhood \( W \) of \( u_0 \) in \( H^s(\mathbb{R}^2) \) such that the map \( \tilde{u}_0 \mapsto \tilde{u}(t) \) from \( W \) into the class defined by \((3)\)–\((6)\) is smooth.
Consider $\varphi$ the unique (up to translation) positive radial solution of the equation
\[-\Delta \varphi + \varphi - \varphi^3 = 0.\] (7)

Then we have the next global well-posedness result:

**Theorem 2.** Let $u_0 \in H^1(\mathbb{R}^2)$. If $\|u_0\|_{L^2} < \sqrt{3} \|\varphi\|_{L^2}$, $\varphi$ as in (7), then the local solution given in Theorem 1 can be extended to any time interval $[0, T]$.

**Remark 1.** One can prove that if the initial data $u_0$ belongs to $H^s(\mathbb{R}^2)$, $s > 19/21$, and satisfies $\|u_0\|_{L^2} < \sqrt{3} \|\varphi\|_{L^2}$, then the local solution given in Theorem 1 can also be extended globally in time. To prove this one can follow the argument used by Fonseca, L- and Ponce, following the ideas introduced in Bourgain, to established a global result for the critical KdV equation,
\[v_t + v^4v_x + v_{xxx} = 0.\]
We show that the minimum index of local well-posedness cannot be achieved. Actually, we will establish that we cannot have local well-posedness for data in $H^s(\mathbb{R}^2)$, $s \leq 0$ in the sense that the map data-solution, $u_0 \mapsto u(t)$, where $u(t)$ solves the IVP ($\theta$), is not uniformly continuous. In other words, we prove the following result:

**Theorem 3.** The IVP ($\theta$) is ill-posed for data in $H^s(\mathbb{R}^2)$, $s \leq 0$. 
Ingredients

– Local well-posedness

we consider the linear initial value problem

\[
\begin{aligned}
& u_t + \partial_x \Delta u = 0, \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \\
& u(x, y, 0) = u_0(x, y).
\end{aligned}
\] (8)

The solution of (8) is given by the unitary group \( \{U(t)\}_{t=-\infty}^{\infty} \) such that

\[
u(t) = U(t)u_0(x, y)
\]

\[
= \int_{\mathbb{R}^2} e^{i(t(\xi^3 + \xi \eta^2) + x \xi + y \eta)} \hat{u}_0(\xi, \eta) \, d\xi \, d\eta.
\]
Proposition 1. Let $0 \leq \varepsilon < 1/2$ and $0 \leq \theta \leq 1$. Then the group $\{U(t)\}_{t=-\infty}^{\infty}$ satisfies

$$\|D_{x}^{\theta \varepsilon/2} U(t) f\|_{L_t^q L_{xy}^p} \leq c \|f\|_{L_{xy}^2},$$

$$\|D_{x}^{\theta \varepsilon} \int_{-\infty}^{\infty} U(t - t') g(\cdot, t') dt'\|_{L_t^q L_{xy}^p} \leq c \|g\|_{L_t^{q'} L_{xy}^{p'}},$$

$$\|D_{x}^{\theta \varepsilon} \int_{-\infty}^{\infty} U(t) g(\cdot, t) dt\|_{L_{xy}^2} \leq c \|g\|_{L_t^{q'} L_{xy}^{p'}},$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ with

$$p = \frac{2}{1 - \theta} \quad \text{and} \quad \frac{2}{q} = \frac{\theta(2 + \varepsilon)}{3}.$$
As a consequence of Proposition 1 we have

Let $0 \leq \varepsilon < 1/2$. Then the group $\{U(t)\}_{t=-\infty}^{\infty}$ satisfies

$$\|U(t)f\|_{L_T^2 L_{xy}^\infty} \leq cT^\gamma \|D_x^{-\varepsilon/2} f\|_{L_{xy}^2} \quad (9)$$

and

$$\|U(t)f\|_{L_T^{9/4} L_{xy}^\infty} \leq cT^\delta \|D_x^{-\varepsilon/2} f\|_{L_{xy}^2}, \quad (10)$$

where $\gamma = (1 - \varepsilon)/6$ and $\delta = (2 - 3\varepsilon)/18$. 

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Smoothing Effect

**Lemma 1.** Let $u_0 \in L^2(\mathbb{R}^2)$. Then,

$$\|\partial_x U(t)u_0\|_{L_x^\infty L_y^2} \leq c\|u_0\|_{L^2_{xy}}.$$

Maximal Function Estimate

**Lemma 2.** Let $u_0 \in H^s(\mathbb{R}^2), s > 3/4$. Then,

$$\|U(t)u_0\|_{L_x^2 L_y^\infty} \leq c(s, T)\|u_0\|_{H^s_{xy}},$$

where $c(s, T)$ is a constant depending on $s$ and $T$.

Leibniz rule for fractional derivatives:

**Lemma 3.** Let $0 < \alpha < 1$ and $1 < p < \infty$. Then,

$$\|D^\alpha(fg) - fD^\alpha g - gD^\alpha f\|_{L^p(\mathbb{R})} \leq c\|g\|_{L^\infty(\mathbb{R})}\|D^\alpha f\|_{L^p(\mathbb{R})},$$

where $D^\alpha$ denotes either $D^\alpha_x$ or $D^\alpha_y$. 

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Proof of Theorem 1

Consider the integral operator

\[ \Psi(u)(t) = \Psi_{u_0}(u)(t) = U(t)u_0 + \int_0^t U(t-t')(u^2 u_x)(t')dt' \]

and define the metric spaces

\[ Y_T = \{ u \in C([0, T]; H^s(\mathbb{R}^2)); \| u \| < \infty \} \]

and

\[ Y^a_T = \{ u \in X_T; \| u \| \leq a \}, \]

with

\[ \| u \| := \| u \|_{L^\infty_T H^{s}_{xy}} + \| u \|_{L^3_T L^\infty_{xy}} + \| u_x \|_{L^{9/4}_T L^\infty_{xy}} \]
\[ + \| D^s_x u_x \|_{L^\infty_T L^2_{yT}} + \| D^s_y u_x \|_{L^\infty_T L^2_{yT}} + \| u \|_{L^2_T L^\infty_{xy}}, \]

where \( a, T > 0 \) will be chosen later.

We assume that \( 3/4 < s < 1 \) and \( T \leq 1 \).
We only sketch the estimate of the $H^s$-norm of $\Psi(u)$. Let $u \in \mathcal{Y}_T$. By using Minkowski’s inequality, group properties and Hölder inequality, we have

$$\|\Psi(u)(t)\|_{L^2_{xy}} \leq c\|u_0\|_{H^s} + c\int_0^T \|u\|_{L^2_{xy}} \|uu_x\|_{L^\infty_{xy}} dt' \leq c\|u_0\|_{H^s} + cT^{2/9}\|u\|_{L^\infty_T L^2_{xy}} \|u\|_{L^3_T L^\infty_{xy}} \|ux\|_{L^{9/4}_T L^\infty_{xy}}.$$
Using group properties, Minkowski’s inequality, Leibniz rule for fractional derivatives and Hölder’s inequality, we have

\[
\| D^s_x \Psi(u)(t) \|_{L^{2}_{xy}} \\
\leq \| D^s_x u_0 \|_{L^{2}_{xy}} + \int_0^T \| D^s_x (u^2 u_x)(t') \|_{L^{2}_{xy}} dt' \\
\leq c \| u_0 \|_{H^s} + c \int_0^T \| u_x \|_{L^{\infty}_{xy}} \| u \|_{L^{\infty}_{xy}} \| D^s_x u \|_{L^{2}_{xy}} dt' \\
+ \int_0^T \| u^2 D^s_x u_x \|_{L^{2}_{xy}} dt' \\
\leq c \| u_0 \|_{H^s} + c T^{2/9} \| u \|_{L^{\infty}_T H^s_{xy}} \| u \|_{L^{3}_T L^{\infty}_{xy}} \| u_x \|_{L^{9/4}_T L^{\infty}_{xy}} \\
+ \int_0^T \| u^2 D^s_x u_x \|_{L^{2}_{xy}} dt'.
\]
From Hölder’s inequality we get

\[
\int_0^T \| u^2 D_x^s u_x \|_{L_{xy}^2} dt' \leq \int_0^T \| u \|_{L_{xy}^\infty} \| u D_x^s u_x \|_{L_{xy}^2} dt'
\]

\[
\leq \| u \|_{L_{xy}^2}^2 \| D_x^s u_x \|_{L_{xy}^2 T}^2
\]

\[
\leq c T^{1/6} \| u \|_{L_T^3 L_{xy}^\infty} \| u \|_{L_T^2 L_{xy}^\infty} \| D_x^s u_x \|_{L_T^2 L_{xy}^\infty}.
\]

Thus,

\[
\| D_x^s \Psi(u)(t) \|_{L_{xy}^2}
\]

\[
\leq c \| u_0 \|_{H^s} + c T^{2/9} \| u \|_{L_T^\infty H_{xy}^s} \| u \|_{L_T^3 L_{xy}^\infty} \| u_x \|_{L_T^9/4 L_{xy}^\infty}
\]

\[ + c T^{1/6} \| u \|_{L_T^3 L_{xy}^\infty} \| u \|_{L_T^2 L_{xy}^\infty} \| D_x^s u_x \|_{L_T^2 L_{xy}^\infty}.
\]
Similarly,
\[
\|D^s_y \Psi(u)(t)\|_{L^2_{xy}} \\
\leq c \|u_0\|_{H^s} + cT^{2/9} \|u\|_{L^\infty_T H^s_{xy}} \|u\|_{L^3_T L^\infty_{xy}} \|u_x\|_{L^9_T L^\infty_{xy}} \\
+ cT^{1/6} \|u\|_{L^3_T L^\infty_{xy}} \|u\|_{L^2_x L^\infty_y} \|D^s_y u_x\|_{L^2_x L^\infty_y}.
\]

Therefore,
\[
\|\Psi(u)\|_{L^\infty_T H^s} \leq c \|u_0\|_{H^s} + cT^{1/6} \|u\|^3.
\]
Proof of Theorem 2

The main ingredients are the conserved quantities $I_1$, $I_2$ and the following Gagliardo-Nirenberg interpolation inequality

$$\frac{1}{6} \|u(t)\|_{L^4}^4 \leq \frac{1}{3} \left( \frac{\|u(t)\|_{L^2}}{\|\varphi\|_{L^2}} \right)^2 \|\nabla u(t)\|_{L^2}^2,$$

where $\varphi$ is the solution of (7). Thus we can estimate

$$\|u(t)\|_{H^1}^2 \leq \|u(t)\|_{L^2}^2 + I_2(u(t)) + \frac{1}{6} \|u(t)\|_{L^4}^4$$

$$= \|u_0\|_{L^2}^2 + I(u_0) + \frac{1}{6} \|u(t)\|_{L^4}^4$$

$$\leq \|u_0\|_{L^2}^2 + c \|u_0\|_{H^1}^2 + \frac{1}{3} \left( \frac{\|u_0\|_{L^2}}{\|\varphi\|_{L^2}} \right)^2 \|\nabla u(t)\|_{L^2}^2.$$

Hence using the hypothesis we obtain

$$\|u(t)\|_{H^1} \leq c \|u_0\|_{H^1}.$$

This a priori estimate and a standard argument yield the desired result.
Proof of Theorem 3

As we have already pointed out, a scaling argument suggests the IVP being locally well-posed for data in $H^s(\mathbb{R}^2)$, $s \geq 0$. We will show using an example that the IVP (1) is ill-posed in $H^s(\mathbb{R}^2)$, $s \leq 0$, in the sense that the map data-solution is not uniformly continuous.

Let $f(x, y)$ be the positive radial solution in $H^1(\mathbb{R}^2)$ of

$$\Delta f - f + f^3 = 0.$$ 

So, $f(x, y) = f(r)$, $r^2 = x^2 + y^2$. Now we define $g = g_c(r) = \sqrt{c}f(\sqrt{cr})$. It is easy to see that $g$ satisfies

$$\Delta g - cg + g^3 = 0.$$
Next we define $u = u_c(x, y, t) = g_c(\tilde{r})$, where
\[ \tilde{r}^2 = \xi^2 + y^2 \] with $\xi = x - ct$. Then it is easy to check that $u$ is a solution of (1) with initial datum $u_0(x, y) = \sqrt{c} f(\sqrt{cx}, \sqrt{cy})$. Moreover,
\[
\widehat{u_c(0)}(\xi, \eta) = \hat{u}_c(\xi, \eta, 0) = \frac{1}{\sqrt{c}} \hat{f} \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right).
\]

Let $c_1, c_2 > 0$. Let us evaluate the $L^2$-norm of the difference $u_{c_1}(0) - u_{c_2}(0)$. First, we note that if $\langle \cdot, \cdot \rangle_0$ denotes the inner product in $L^2(\mathbb{R}^2)$ then
\[
\langle u_{c_1}(0), u_{c_2}(0) \rangle_0 = \frac{1}{\sqrt{c_1 c_2}} \int_{\mathbb{R}^2} \hat{f} \left( \frac{\xi}{\sqrt{c_1}}, \frac{\eta}{\sqrt{c_1}} \right) \overline{\hat{f} \left( \frac{\xi}{\sqrt{c_2}}, \frac{\eta}{\sqrt{c_2}} \right)} \, d\xi d\eta
\]
\[
= \frac{\sqrt{c_1}}{\sqrt{c_2}} \int_{\mathbb{R}^2} \hat{f}(\xi_1, \eta_1) \overline{\hat{f} \left( \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1, \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_1 \right)} \, d\xi_1 d\eta_1.
\]

Hence, when $\theta := \frac{c_1}{c_2} \to 1$, we have
\[
\lim_{\theta \to 1} \langle u_{c_1}(0), u_{c_2}(0) \rangle_0 = \|f\|_0^2.
\]

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Also,
\[ \| u_c(0) \|_0^2 = \int_{\mathbb{R}^2} \left| \sqrt{c} f(\sqrt{c}x, \sqrt{c}y) \right|^2 \, dx \, dy = \int_{\mathbb{R}^2} |f(x, y)|^2 \, dx \, dy = \| f \|_0^2. \]

Therefore, we get
\[ \lim_{\theta \to 1} \| u_{c_1}(0) - u_{c_2}(0) \|_0^2 = 0. \]

Now for \( t > 0 \),
\[ u_c(x, y, t) = \sqrt{c} f \left( \sqrt{c}(x - ct, y) \right) \]
\[ = \sqrt{c} f \left( (\sqrt{c}x, \sqrt{c}y) - (c^{3/2}t, 0) \right) \]
\[ = \sqrt{c} \, \delta_{\sqrt{c} \tau_h f}(x, y), \]

where
\[ h = (c^{3/2}t, 0), \]
\[ \tau_{(a,b)} f(x, y) = f((x, y) - (a, b)) \]
\[ \delta_a f(x, y) = f(ax, ay). \]
Thus,

\[ \hat{u}_c(t)(\xi, \eta) = \hat{u}_c(\xi, \eta, t) = \frac{1}{\sqrt{c}} \tau_h \hat{f} \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right) = \frac{1}{\sqrt{c}} e^{-2\pi i (c^{3/2}t, 0) \cdot (c^{-1/2}\xi, c^{-1/2}\eta)} \hat{f} \left( \frac{\xi}{\sqrt{c}}, \frac{\eta}{\sqrt{c}} \right) \]

Next, we evaluate the \( L^2 \)-norm of \( u_{c_1}(t) - u_{c_2}(t) \):

\[
\langle u_{c_1}(t), u_{c_2}(t) \rangle_0 = \int_{\mathbb{R}^2} \hat{u}_{c_1}(\xi, \eta, t) \hat{u}_{c_2}(\xi, \eta, t) d\xi d\eta
= \frac{1}{\sqrt{c_1 c_2}} \int e^{2\pi i t (c_2 - c_1)} \hat{f} \left( \frac{\xi}{\sqrt{c_1}}, \frac{\eta}{\sqrt{c_1}} \right) \hat{\overline{f}} \left( \frac{\xi}{\sqrt{c_2}}, \frac{\eta}{\sqrt{c_2}} \right)
= \frac{\sqrt{c_1}}{\sqrt{c_2}} \int e^{2\pi i t \sqrt{c_1} (c_2 - c_1)} \hat{f} (\xi_1, \eta_1) \hat{\overline{f}} \left( \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1, \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_1 \right)
\]
Taking $c_1 = n + 1$ and $c_2 = n$ we have, by the Riemann-Lebesgue lemma,

$$\lim_{n \to \infty} \langle u_{n+1}(t), u_n(t) \rangle_0 = 0.$$ 

But, since

$$\|u_{n+1}(t)\|_0 = \|u_n\|_0 = \|f\|_0$$

we have

$$\lim_{\theta \to 1} \|u_{c_1}(t) - u_{c_2}(t)\|_0 = \lim_{n \to \infty} \|u_{n+1}(t) - u_n(t)\|_0$$

$$= \sqrt{2} \|f\|_0 \neq 0.$$ 

This finishes the proof of Theorem 3.
Final Remarks

We observe that Bourgain’s approach to deal with the KdV equation, does not seem to work in our case. Indeed, it is well-known that to obtain “good bounds” by using the Fourier restriction method we need to know very well the behavior of the resonant function, or equivalently, the geometry of the resonant set, which is the zero set of the resonant function. In general, if the geometry of the resonant set is too “complicated” then it is not clear how to perform dyadic decompositions to get the needed estimates. This is the situation in our case where the resonant function is given by

\[ h(\xi, \xi_1, \eta, \eta_1) = (\xi - \xi_1)(3\xi\xi_1 + \eta\eta_1) + (\eta - \eta_1)(\xi\eta_1 + \xi_1\eta). \]
In trying to improve the local well-posedness result in Theorem 1 we note that for the mKdV equation, by using a $L^4$-maximal function estimate, Kenig, Ponce and Vega obtained sharp well-posedness results for that equation. So, in lights of the mKdV we may ask if, taking into account a $L^4_x$-maximal function estimate, we can improve the result in Theorem 1. We establish the following sharp maximal function estimate for solutions of the linear problem associated to (\ref{mKdV}):

**Proposition 2.** For any $s > 1/4$, $r > 1/2$ and $0 \leq T \leq 1$, we have

$$
\|U(t)u_0\|_{L^4_xL^\infty_yT} \leq C\|(1 + D_x)^s(1 + D_y)^r u_0\|_{L^2_{xy}}.
$$

A similar estimate was proved by Kenig and Ziesler for solutions of the linear problem associated to the modified KP equation.
The Zakharov-Kuznetsov (ZK) equation

\[ u_t + u \partial_x u + \partial_x \Delta u = 0, \quad (11) \]

admits as a solution the well-known KdV solitary wave \( \phi_\omega(x, t) = \phi_\omega(x - \omega t) \), where

\[ \phi_\omega(\xi) = 3\omega \text{sech}^2 \left( \frac{\sqrt{\omega}}{2} \xi \right), \quad \omega > 0. \]

More generally, the \( N \)-soliton \( \phi^N \) of the KdV equation is also a particular solution of the ZK equation which is smooth and bounded together with its time and space derivatives and behaves as a sum of solitons of velocities \( 4n^2, 1 \leq n \leq N \) when \( t \to \infty \).

For instance, the 2-soliton \( \phi^2 \) is given by

\[ \phi^2(x, t) = 72 \frac{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)}{\{3 \cosh(x - 28t) + \cosh(3x - 36t)\}^2} \]
A fundamental issue is that of the transverse stability/instability of those one-dimensional “localized” solutions of the KdV equation (such as the solitary wave) with respect to transverse perturbations governed by the ZK equation. This question was rigorously addressed recently by Rousset and Tzvetkov who developed a general theory which applies in particular to one-dimensional transverse perturbations of the KdV solitary wave.

Functional framework for the Cauchy problem which should be suitable to describe the aforementioned transverse perturbations.

This framework cannot be the classical Sobolev spaces $H^s(\mathbb{R}^d)$ since the KdV soliton or multi-soliton do not belong to this class of spaces. A natural space to study the transverse stability of localized one-dimensional solutions should contain those solutions.
A first possibility consists in functions which are “localized” in $x$ and periodic in $y$. This leads to our study of the Cauchy problem for the ZK equation in $H^s(\mathbb{R} \times \mathbb{T})$.

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one-dimensional torus. We will thus consider the IVP associated to the ZK equation in a cylinder

$$\begin{cases} 
\partial_t u + \partial_x \Delta u + u \partial_x u = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T}, \quad t > 0, \\
u(x, y, 0) = u_0(x, y)
\end{cases}$$
A second possibility is to consider two-dimensional “localized” perturbations of the one-dimensional solution $\phi$. This motivates the study of the Cauchy problem,

$$\begin{cases}
    u_t + \partial_x \Delta u + u \partial_x u + \partial_x (\phi u) = 0, & (x, y) \in \mathbb{R}^2, \ t > 0 \\
    u(x, y, 0) = u_0(x, y).
\end{cases}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator and $\phi$ is the KdV solitary wave solitary wave or more generally any $N$-soliton of the KdV equation. Actually, we will only use that $\phi = \phi(x, t)$ is a solution of the KdV equation which is smooth and bounded together with its time and space derivatives, and furthermore belongs to the space $L^2_x L^\infty_T$. Those assumptions are obviously satisfied by the $N$-soliton solution of the KdV equation.