C^1 DENSITY OF AXIOM A FOR 1D DYNAMICS

DAVID DIICA

Abstract. We outline a proof of the C^1 density of Axiom A systems among the set of unimodal maps of the interval.

1. Introduction

The amazing theory of uniformly hyperbolic (i.e., *Axiom A*) systems developed by Smale, Anosov, Bowen, Ruelle (among several others) during the 60s and 70s provides a unified framework to completely understand the dynamics of Axiom A system from both topological and probabilistic point of view. Motivated by this astonishing success, Smale conjectured that Axiom A is open and *dense* among the set of dynamical systems.[1] This conjecture is nowadays known to be *false* in general: for instance, in the 70s, Newhouse exhibited an open set of nonhyperbolic systems by constructing a locally residual set of *surface* diffeomorphisms with infinitely many sinks/sources (via the bifurcation of *homoclinic tangencies*). On the other hand, in dimension 1, the situation is more favorable to Smale's conjecture. In particular, Jakobson (1971) proved that Smale's conjecture for one-dimensional systems holds in the C^1-topology. More recently, Kozlovski, Shen and van Strien (2007) settled Smale's conjecture in the C^r-topology for any $r \geq 2$.

The bulk of this note is the discussion of a proof of one interesting particular case of Jakobson's result quoted above. More precisely, we would like to sketch the proof of the following theorem:

Theorem 1.1 (Jakobson [J])**.** *Axiom A is C^1-dense among the set of* unimodal[2] *maps of the interval $I = [-1, 1]$.*

The original argument of Jakobson is a little bit long because he deals with multimodal maps (i.e., intervals maps with several critical points) and his criterion of uniform hyperbolicity is somewhat technical. For these reasons, we shall adopt here a more modern strategy outlined in the excelent book [dMS] of W. de Melo and S. van Strien: using Mañé's criterion [M] of hyperbolicity for intervals maps[3], we know that uniform hyperbolicity holds for the set of points whose orbits stay "away" from the critical point. Thus, it remains only to analyze the dynamics nearby the criticality. At this point, the idea is the following: after a C^1-small perturbation, it is possible to put the orbit of the critical point inside the basin of

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[1] It is well known that Axiom A is an open property, so that the non-trivial assertion of Smale's conjeture is the denseness claim.

[2] Recall that a map f of the interval is called *unimodal* if f has exactly one critical point c.

[3] Certainly not available at the time of writing of the paper [J].

attraction of a *sink* (i.e., an attracting periodic point). Once we get this fact, the proof of theorem 1.1 is complete since we are showing that the points of the non-wandering set should stay far away from the critical point[4]. By Mañé's criterion, it follows that the non-wandering set is the union of finitely many sinks and a compact invariant hyperbolic set (i.e., the map is Axiom A).[5] Bearing this plan in mind, let us begin the proof of Jakobson's C^1-density result.

2. PROOF OF THEOREM 1.1

We start with a fundamental criterion of hyperbolicity due to Mañé:

Theorem 2.1 (Mañé). *Let f be a C^2 endomorphism of the interval I and U be an open neighborhood of the set $C(f)$ of critical points of f. Denote by $\Sigma(f)$ the union of the basins of attractions of the sinks of f. It holds:*

- *any periodic orbit of f inside $I - U$ with* sufficiently high period *is a source;*
- *if* every *periodic point of f inside $I - U$ is a source, then there are some constants $0 < \lambda < 1$ and $C > 0$ such that $\|Df^n(x)\| \geq C\lambda^n$ for any $\{x, f(x), \dots, f^n(x)\} \subset I - (U \cup \Sigma(f))$.*

Corollary 2.1 (Mañé). *Let f be a C^2 endomorphism of the interval I and $\Lambda \subset I$ be a compact f-invariant set. Assume that all periodic point of Λ are sources and Λ does not contain critical points of f. Then, Λ is hyperbolic*[6].

The reader should pay attention to the C^2 regularity hypothesis in the statements above (which is a necessary assumption). For a proof of these results see [M] (if you can read Portuguese, see also [CM]).

Keeping these results in our toolbox, we are ready to begin the proof of the C^1 density of Axiom A among unimodal maps. Let f be a C^1 unimodal map of the interval $I = [-1, 1]$. Without loss of generality, we can suppose that:

- f is C^2
- 0 is the critical point of f and
- f is Kupka-Smale.

For sake of simplicity, we denote by $\mathcal{D}^2$ the set of endomorphisms of $I = [-1, 1]$ verifying the three conditions above.

Proposition 2.1. *For a typical (Baire generic) $f \in \mathcal{D}^2$, the critical point 0 is recurrent or it falls into the basin of some sink.*

Proof. For each $\varepsilon > 0$, let $R(\varepsilon)$ be the set of $g \in \mathcal{D}^2$ such that either $g^n(0) \in (-\varepsilon, \varepsilon)$ for some $n \geq 1$ or 0 falls into the basin of a sink of g. We claim that $R(\varepsilon)$ is open and dense (for every $\varepsilon > 0$). Indeed, since $R(\varepsilon)$ is clearly open, our task is reduced to show that $R(\varepsilon)$ is dense.

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[4] If the orbit of a point of the non-wandering set is very close to the critical point, it falls into the basin of a sink (since we are assuming that the critical point is absorbed by a sink). In particular the orbit of this point is wandering, a contradiction.

[5] Usually, Axiom A requires that the periodic points are dense in the non-wandering set and the non-wandering set is hyperbolic. However, since the endomorphisms considered above are not invertible, generally speaking the non-wandering set is not invariant by backward iteration, so that a slight modification of the Axiom A is needed.

[6] I.e., there are constants $0 < \lambda < 1$ and $C > 0$ such that $\|Df^n(x)\| \geq C\lambda^n$ for all $n \geq 0$ and $x \in \Lambda$.

Given $\varepsilon > 0$ and $g \in \mathcal{D}^2 - R(\varepsilon)$, consider

$$\Lambda_\varepsilon(g) := \bigcap_{n\geq 0} g^{-n}(I - (-\varepsilon, \varepsilon)) - \Sigma(g).$$

Note that $\Lambda_\varepsilon(g)$ is a compact invariant set without critical points and sinks. Since g is Kupka-Smale, every periodic point inside $\Lambda_\varepsilon(g)$ must be a source, so that the corollary 2.1 implies that $\Lambda_\varepsilon(g)$ is hyperbolic, i.e., for some $0 < \lambda < 1$, $C > 0$, we have $\|Df^n(x)\| \geq C\lambda^n$ for all $x \in \Lambda_\varepsilon(g)$. In particular, $\Lambda_\varepsilon(g)$ is a *Cantor set* (i.e., a compact set with empty interior).[7] On the other hand, since $g \notin R(\varepsilon)$, it follows that $g(0) \in \Lambda_\varepsilon(g)$.

At this point, we use the following very simple idea: because $\Lambda_\varepsilon(g)$ is a Cantor set, the condition $g(0) \in \Lambda_\varepsilon(g)$ can not be generic (so that it can be destroyed by small perturbation). More precisely, we consider $f \in \mathcal{D}^2$ a C^1-small perturbation of g supported on $(-\varepsilon, \varepsilon)$ such that $f(0) \notin \Lambda_\varepsilon(g)$. We claim that $f \in R(\varepsilon)$. Indeed, since f coincides with g outside $(-\varepsilon, \varepsilon)$, we get

$$\bigcap_{n\geq 0} f^{-n}(I - (-\varepsilon, \varepsilon)) = \bigcap_{n\geq 0} g^{-n}(I - (-\varepsilon, \varepsilon)).$$

Thus, if $f \notin R(\varepsilon)$, we have

$$f(0) \in \bigcap_{n\geq 0} f^{-n}(I - (-\varepsilon, \varepsilon)) = \bigcap_{n\geq 0} g^{-n}(I - (-\varepsilon, \varepsilon)).$$

Hence, using that $f(0) \notin \Lambda_\varepsilon(g)$ by construction, we obtain $f(0) \in \Sigma(g)$, that is, $f(0)$ is attracted by a sink of g. However, since the orbit of $f(0)$ under f (and consequently g) never touches the interval $(-\varepsilon, \varepsilon)$, one sees that the orbit of sink of g attracting $f(0)$ belongs to $I - (-\varepsilon, \varepsilon)$. Therefore, the sink of g attracting $f(0)$ is also a sink of f. In other words, $0 \in \Sigma(f)$, i.e., $f \in R(\varepsilon)$, an absurd. This shows that $R(\varepsilon)$ is open and dense for any $\varepsilon > 0$.

Finally, we complete the proof of the proposition by taking the residual set $R := \bigcap\limits_{n\in\mathbb{N}} R(1/n)$ of $\mathcal{D}^2$. Clearly, any $f \in R$ verifies the statement of the proposition. This finishes the proof. $\square$

Remark 2.1. *The argument used in the proof of this proposition does not rely on the C^1-topology so that a similar statement holds with respect to the C^r-topology for any $r \geq 2$.*

Up to now, we do not used the C^1-topology in our discussion. However, the next proposition strongly relies on the C^1-topology and it is the main obstruction for an extension of Jakobson's result to the C^2-topology, for instance.

Proposition 2.2 (Flatness perturbation)**.** *Let $f \in \mathcal{D}^2$ such that 0 is recurrent, i.e., $0 \in \omega(0)$. Then, there exists an arbitrarily small C^1-perturbation $g \in \mathcal{D}^2$ of f such that 0 belongs to the basin of attraction of a sink of g, i.e., $0 \in \Sigma(g)$.*

Proof. Since the $\varepsilon - \delta$ management of this argument is a little bit hard (while the idea behind it is quite clear), we'll just sketch the proof of the proposition. We start by selecting a large integer n such that $f^n(0)$ is very close to 0 and $|f^k(0)| = d(f^k(0), 0) > d(f^n(0), 0) = |f^n(0)|$ for all $0 < k < n$. This is always possible because we are assuming that 0 is recurrent.

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[7] In fact, if the interior of $\Lambda_\varepsilon(g)$ were non-empty, say $J \subset \Lambda_\varepsilon(g)$ where J is a non-trivial, it would follow that $g^n(J) \subset \Lambda_\varepsilon(g)$ so that the condition of hyperbolicity implies $\ell(g^n(J)) \geq C\lambda^n\ell(J)$ for all $n \geq 1$. Thus, $\ell(g^n(J)) \to \infty$ when $n \to \infty$, a contradiction with $g^n(J) \subset I$ and $\ell(I) = 2$.

Denote by $J = [f^n(0), 0]$. Since $0 \in \omega(0)$, we can made J arbitrarily small and we can select $m > n$ such that $f^m(0) \in J$. For sake of convenience, we take $m > n$ minimal for the property $f^m(0) \in J$. We have two possibilities:

- (a) $f^m(0)$ is more close to 0 than $f^n(0)$, i.e., $d(f^m(0), 0) \leq d(f^m(0), f^n(0))$;
- (b) $f^m(0)$ is more close to $f^n(0)$ than 0, i.e., $d(f^m(0), f^n(0)) \leq d(f^m(0), 0)$.

In the first case (a), we apply a "Closing Lemma" argument: take z to be the middle point between $f^m(0)$ and $f^n(0)$ and U a small open neighborhood of $[z, 0]$ with $f^k(0) \notin \overline{U}$ for all $0 < k < m$. See the figure 1 below.

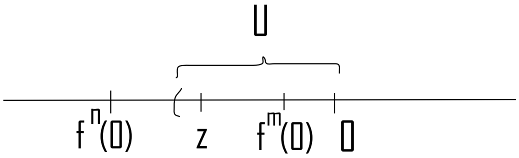

Figure 1. Case (a): $f^m(0)$ is more close to 0 than $f^n(0)$.

In this situation, we modify f inside U as follows: denoting by $c_m = f^m(0)$, take h C^1-close to f so that $h(c_m) = f(0)$, $h|U$ has an unique critical point at c_m and $h = f$ outside U. It follows that h is unimodal with critical point c_m such that c_m is periodic[8]. By a C^1-perturbation of h, we obtain g verifying the conclusion of the proposition (in fact, it turns out that here the critical point itself is a super-attracting sink).

Finally, in the second case (b), we introduce the "flatness perturbation": take z to be the middle point between 0 and $f^m(0)$ and U a small open neighborhood of $[f^n(z), f^m(z)]$ such that $f^k(0) \notin \overline{U}$ for all $0 < k < m$ with $k \neq n$. See the figure 2 below.

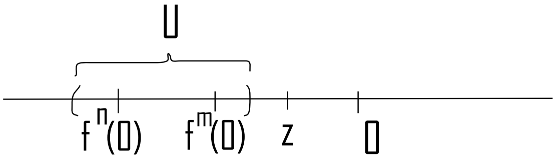

Figure 2. Case (b): $f^m(0)$ is more close to $f^n(0)$ than 0.

In this context, we modify f as follows: take h C^1-close to f so that h is *constant*[9] on $[f^n(0), f^m(0)]$ and $h(f^n(0)) = f(f^n(c))$. Note that $f^n(0)$ is a periodic point of h with period $m-n$. Moreover, $f^n(0)$ is a super-attracting sink of h (because h is constant on $[f^n(0), f^m(0)]$ and thus $h'(f^n(0)) = 0$). This allows us to make a C^1-perturbation g of h so that g is unimodal with derivative g' almost zero (but never vanishing) on U and g possesses a sink whose basin contains $f^n(0) = g^n(0)$. See the figure 3 below.

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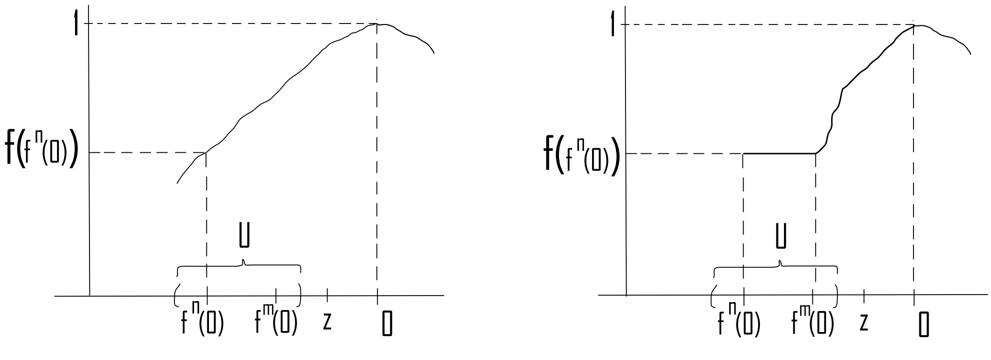
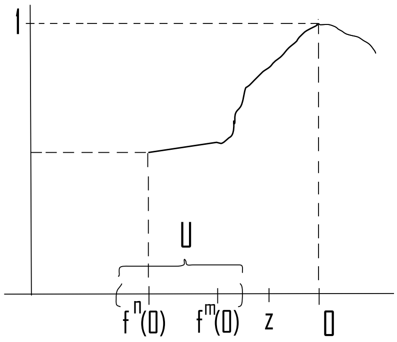
[8] The fact that the recurrent point c_m becomes periodic justifies the name "Closing Lemma" for this argument.

[9] Note that this perturbation can be done (*only*) in the C^1-topology since we know that 0 is a critical point of f and $f^n(0), f^m(0)$ are fairly close to 0.



Graph of f(x)

Graph of h(x)



Graph of g(x)

FIGURE 3. Flatness perturbation – Case (b).

Consequently, $g \in \mathcal{D}^2$ is C^1-close to f and $0 \in \Sigma(g)$. This completes the proof. $\square$

Once these two propositions are proved, the theorem 1.1 follows directly. Indeed, given f a C^1 endormorphism of the interval I, we can assume that $f \in \mathcal{D}^2$ (as discussed before). Next, we approximate f by a "typical" $g \in \mathcal{D}^2$ so that the proposition 2.1 says that either $0 \in \Sigma(g)$ or $0 \in \omega(0)$.

If $0 \in \Sigma(g)$, say the critical point 0 falls by iteration into the basin of a sink p of g), we see that there exists $\varepsilon > 0$ such that the neighborhood $(-\varepsilon, \varepsilon)$ falls into the basin of attraction of the sink p. On the other hand, since $g \in \mathcal{D}^2$, we know that $\Lambda_\varepsilon(g) = \bigcap\limits_{n \geq 0} g^n(I - (-\varepsilon, \varepsilon))$ is the union of an uniformly expanding hyperbolic set and a finite number of sinks. In particular, we get that g is an Axiom A endomorphism arbitrarily C^1-close to f.

If $0 \in \omega(0)$ (i.e., the critical point 0 is recurrent), we apply the proposition 2.2 to get an endomorphism $h \in \mathcal{D}^2$ arbitrarily C^1-close to g so that $0 \in \Sigma(h)$. Therefore, the discussion of the previous paragraph gives us some Axiom A endomorphism k arbitrarily C^1-close to h.

Thus, in any case, f can be C^1-approximated by an Axiom A endomorphism. This completes the proof of theorem 1.1.

REFERENCES

[CM] C. MATHEUS, A prova de Pujals-Sambarino da conjectura de Palis em superfícies II: o teorema de Mañé, *available at the webpage* www.impa.br/∼cmateus, 2007.

[dMS] W. DE MELO AND S. VAN STRIEN, One-dimesional dynamics, *Springer-Verlag*, New York, vol.25, 1993.

[J] M. JAKOBSON, Smooth mappings of the circle into itself, *Math. USSR-Sb.*, vol. 14, p. 161–185, 1971.

[M] R. MAÑÉ, Hyperbolicity, sinks and measure in one-dimensional dynamics, *Comm. Math. Phys.*, vol. 100, p. 495–524, 1985.