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# Rough solutions for the periodic Schrödinger–Korteweg–de Vries system

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**Abstract**

We prove two new mixed sharp bilinear estimates of Schrödinger–Airy type. In particular, we obtain the local well-posedness of the Cauchy problem of the Schrödinger–Kortweg–de Vries (NLS–KdV) system in the *periodic setting*. Our lowest regularity is $H^{1/4} \times L^2$, which is somewhat far from the naturally expected endpoint $L^2 \times H^{-1/2}$. This is a novel phenomena related to the periodicity condition. Indeed, in the continuous case, Corcho and Linares proved local well-posedness for the natural endpoint $L^2 \times H^{-3/4+}$.

Nevertheless, we conclude the global well-posedness of the NLS–KdV system in the energy space $H^1 \times H^1$ using our local well-posedness result and three conservation laws discovered by M. Tsutsumi.


*Keywords:* Local and global well-posedness; Schrödinger–Korteweg–de Vries system

## 1. Introduction

The interaction of a short-wave $u = u(x,t)$ and a long-wave $v = v(x,t)$ in fluid mechanics (and plasma physics) is governed by the Schrödinger–Korteweg–de Vries (NLS–KdV) system

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$$\begin{cases} i\partial_t u + \partial_x^2 u = \alpha u v + \beta |u|^2 u, \quad t \in \mathbb{R}, \\ \partial_t v + \partial_x^3 v + \frac{1}{2}\partial_x(v^2) = \gamma \partial_x(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex-valued function, $v = v(x,t)$ is a real-valued function and $\alpha$, $\beta$, $\gamma$ are real constants.[1]

This physical model of interaction of waves motivates the mathematical study of the local and global well-posedness of the Cauchy problem for the NLS–KdV system with rough initial data.[2]

The central theme of this paper is the local and global well-posedness theory of the NLS–KdV system in the periodic setting (i.e., $x \in \mathbb{T}$); but, in order to motivate our subsequent results, we recall some known theorems in the nonperiodic setting.

In the continuous context (i.e., $x \in \mathbb{R}$), Corcho and Linares [7] showed the local well-posedness of the NLS–KdV for initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \geqslant 0$, $s > -3/4$ provided that $k - 1 \leqslant s \leqslant 2k - 1/2$ for $k \leqslant 1/2$ and $k - 1 \leqslant s < k + 1/2$ for $k > 1/2$. It is worth to point out that the lowest regularity obtained by Corcho and Linares is $k = 0$ and $s = -3/4+$. In the nonresonant case $\beta \neq 0$, it is reasonable to expect that the NLS–KdV is locally well-posed in $L^2 \times H^{-3/4+}$: the nonlinear Schrödinger (NLS) equation with cubic term ($|u|^2 u$) is globally well-posed in $H^s(\mathbb{R})$ for $s \geqslant 0$ and ill-posed below $L^2(\mathbb{R})$; similarly, the Kortweg–de Vries (KdV) equation is globally well-posed in $H^s(\mathbb{R})$ for $s > -3/4$ and ill-posed in $H^s(\mathbb{R})$ for $-1 \leqslant s < -3/4$. Also, using three conserved quantities for the NLS–KdV flow, M. Tsutsumi [15] showed global well-posedness for initial data $(u_0, v_0) \in H^{s+1/2}(\mathbb{R}) \times H^s(\mathbb{R})$ with $s \in \mathbb{Z}_+$, and Corcho and Linares [7], assuming $\alpha\gamma > 0$, showed global well-posedness in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.[3] See also [2–4,9] for some related references.

The point of view adopted by Corcho and Linares in order to prove their local well-posedness result is to use a basic strategy to treat, in both continuous and periodic contexts, the low-regularity study of dispersive equations (such as NLS and KdV): one considers the Fourier restriction norm method introduced by Bourgain in [5]; then, they showed two new mixed bilinear estimates for the coupling terms of the NLS–KdV system (namely, $uv$ and $\partial_x(|u|^2)$) in certain Bourgain's spaces, which implies that an equivalent integral equation can be solved by Picard's fixed point method (in other words, the operator associated to the integral equation is a contraction in certain Bourgain spaces). Coming back to the periodic setting, before stating our results, we advance that, although our efforts are to obtain similar well-posedness theorems, the periodic case is more subtle than the continuous context: since the cubic NLS is globally well-posed (respectively, ill-posed) in $H^s(\mathbb{T})$ for $s \geqslant 0$ (respectively, $s < 0$) and the KdV is globally well-posed (respectively, ill-posed) in $H^s(\mathbb{T})$ for $s \geqslant -1/2$ (respectively, $s < -1/2$), it is reasonable to expect $L^2(\mathbb{T}) \times H^{-1/2}(\mathbb{T})$ as the lowest regularity for the local well-posedness results; but, surprisingly enough, the endpoint of the bilinear estimates for the coupling terms

[1] The case $\beta = 0$ of the NLS–KdV system occurs in the study of the resonant interaction between short and long capillary-gravity waves on water channels of uniform finite depth and in a diatomic lattice system. For more details about these physical applications, see [8,10,11,14].

[2] Benilov and Burtsev in [1] showed that the NLS–KdV is not completely integrable. In particular, the solvability of (1.1) depends on the theory of nonlinear dispersive equations.

[3] Pecher [13] announced the global well-posedness of the NLS–KdV system (with $\alpha\gamma > 0$) in the continuous setting for initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, for $3/5 < s < 1$ in the resonant case $\beta = 0$ and $2/3 < s < 1$ in the nonresonant case $\beta \neq 0$. The proof is based on two refined bilinear estimates and the I-method of Colliander, Keel, Staffilani, Takaoka and Tao.

$uv$, $\partial_x(|u|^2)$ in the periodic setting is $(k, s) = (1/4, 0)$, i.e., our lowest regularity is $H^{1/4} \times L^2$ (see Propositions 1.1, 1.2, Theorem 1.1 and Remark 1.1 below). We refer the reader to the Section 6 for a more detailed comparison between the well-posedness results for the NLS–KdV system in the periodic and nonperiodic settings (as well as a couple of questions motivated by this discussion).

Now, we introduce some notations. Let $U(t) = e^{it\partial_x^2}$ and $V(t) = e^{-t\partial_x^3}$ be the unitary groups associated to the linear Schrödinger and the Airy equations, respectively. Given $k, s, b \in \mathbb{R}$, we define the spaces $X^{k,b}$ and $Y^{s,b}$ via the norms

$$\|f\|_{X^{k,b}} := \left(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \langle n\rangle^{2k} \langle \tau + n^2\rangle^{2b} \left|\hat{f}(n,\tau)\right|^2 d\tau\right)^{1/2} = \left\|U(-t)f\right\|_{H_t^b(\mathbb{R},H_x^k)},$$

$$\|g\|_{Y^{s,b}} := \left(\sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} \langle n\rangle^{2s} \langle \tau - n^3\rangle^{2b} \left|\hat{g}(n,\tau)\right|^2 d\tau\right)^{1/2} = \left\|V(-t)g\right\|_{H_t^b(\mathbb{R},H_x^s)},$$

where $\langle\cdot\rangle := 1 + |\cdot|$ and $\hat{f}$ is the Fourier transform of $f$ in both variables $x$ and $t$:

$$\hat{f}(n,\tau) = (2\pi)^{-1} \int_{\mathbb{R}\times\mathbb{T}} e^{-it\tau} e^{-ixn} f(x,t)\, dt\, dx$$

and, given a time interval $I$, we define $X^{k,b}(I)$ and $Y^{s,b}(I)$ via the (restriction in time) norms

$$\|f\|_{X^{k,b}(I)} = \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} = \inf_{\tilde{g}|_I = g} \|\tilde{g}\|_{Y^{s,b}}.$$

The study of the periodic dispersive equations (e.g., KdV) has been based around iteration in the Bourgain spaces (e.g., $Y^{s,b}$) with $b = 1/2$. Since we are interested in the continuity of the flow associated to the NLS–KdV system and the Bourgain spaces with $b = 1/2$ do not control the $L_t^\infty H_x^s$, we consider the slightly smaller spaces $\widetilde{X}^k$, $\widetilde{Y}^s$ defined by the norms

$$\|u\|_{\widetilde{X}^k} := \|u\|_{X^{k,1/2}} + \left\|\langle n\rangle^k \hat{u}(n,\tau)\right\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{\widetilde{Y}^s} := \|v\|_{Y^{s,1/2}} + \left\|\langle n\rangle^s \hat{v}(n,\tau)\right\|_{L_n^2 L_\tau^1}$$

and, given a time interval $I$, we define the spaces $\widetilde{X}^k(I)$, $\widetilde{Y}^s(I)$ via the restriction in time norms

$$\|f\|_{\widetilde{X}^k(I)} = \inf_{\tilde{f}|_I = f} \|\tilde{f}\|_{\widetilde{X}^k} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} = \inf_{\tilde{g}|_I = g} \|\tilde{g}\|_{\widetilde{Y}^s}.$$

Also, we introduce the companion spaces $Z^k$ and $W^s$ via the norms

$$\|u\|_{Z^k} := \|u\|_{X^{k,-1/2}} + \left\|\frac{\langle n\rangle^k \hat{u}(n,\tau)}{\langle \tau + n^2\rangle}\right\|_{L_n^2 L_\tau^1} \quad \text{and} \quad \|v\|_{W^s} := \|v\|_{Y^{s,-1/2}} + \left\|\frac{\langle n\rangle^s \hat{v}(n,\tau)}{\langle \tau - n^3\rangle}\right\|_{L_n^2 L_\tau^1}.$$

Denote by $\psi$ a non-negative smooth bump function supported in $[-2, 2]$ with $\psi = 1$ on $[-1, 1]$ and $\psi_\delta(t) := \psi(t/\delta)$ for any $\delta > 0$. Also, let $a\pm$ be a number slightly larger (respectively, smaller) than $a$. At this point, we are ready to state our main results. The fundamental technical propositions are the following two sharp bilinear for the coupling terms of the NLS–KdV system:

**Proposition 1.1.** *For any $s \geqslant 0$ and $k - s \leqslant 3/2$,*

$$\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,1/2-}} \|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}} \|v\|_{Y^{s,1/2-}}. \tag{1.2}$$

*Furthermore, estimate* (1.2) *fails if either $s < 0$ or $k - s > 3/2$. More precisely, if the bilinear estimate $\|uv\|_{X^{k,b-1}} \lesssim \|u\|_{X^{k,b}} \|v\|_{Y^{s,b}}$ with $b = 1/2$ holds then $s \geqslant 0$ and $k - s \leqslant 3/2$.*

**Proposition 1.2.** *For any $k > 0$, $1 + s \leqslant 4k$ and $-1/2 \leqslant k - s$,*

$$\left\|\partial_x(u_1\overline{u_2})\right\|_{W^s} \lesssim \|u_1\|_{X^{k,1/2-}} \|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}} \|u_2\|_{X^{k,1/2-}}. \tag{1.3}$$

*Furthermore, estimate* (1.3) *fails if either $1 + s > 4k$ or $k - s < -1/2$. More precisely, if the bilinear estimate $\|\partial_x(u_1\overline{u_2})\|_{Y^{s,-1/2}} \lesssim \|u_1\|_{X^{k,1/2}} \|u_2\|_{X^{k,1/2}}$ holds then $1 + s \leqslant 4k$ and $-1/2 \leqslant k - s$.*

Using these bilinear estimates for the coupling terms $uv$ and $\partial_x(|u|^2)$, we show the main theorem of this paper, namely, we prove the following local well-posedness result:

**Theorem 1.1.** *The periodic NLS–KdV* (1.1) *is locally well-posed in $H^k(\mathbb{T}) \times H^s(\mathbb{T})$ whenever $s \geqslant 0$, $-1/2 \leqslant k - s \leqslant 3/2$ and $1 + s \leqslant 4k$. I.e., for any $(u_0, v_0) \in H^k(\mathbb{T}) \times H^s(\mathbb{T})$, there exist a positive time $T = T(\|u_0\|_{H^k}, \|v_0\|_{H^s})$ and a unique solution $(u(t), v(t))$ of the NLS–KdV system* (1.1) *satisfying*

$$\big(\psi_T(t)u, \psi_T(t)v\big) \in \widetilde{X}^k \times \widetilde{Y}^s,$$
$$(u, v) \in C\big([0, T], H^k(\mathbb{T}) \times H^s(\mathbb{T})\big).$$

*Moreover, the map $(u_0, v_0) \mapsto (u(t), v(t))$ is locally Lipschitz from $H^k(\mathbb{T}) \times H^s(\mathbb{T})$ into $C([0, T], H^k(\mathbb{T}) \times H^s(\mathbb{T}))$, whenever $k, s \geqslant 0$, $-1/2 \leqslant k - s \leqslant 3/2$ and $1 + s \leqslant 4k$.*

**Remark 1.1.** As we pointed out before, the endpoint of our sharp bilinear estimates and, consequently, our local well-posedness result is $H^{1/4} \times L^2$. Since the endpoint of the sharp well-posedness theory for the periodic NLS is $L^2$ and for the periodic KdV is $H^{-1/2}$, we are somewhat far from the naturally expected endpoint $L^2 \times H^{-1/2}$ for the local in time theory for the NLS–KdV system (although, our bilinear estimates are optimal). This leads us to ask about possible ill-posedness results in this gap between $H^{1/4} \times L^2$ and $L^2 \times H^{-1/2}$. For precise statements and some comparison with the continuous setting, see Section 6.

**Remark 1.2.** It is easy to see that the NLS–KdV system (1.1) is ill-posed for $k < 0$. Indeed, if we put

$$\begin{cases} u := e^{-it} w, \\ v \equiv \alpha^{-1} \in H^s(\mathbb{T}), \quad \forall s \in \mathbb{R}, \end{cases}$$

system (1.1) becomes into the equation

$$\begin{cases} i w_t + \partial_x^2 w = \beta |w|^2 w, \\ \partial_x (|w|^2) = 0, \\ w_0(x) = u_0 \in H^k(\mathbb{T}), \end{cases}$$

which is not locally-well posed (ill-posed) below $L^2(\mathbb{T})$ in the sense that the data-solution map is not uniformly continuous.

Using this local well-posedness result and three conserved quantities for the NLS–KdV flow, it will be not difficult to prove the following global well-posedness theorem in the energy space $H^1(\mathbb{T}) \times H^1(\mathbb{T})$:

**Theorem 1.2.** *Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that $\alpha\gamma > 0$ and $(u_0, v_0) \in H^1(\mathbb{T}) \times H^1(\mathbb{T})$. Then, the unique solution in Theorem* 1.1 *can be extended to the time interval* $[0, T]$ *for any* $T > 0$.

To close this introduction, we give the outline of the paper. In Section 2 we give counter-examples for the bilinear estimates of the coupling terms, when the indices $k$ and $s$ satisfy (at least) one of the following inequalities: $s < 0$, $k - s > 3/2$, $1 + s > 4k$ or $k - s < -1/2$. In Section 3 we complete the proof of Propositions 1.1 and 1.2 by establishing the claimed bilinear estimates for the terms $uv$ and $\partial_x(|u|^2)$. In Section 4 we use Propositions 1.1 and 1.2 to show that the integral operator associated to the NLS–KdV system is a contraction in the space $\widetilde{X}^k([0, T]) \times \widetilde{Y}^s([0, T])$ (for sufficiently small $T > 0$) when $k, s \geqslant 0$, $-1/2 \leqslant k - s \leqslant 3/2$ and $1 + s \leqslant 4k$. In particular, we obtain the desired local well-posedness statement in Theorem 1.1. In Section 5 we make a standard use of three conserved quantities for the NLS–KdV flow to obtain the global well-posedness result of Theorem 1.2 in the energy space $H^1(\mathbb{T}) \times H^1(\mathbb{T})$. In Section 6 we make some questions related to the gap between the expected $L^2 \times H^{-1/2}$ end-point regularity and our lowest regularity $H^{1/4} \times L^2$ for the local well-posedness for the periodic NLS–KdV system; also, we compare the known theorems in the continuous setting with the periodic setting. Finally, in Appendix A, we collect some standard facts about linear and multilinear estimates associated to the cubic NLS and the KdV equations (which were used in the proof of Theorem 1.1) and we show that the NLS–KdV flow preserves three quantities controlling the $H^1$ norms of $u(t)$ and $v(t)$ (this is the heart of the proof of Theorem 1.2).

## 2. Counter-examples

We start with some counter-examples for the bilinear estimate in Proposition 1.1 when $s < 0$ or $k - s > 3/2$:

**Lemma 2.1.** $\|uv\|_{X^{k,b-1}} \leqslant \|u\|_{X^{k,b}} \cdot \|v\|_{Y^{s,b}}$ *(with $b = 1/2$) implies $s \geqslant 0$ and $k - s \leqslant 3/2$.*

**Proof.** Fix $N \gg 1$ a large integer. First, we show that $\|uv\|_{X^{k,b-1}} \leqslant \|u\|_{X^{k,b}} \cdot \|v\|_{Y^{s,b}}$ (with $b = 1/2$) implies $s \geqslant 0$. Define

$$b_n = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases}$$

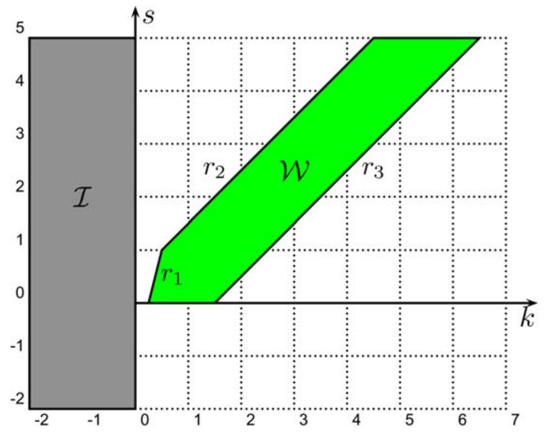


Fig. 1. Well-posedness results for periodic NLS–KdV system. The region $\mathcal{W}$, limited for the lines $r_1$: $s = 4k - 1$, $r_2$: $s = k + 1/2$ and $r_3$: $s = k - 3/2$, contains indices $(k, s)$ for which local well-posedness is achieved in Theorem 1.1. The region $\mathcal{I}$ show the ill-posedness results commented in Remark 1.2.

and

$$a_n = \begin{cases} 1 & \text{if } n = (-N^2 - N)/2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u$ and $v$ be defined by $\hat{u}(n,\tau) = a_n \chi_1(\tau + n^2)$ and $\hat{v}(n,\tau) = b_n \chi_1(\tau - n^3)$, where $\chi_1$ is the characteristic function of the interval $[-1, 1]$. Now let us go to the calculations. By definition of the Bourgain space $X^{k,b}$,

$$\|uv\|_{X^{k,b-1}} = \left\| \frac{\langle n \rangle^k}{\langle \tau + n^2 \rangle^{1/2}} \hat{u} * \hat{v} \right\|_{L^2_{n,\tau}}.$$

Hence,

$$\|uv\|_{X^{k,b-1}} = \left\| \frac{\langle n \rangle^k}{\langle \tau + n^2 \rangle^{1/2}} \sum_{n_1} \int d\tau_1 \, a_{n-n_1} \chi_1\big((\tau - \tau_1) + (n - n_1)^2\big) b_{n_1} \chi_1\big(\tau_1 - n_1^3\big) \right\|_{L^2_{n,\tau}}.$$

Recall the following numerical expression:

$$\big(\tau_1 - n_1^3\big) + \big((\tau - \tau_1) + (n - n_1)^2\big) - \big(\tau + n^2\big) = -n_1^3 + n_1^2 - 2nn_1. \tag{2.1}$$

Taking into account that $b_{n_1} \neq 0$ iff $n_1 = N$, $a_{n-n_1} \neq 0$ iff $n = (-N^2 + N)/2$, $\chi_1(\tau_1 - n_1^3) \neq 0$ iff $|\tau_1 - n_1^3| \leqslant 1$ and $\chi_1((\tau - \tau_1) + (n - n_1)^2) \neq 0$ iff $|(\tau - \tau_1) + (n - n_1)^2| \leqslant 1$, we conclude, from a direct substitution of these data into (2.1), that

$$\|uv\|_{X^{k,b-1}} \approx N^{2k}. \tag{2.2}$$

On the other hand, it is not difficult to see that

$$\|u\|_{X^{k,b}} = \big\| \langle n\rangle^k \big\langle \tau + n^2\big\rangle^{1/2} a_n \chi_1\big(\tau+n^2\big)\big\|_{L^2_{n,\tau}} \approx N^{2k} \quad \text{and} \tag{2.3}$$

$$\|v\|_{Y^{s,b}} = \big\| \langle n\rangle^s \big\langle \tau - n^3\big\rangle^{1/2} b_n \chi_1\big(\tau-n^3\big)\big\|_{L^2_{n,\tau}} \approx N^{s}. \tag{2.4}$$

Putting together Eqs. (2.2)–(2.4), we obtain that the bilinear estimate implies

$$N^{2k} \lesssim N^{2k} \cdot N^s,$$

which is possible only if $s \geqslant 0$.

Secondly, we prove that $\|uv\|_{X^{k,b-1}} \leqslant \|u\|_{X^{k,b}} \cdot \|v\|_{Y^{s,b}}$ (with $b = 1/2$) implies $k - s \leqslant 3/2$. Define

$$b_n = \begin{cases} 1 & \text{if } n = N, \\ 0 & \text{otherwise} \end{cases}$$

and

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u$ and $v$ be defined by $\hat{u}(n,\tau) = a_n\chi_1(\tau+n^2)$ and $\hat{v}(n,\tau) = b_n\chi_1(\tau-n^3)$, where $\chi_1$ is the characteristic function of the interval $[-1,1]$.

Using the definitions of the Bourgain $X^{k,b}$ and $Y^{s,b}$ spaces and the algebraic relation (2.1), we have

$$\|uv\|_{X^{k,b-1}} \approx \frac{N^k}{N^{3/2}}, \qquad \|u\|_{X^{k,b}} \approx 1 \quad \text{and} \quad \|v\|_{Y^{s,b}} \approx N^s.$$

Hence, the bilinear estimate says

$$N^k \lesssim N^s N^{3/2},$$

which is only possible if $k - s \leqslant 3/2$. □

We consider now some counter-examples for the bilinear estimate in Proposition 1.2 when $1 + s > 4k$ or $k - s < -1/2$:

**Lemma 2.2.** $\|\partial_x(u_1\overline{u_2})\|_{Y^{s,b-1}} \leqslant \|u_1\|_{X^{k,b}} \cdot \|u_2\|_{X^{k,b}}$ *(with $b = 1/2$) implies $1 + s \leqslant 4k$ and $k - s \geqslant -1/2$.*

**Proof.** Fix $N \gg 1$ a large integer. First, we prove that $\|\partial_x(u_1\overline{u_2})\|_{Y^{s,b-1}} \leqslant \|u_1\|_{X^{k,b}} \cdot \|u_2\|_{X^{k,b}}$ (with $b = 1/2$) implies $1 + s \leqslant 4k$. Define

$$b_n = \begin{cases} 1 & \text{if } n = \frac{-N^2-N}{2}, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a_n = \begin{cases} 1 & \text{if } n = \frac{-N^2+N}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $u_1$ and $u_2$ be defined by $\widehat{u_1}(n,\tau)=a_n\chi_1(\tau+n^2)$ and $\widehat{u_2}(n,\tau)=b_n\chi_1(\tau+n^2)$, where $\chi_1$ is the characteristic function of the interval $[-1,1]$.

By definition of the Bourgain space $Y^{s,b}$,

$$\big\|\partial_x(u_1\overline{u_2})\big\|_{Y^{s,b-1}}=\left\|\frac{\langle n\rangle^s}{\langle\tau-n^3\rangle^{1/2}}n\big(\widehat{u_1}*\widehat{\overline{u_2}}\big)\right\|_{L^2_{n,\tau}}.$$

Hence, if one uses that $\hat{\bar{u}}(n,\tau)=\overline{\hat{u}(-n,-\tau)}$, it is not difficult to see that

$$\begin{aligned}&\big\|\partial_x(u_1\overline{u_2})\big\|_{Y^{s,b-1}}\\&\quad=\left\|\frac{n\langle n\rangle^s}{\langle\tau-n^3\rangle^{1/2}}\sum_{n_1}\int d\tau_1\,a_{n-n_1}\chi_1\big((\tau-\tau_1)+(n-n_1)^2\big)b_{-n_1}\chi_1\big(-\tau_1+n_1^2\big)\right\|_{L^2_{n,\tau}}.\end{aligned}$$

Note the following numerical expression:

$$\big(\tau-n^3\big)-\big((\tau-\tau_1)+(n-n_1)^2\big)+\big(-\tau_1+n_1^2\big)=-n^3-n^2+2n_1n. \tag{2.5}$$

Taking into account that $b_{-n_1}\neq 0$ iff $n_1=(N^2+N)/2$, $a_{n-n_1}\neq 0$ iff $n=N$, $\chi_1(-\tau_1+n_1^2)\neq 0$ iff $|-\tau_1+n_1^2|\leqslant 1$ and $\chi_1((\tau-\tau_1)+(n-n_1)^2)\neq 0$ iff $|(\tau-\tau_1)+(n-n_1)^2|\leqslant 1$, we conclude, from a direct substitution of these data into (2.5), that

$$\big\|\partial_x(u_1\overline{u_2})\big\|_{Y^{s,b-1}}\approx N^{1+s}. \tag{2.6}$$

On the other hand, it is not difficult to see that

$$\|u_1\|_{X^{k,b}}=\big\|\langle n\rangle^k\big\langle\tau+n^2\big\rangle^{1/2}a_n\chi_1\big(\tau+n^2\big)\big\|_{L^2_{n,\tau}}\approx N^{2k}\quad\text{and} \tag{2.7}$$

$$\|u_2\|_{X^{k,b}}=\big\|\langle n\rangle^k\big\langle\tau+n^2\big\rangle^{1/2}b_n\chi_1\big(\tau+n^2\big)\big\|_{L^2_{n,\tau}}\approx N^{2k}. \tag{2.8}$$

Putting together Eqs. (2.6)–(2.8), we obtain that the bilinear estimate implies

$$N^{1+s}\lesssim N^{2k}\cdot N^{2k},$$

which is possible only if $1+s\leqslant 4k$.

Second, we obtain that $\|\partial_x(u_1\overline{u_2})\|_{Y^{s,b-1}}\leqslant\|u_1\|_{X^{k,b}}\cdot\|u_2\|_{X^{k,b}}$ (with $b=1/2$) implies $k-s\geqslant -1/2$.

Define

$$b_n=\begin{cases}1 & \text{if } n=-N,\\ 0 & \text{otherwise}\end{cases}\quad\text{and}\quad a_n=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{otherwise.}\end{cases}$$

Let $u_1$ and $u_2$ be defined by $\widehat{u_1}(n,\tau)=a_n\chi_1(\tau+n^2)$ and $\widehat{u_2}(n,\tau)=b_n\chi_1(\tau+n^2)$, where $\chi_1$ is the characteristic function of the interval $[-1,1]$.

Using the definitions of the Bourgain $X^{k,b}$ and $Y^{s,b}$ spaces and the algebraic relation (2.1), we have

$$\left\|\partial_x(u_1\overline{u_2})\right\|_{Y^{s,b-1}} \approx \frac{N^{1+s}}{N^{3/2}}, \qquad \|u_1\|_{X^{k,b}} \approx 1, \quad \text{and} \quad \|u_2\|_{Y^{s,b}} \approx N^k.$$

Hence, the bilinear estimate says

$$N^{1+s} \lesssim N^k N^{3/2},$$

which is only possible if $k - s \geqslant -1/2$. □

## 3. Bilinear estimates for the coupling terms

This section is devoted to the proof of our basic tools, that is, the sharp bilinear estimates 1.1, 1.2 for the coupling terms of the NLS–KdV system. We begin by showing some elementary calculus lemmas; next, using Plancherel and duality, the claimed bilinear estimates reduce to controlling some weighted convolution integrals, which is quite easy from these lemmas.

### *3.1. Preliminaries*

The first elementary calculus lemma to be used later is:

**Lemma 3.1.**

$$\int_{-\infty}^{+\infty} \frac{d\kappa}{\langle\kappa\rangle^{\theta}\langle\kappa-a\rangle^{\tilde\theta}} \lesssim \frac{\log(1+\langle a\rangle)}{\langle a\rangle^{\theta+\tilde\theta-1}},$$

*where* $0<\theta, \tilde\theta<1$ *and* $\theta+\tilde\theta>1$.

**Proof.** Clearly we can assume that $|a| \gg 1$. In this case, we divide the domain of integration into the regions $I_1 := \{|\kappa| \ll |a|\}$, $I_2 := \{|\kappa| \sim |a|\}$ and $I_3 := \{|\kappa| \gg |a|\}$. Since $\kappa \in I_1$ implies $\langle\kappa-a\rangle \gtrsim \langle a\rangle \geqslant \langle\kappa\rangle$, $\kappa \in I_2$ implies $\langle\kappa\rangle \sim \langle a\rangle$ and $x \in I_3$ implies $\langle\kappa-a\rangle \gtrsim \langle\kappa\rangle$, we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d\kappa}{\langle\kappa\rangle^{\theta}\langle\kappa-a\rangle^{\tilde\theta}} &= \int_{I_1} \frac{d\kappa}{\langle\kappa\rangle^{\theta}\langle\kappa-a\rangle^{\tilde\theta}} + \int_{I_2} \frac{d\kappa}{\langle\kappa\rangle^{\theta}\langle\kappa-a\rangle^{\tilde\theta}} + \int_{I_3} \frac{d\kappa}{\langle\kappa\rangle^{\theta}\langle\kappa-a\rangle^{\tilde\theta}} \\ &\lesssim \frac{1}{\langle a\rangle^{\theta+\tilde\theta-1}} \int_{I_1} \frac{d\kappa}{\langle\kappa\rangle} + \frac{1}{\langle a\rangle^{\theta+\tilde\theta-1}} \int_{I_2} \frac{d\kappa}{\langle\kappa-a\rangle} + \int_{I_3} \frac{d\kappa}{\langle\kappa-a\rangle^{\theta+\tilde\theta}} \\ &\lesssim \frac{\log(1+\langle a\rangle)}{\langle a\rangle^{\theta+\tilde\theta-1}}. \qquad \square\end{aligned}$$

The second lemma is a well-known fact concerning the convergence of series whose terms are the values of certain polynomials along the integer numbers:[4]

**Lemma 3.2.** *For any constant $\theta > 1/3$,*

$$\sum_{m\in\mathbb{Z}} \frac{1}{\langle p(m)\rangle^{\theta}} \leqslant C(\theta) < \infty,$$

*where $p(x)$ is a cubic polynomial of the form $p(x) := x^3 + ex^2 + fx + g$ with $e, f, g \in \mathbb{R}$.*

**Proof.** We start the proof of Lemma 3.2 with two simple observations: defining

$$\mathcal{E} := \bigl\{m \in \mathbb{Z}\colon |m-\alpha| \geqslant 2,\ |m-\beta| \geqslant 2 \text{ and } |m-\gamma| \geqslant 2\bigr\} \quad \text{and} \quad \mathcal{F} := \mathbb{Z} - \mathcal{E},$$

then

$$\#\mathcal{F} \leqslant 12 \quad \text{and} \quad \bigl\langle (m-\alpha)(m-\beta)(m-\gamma)\bigr\rangle \gtrsim \langle m-\alpha\rangle\langle m-\beta\rangle\langle m-\gamma\rangle$$

for any $m \in \mathcal{E}$.

In particular, writing $p(x) = (x-\alpha)(x-\beta)(x-\gamma)$, we can estimate

$$\sum_m \frac{1}{p(m)^{\theta}} \leqslant \sum_{m\in\mathcal{F}} \frac{1}{p(m)^{\theta}} + \sum_{m\in\mathcal{E}} \frac{1}{p(m)^{\theta}} \leqslant 12 + \sum_{m\in\mathcal{E}} \frac{1}{p(m)^{\theta}}$$
$$\lesssim 12 + \sum_{m\in\mathcal{E}} \frac{1}{\langle m-\alpha\rangle^{\theta}\langle m-\beta\rangle^{\theta}\langle m-\gamma\rangle^{\theta}}.$$

Now, by Hölder's inequality

$$\sum_m \frac{1}{p(m)^{\theta}} \lesssim 12 + \left(\sum_m \frac{1}{\langle m-\alpha\rangle^{3\theta}}\right)^{1/3} \left(\sum_m \frac{1}{\langle m-\beta\rangle^{3\theta}}\right)^{1/3} \left(\sum_m \frac{1}{\langle m-\gamma\rangle^{3\theta}}\right)^{1/3}.$$

So, the hypothesis $3\theta > 1$ implies

$$\sum_m \frac{1}{p(m)^{\theta}} \leqslant C(\theta) < \infty.$$

This completes the proof of Lemma 3.2. □

Finally, the third lemma is a modification of the previous one for linear polynomials with large coefficients:

[4] This lemma is essentially contained in the work [12] of Kenig, Ponce and Vega on bilinear estimates related to the KdV equation.

**Lemma 3.3.** *For any constant $\theta > 1/2$, whenever $n_1 \in \mathbb{Z} - \{0\}$, $|n_1| \gg 1$,*

$$\sum_{n\in\mathbb{Z},\, |n|\sim|n_1|} \frac{1}{q(n)^\theta} \leqslant C(\theta) < \infty,$$

*where $q(x) := 2n_1 x - n_1^2 + r$ with $r \in \mathbb{R}$.*

**Proof.** The strategy of the proof is the same as before, but since now the polynomial $q$ is linear, we have to take a little bit of care. The idea is: although the polynomial $q$ has degree 1, the fact that $|n_1| \sim |n|$ means morally that $q$ has degree 2 in this range. So, the exponent of $n$ in the summand is morally $2\theta > 1$ and, in particular, the series is convergent. This intuition can be formalized as follows: we write $q(x) := r - n_1^2 + 2n_1 x = 2n_1(x+\delta)$, where $\delta = (r - n_1^2)/(2n_1)$ (of course the assumption $n_1 \neq 0$ enters here). If we define

$$\mathcal{G} := \bigl\{n \in \mathbb{Z}\colon\ |n+\delta| \geqslant 2\bigr\} \quad \text{and} \quad \mathcal{H} := \mathbb{Z} - \mathcal{G},$$

then

$$\#\mathcal{H} \leqslant 4 \quad \text{and} \quad \bigl\langle 2n_1(n+\delta)\bigr\rangle \gtrsim \langle n\rangle\langle n-\delta\rangle$$

for any $n \in \mathcal{G}$, since $|n_1| \sim |n|$.

In particular, we can estimate

$$\begin{aligned}\sum_{|n|\sim|n_1|} \frac{1}{q(n_1)^\theta} &\leqslant \sum_{n\in\mathcal{H}} \frac{1}{q(n_1)^\theta} + \sum_{n\in\mathcal{G},\, |n|\sim|n_1|} \frac{1}{q(n_1)^\theta} \\ &\leqslant 4 + \sum_{n\in\mathcal{G},\, |n|\sim|n_1|} \frac{1}{q(n_1)^\theta} \lesssim 4 + \sum_{n\in\mathcal{G},\, |n|\sim|n_1|} \frac{1}{\langle n\rangle^\theta \langle n+\delta\rangle^\theta}.\end{aligned}$$

Now, by Hölder's inequality

$$\sum_{|n|\sim|n_1|} \frac{1}{q(n_1)^\theta} \lesssim 4 + \biggl(\sum_n \frac{1}{\langle n\rangle^{2\theta}}\biggr)^{1/2} \biggl(\sum_n \frac{1}{\langle n+\delta\rangle^{2\theta}}\biggr)^{1/2}.$$

So, the hypothesis $2\theta > 1$ implies

$$\sum_{|n|\sim|n_1|} \frac{1}{q(n)^\theta} \leqslant C(\theta) < \infty.$$

This completes the proof of Lemma 3.3. □

*3.2. Proof of Proposition 1.1: bilinear estimates for the coupling term $uv$*

In view of Lemma 2.1, it suffices to show the bilinear estimate:

**Lemma 3.4.** $\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}}\|v\|_{Y^{s,1/2-}}$ *whenever* $s \geqslant 0$ *and* $k - s \leqslant 3/2$.

**Proof.** From the definition of $Z^k$, we must show that

$$\|uv\|_{X^{k,-1/2}} \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}}\|v\|_{Y^{s,1/2-}} \quad \text{and} \tag{3.1}$$

$$\left\| \frac{\langle n\rangle^k \widehat{uv}}{\langle \tau + n^2\rangle} \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}}\|v\|_{Y^{s,1/2-}}. \tag{3.2}$$

We begin with estimate (3.1). By the definition of Bourgain's space,

$$\|uv\|_{X^{k,-a}} = \left\|\left\langle \tau + n^2\right\rangle^{-a} \langle n\rangle^k \widehat{uv}(n,\tau)\right\|_{L^2_\tau L^2_n} = \left\| \frac{\langle n\rangle^k}{\langle \tau + n^2\rangle^a} \hat{u} * \hat{v}(n,\tau)\right\|_{L^2_\tau L^2_n}.$$

Let

$$f(\tau,n) = \left\langle \tau + n^2\right\rangle^b \langle n\rangle^k \hat{u}(n,\tau) \quad \text{and} \quad g(\tau,n) = \left\langle \tau - n^3\right\rangle^c \langle n\rangle^s \hat{v}(n,\tau).$$

In particular, by duality, we obtain

$$\begin{aligned}
\|uv\|_{X^{k,-a}} &= \sup_{\|\varphi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n\in\mathbb{Z}} \int d\tau \, \frac{\langle n\rangle^k}{\langle \tau + n^2\rangle^a} \bar{\varphi}(n,\tau) \left( \frac{f}{\langle \tau + n^2\rangle^b \langle n\rangle^k} * \frac{g}{\langle \tau - n^3\rangle^c \langle n\rangle^s} \right) \\
&= \sup_{\|\varphi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n\in\mathbb{Z}} \int d\tau \sum_{n_1\in\mathbb{Z}} \int d\tau_1 \, \frac{\langle \tau + n^2\rangle^{-a} \langle n\rangle^k g(n_1,\tau_1) f(n-n_1,\tau-\tau_1)\bar{\varphi}(\tau,n)}{\langle \tau_1 - n_1^3\rangle^c \langle n_1\rangle^s \langle \tau - \tau_1 + (n-n_1)^2\rangle^b \langle n - n_1\rangle^k} \\
&= \sum \int \sum \int_{(n,n_1,\tau,\tau_1)\in\mathcal{R}_0} + \sum \int \sum \int_{(n,n_1,\tau,\tau_1)\in\mathcal{R}_1} + \sum \int \sum \int_{(n,n_1,\tau,\tau_1)\in\mathcal{R}_2} \\
&\equiv W_0 + W_1 + W_2,
\end{aligned} \tag{3.3}$$

whenever $\mathbb{Z}^2 \times \mathbb{R}^2 = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$.

Now, taking into account the previous calculation, we look at three general simple ways to reduce the problem of goods bounds on the expressions $W_i$ into some multiplier estimates. In the sequel, $\chi_{\mathcal{R}}$ denotes the characteristic function of the set $\mathcal{R}$. So, we consider the expression

$$W = \sup_{\|\varphi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n\in\mathbb{Z}} \int d\tau \sum_{n_1\in\mathbb{Z}} \int d\tau_1 \, \frac{\langle \tau + n^2\rangle^{-a} \langle n\rangle^k g(n_1,\tau_1) f(n-n_1,\tau-\tau_1)\bar{\varphi}(\tau,n)\chi_{\mathcal{R}}}{\langle \tau_1 - n_1^3\rangle^b \langle n_1\rangle^s \langle \tau - \tau_1 + (n-n_1)^2\rangle^b \langle n - n_1\rangle^k}. \tag{3.4}$$

The first way to bound $W$ is: integrate over $\tau_1$ and $n_1$, and then use the Cauchy–Schwarz and Hölder inequalities to obtain

$$
\begin{aligned}
|W|^2 &\leqslant \|\varphi\|^2_{L^2_\tau L^2_n} \left\| \frac{\langle n\rangle^k}{\langle \tau+n^2\rangle^a} \iint \frac{g(n_1,\tau_1) f(n-n_1,\tau-\tau_1)\chi_{\mathcal{R}}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^c \langle n_1\rangle^s \langle \tau-\tau_1+(n-n_1)^2\rangle^b \langle n-n_1\rangle^k} \right\|^2_{L^2_\tau L^2_n} \\
&\leqslant \iint \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \left| \iint \frac{g(n_1,\tau_1) f(n-n_1,\tau-\tau_1)\chi_{\mathcal{R}}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^c \langle n_1\rangle^s \langle \tau-\tau_1+(n-n_1)^2\rangle^b \langle n-n_1\rangle^k} \right|^2 d\tau\, dn \\
&\leqslant \iint \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \biggl( \iint \frac{\chi_{\mathcal{R}}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^{2c} \langle n_1\rangle^{2s} \langle \tau-\tau_1+(n-n_1)^2\rangle^{2b} \langle n-n_1\rangle^{2k}} \\
&\quad \times \iint \bigl|g(n_1,\tau_1)\bigr|^2 \bigl|f(n-n_1,\tau-\tau_1)\bigr|^2 d\tau_1\, dn_1 \biggr) d\tau\, dn \\
&\leqslant \|f\|^2_{L^2_\tau L^2_n} \|g\|^2_{L^2_{\tau_1} L^2_{n_1}} \\
&\quad \times \left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \iint \frac{\chi_{\mathcal{R}}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^{2c} \langle n_1\rangle^{2s} \langle \tau-\tau_1+(n-n_1)^2\rangle^{2b} \langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \\
&= \|u\|^2_{X^{k,b}} \|v\|^2_{Y^{s,c}} \\
&\quad \times \left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \iint \frac{\chi_{\mathcal{R}}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^{2c} \langle n_1\rangle^{2s} \langle \tau-\tau_1+(n-n_1)^2\rangle^{2b} \langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n}. \qquad (3.5)
\end{aligned}
$$

The second way we can bound $W$ is: put $\tilde{f}(n,\tau) = f(-n,-\tau)$, integrate over $\tau$ and $n$ first and follow the same steps as above to get

$$
\begin{aligned}
|W|^2 &\leqslant \|g\|^2_{L^2_{\tau_1} L^2_{n_1}} \left\| \frac{1}{\langle n_1\rangle^s \langle \tau_1-n_1^3\rangle^c} \iint \frac{\langle n\rangle^k \tilde{f}(n_1-n,\tau_1-\tau)\bar{\varphi}(\tau,n)\chi_{\mathcal{R}}\, d\tau\, dn}{\langle \tau+n^2\rangle^a \langle \tau-\tau_1+(n-n_1)^2\rangle^b \langle n-n_1\rangle^k} \right\|^2_{L^2_{\tau_1} L^2_{n_1}} \\
&\leqslant \|\tilde{f}\|^2_{L^2_{\tau_1} L^2_{n_1}} \|g\|^2_{L^2_{\tau_1} L^2_{n_1}} \\
&\quad \times \left\| \frac{1}{\langle n_1\rangle^{2s} \langle \tau_1-n_1^3\rangle^{2c}} \iint \frac{\langle n\rangle^{2k} \chi_{\mathcal{R}}\, d\tau\, dn}{\langle \tau+n^2\rangle^{2a} \langle \tau-\tau_1+(n-n_1)^2\rangle^{2b} \langle n-n_1\rangle^{2k}} \right\|^2_{L^\infty_{\tau_1} L^\infty_{n_1}} \\
&= \|u\|^2_{X^{k,b}} \|v\|^2_{Y^{s,c}} \\
&\quad \times \left\| \frac{1}{\langle n_1\rangle^{2s} \langle \tau_1-n_1^3\rangle^{2c}} \iint \frac{\langle n\rangle^{2k} \chi_{\mathcal{R}}\, d\tau\, dn}{\langle \tau+n^2\rangle^{2a} \langle \tau-\tau_1+(n-n_1)^2\rangle^{2b} \langle n-n_1\rangle^{2k}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}}. \qquad (3.6)
\end{aligned}
$$

Note that $\tilde{f}(n,\tau) = \langle n\rangle^k \langle \tau-n^2\rangle^b \hat{u}(-n,-\tau)$ and $\|\tilde{f}\|_{L^2_\tau L^2_n} = \|f\|_{L^2_\tau L^2_n} = \|u\|_{X^{k,b}}$. Finally, the third way to estimate $W$ is: using the change of variables $\tau = \tau_1 - \tau_2$ and $n = n_1 - n_2$, the region, $\mathcal{R}$, is transformed into the set $\widetilde{\mathcal{R}}$ such that

$$
\widetilde{\mathcal{R}} = \bigl\{(n_1,n_2,\tau_1,\tau_2) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ (n_1-n_2, n_1, \tau_1-\tau_2, \tau_1) \in \mathcal{R}\bigr\}.
$$

Then, $W$ can be estimated as:

$$
\begin{aligned}
|W|^2 &\leqslant \|\tilde{f}\|^2_{L^2_{\tau_2}L^2_{n_2}} \\
&\quad \times \left\| \frac{1}{\langle n_2\rangle^k \langle \tau_2 - n_2^2\rangle^b} \iint \frac{\langle n_1 - n_2\rangle^k g(n_1,\tau_1)\tilde{\bar{\varphi}}(n_2 - n_1, \tau_2 - \tau_1)\chi_{\widetilde{\mathcal{R}}}\, d\tau_1\, dn_1}{\langle \tau_1 - \tau_2 + (n_1 - n_2)^2\rangle^a \langle \tau_1 - n_1^3\rangle^c \langle n_1\rangle^s} \right\|^2_{L^2_{\tau_2}L^2_{n_2}} \\
&\leqslant \|\tilde{f}\|^2_{L^2_{\tau_2}L^2_{n_2}} \|g\|^2_{L^2_{\tau_1}L^2_{n_1}} \\
&\quad \times \left\| \frac{1}{\langle n_2\rangle^{2k} \langle \tau_2 - n_2^2\rangle^{2b}} \iint \frac{\langle n_1 - n_2\rangle^{2k} \chi_{\widetilde{\mathcal{R}}}\, d\tau_1\, dn_1}{\langle \tau_1 - \tau_2 + (n_1 - n_2)^2\rangle^{2a} \langle \tau_1 - n_1^3\rangle^{2c} \langle n_1\rangle^{2s}} \right\|^2_{L^\infty_{\tau_2}L^\infty_{n_2}} \\
&= \|u\|^2_{X^{k,b}} \|v\|^2_{Y^{s,c}} \\
&\quad \times \left\| \frac{1}{\langle n_2\rangle^{2k} \langle \tau_2 - n_2^2\rangle^{2b}} \iint \frac{\langle n_1 - n_2\rangle^{2k} \chi_{\widetilde{\mathcal{R}}}\, d\tau_1\, dn_1}{\langle \tau_1 - \tau_2 + (n_1 - n_2)^2\rangle^{2a} \langle \tau_1 - n_1^3\rangle^{2c} \langle n_1\rangle^{2s}} \right\|_{L^\infty_{\tau_2}L^\infty_{n_2}}. \qquad (3.7)
\end{aligned}
$$

Next, using Eq. (3.3) and estimates (3.5)–(3.7), we are going to reduce the desired bilinear estimate $\|uv\|_{Z^k} \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}}\|v\|_{Y^{s,1/2-}}$ (whenever $s \geqslant 0$ and $k - s \leqslant 3/2$) into certain $L^\infty$ bounds for multipliers localized in some well-chosen regions $\mathcal{R}_i$, $i = 0, 1, 2$, such that $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 = \mathbb{Z}^2 \times \mathbb{R}^2$. First, if $n_0 := n$, $n_1$, $n_2 := n_1 - n$ are the frequencies of our waves, let $\lambda_0 = \tau + n^2$, $\lambda_1 := \tau_1 - n_1^3$, $\lambda_2 := \tau_2 - n_2^2 := (\tau_1 - \tau) - n_2^2$ be the modulations of our waves. Also, we consider $N_j = |n_j|$, $j = 0, 1, 2$, variables measuring the magnitude of frequencies of the waves, and $L_j = |\lambda_j|$, $j = 0, 1, 2$, variables measuring the magnitude of modulations of the waves. It is convenient to define the quantities $N_{\max} \geqslant N_{\mathrm{med}} \geqslant N_{\min}$ to be the maximum, median and minimum of $N_0$, $N_1$, $N_2$, respectively. Similarly, we define $L_{\max} \geqslant L_{\mathrm{med}} \geqslant L_{\min}$. In order to define the regions $\mathcal{R}_i$, we split $\mathbb{Z}^2 \times \mathbb{R}^2$ into three regions $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$,

$$
\begin{aligned}
\mathcal{A} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ N_1 \leqslant 100\}, \\
\mathcal{B} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ N_1 > 100 \text{ and, either } N_1 \ll N_0 \text{ or } N_1 \gg N_0\}, \\
\mathcal{C} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ N_1 > 100 \text{ and } N_1 \sim N_0\}.
\end{aligned}
$$

Now we separate $\mathcal{C}$ into three parts

$$
\begin{aligned}
\mathcal{C}_0 &= \{(n, n_1, \tau, \tau_1) \in \mathcal{C};\ L_0 = L_{\max}\}, \\
\mathcal{C}_1 &= \{(n, n_1, \tau, \tau_1) \in \mathcal{C};\ L_1 = L_{\max}\}, \\
\mathcal{C}_2 &= \{(n, n_1, \tau, \tau_1) \in \mathcal{C};\ L_2 = L_{\max}\}.
\end{aligned}
$$

At this point, we define the sets $\mathcal{R}_i$, $i = 0, 1, 2$, as:

$$
\mathcal{R}_0 = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}_0, \qquad \mathcal{R}_1 = \mathcal{C}_1, \qquad \mathcal{R}_2 = \mathcal{C}_2
$$

and it is clear that $\mathbb{Z}^2 \times \mathbb{R}^2 = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$. For these regions $\mathcal{R}_i$, we can show the following multiplier estimates.

**Claim 3.1.** *If $k, s \geqslant 0$ and $k - s \leqslant 3/2$,*

$$\left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle} \iint \frac{\chi_{\mathcal{R}_0}\, d\tau_1\, dn_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle n_1\rangle^{2s}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \lesssim 1.$$

**Claim 3.2.** *If $k, s \geqslant 0$ and $k - s \leqslant 3/2$,*

$$\left\| \frac{1}{\langle n_1\rangle^{2s}\langle \tau_1-n_1^3\rangle} \iint \frac{\langle n\rangle^{2k}\chi_{\mathcal{R}_1}\, d\tau\, dn}{\langle \tau+n^2\rangle\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}} \lesssim 1.$$

**Claim 3.3.** *If $k, s \geqslant 0$ and $k - s \leqslant 3/2$,*

$$\left\| \frac{1}{\langle n_2\rangle^{2k}\langle \tau_2-n_2^2\rangle} \iint \frac{\langle n_1-n_2\rangle^{2k}\chi_{\widetilde{\mathcal{R}}_2}\, d\tau_1\, dn_1}{\langle \tau_1-\tau_2+(n_1-n_2)^2\rangle\langle \tau_1-n_1^3\rangle^{1-}\langle n_1\rangle^{2s}} \right\|_{L^\infty_{\tau_2} L^\infty_{n_2}} \lesssim 1,$$

*where $\widetilde{\mathcal{R}}_2$ is the image of $\mathcal{R}_2$ by the change of variables $n_2 := n_1 - n$, $\tau_2 := \tau_1 - \tau$.*

It is easy to show that these facts implies the desired bilinear estimate (3.1). Indeed, by Eqs. (3.5)–(3.7), we see that, for $a = 1/2$ and well-chosen $b, c$, these claims means that, whenever $s \geqslant 0$ and $k - s \leqslant 3/2$, $|W_0| \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2-}}$, $|W_1| \lesssim \|u\|_{X^{k,1/2-}}\|v\|_{Y^{s,1/2}}$ and $|W_2| \lesssim \|u\|_{X^{k,1/2}}\|v\|_{Y^{s,1/2-}}$. Putting these informations into Eq. (3.3), we obtain the bilinear estimate (3.1). So, it remains only to prove these claims. For later use, recall the following algebraic relation:

$$\lambda_0 - \lambda_1 + \lambda_2 = n_1^3 - n_1^2 - 2nn_1. \tag{3.8}$$

**Proof of Claim 3.1.** In the region $\mathcal{A}$, using that $N_1 \leqslant 100$ and $\langle n\rangle \leqslant \langle n_1\rangle\langle n-n_1\rangle$,

$$\left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle} \sum_{n_1} \int \frac{\chi_{\mathcal{A}}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle n_1\rangle^{2s}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n}$$
$$\lesssim \sup_{n,\tau} \sum_{n_1} \int \frac{d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}}.$$

However, Lemma 3.1 (with $\theta = \tilde\theta = 1-$) implies

$$\int \frac{d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}} \leqslant \frac{\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}},$$

where $p(x)$ is the polynomial $p(x) := x^3 - x^2 + 2nx - (\tau + n^2)$. Hence, we can estimate:

$$\sum_{n_1} \int \frac{d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}} \lesssim \sum_{n_1} \frac{\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}}.$$

In particular, Lemma 3.2 can be applied to give

$$\left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int \frac{\chi_{\mathcal{A}}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{2b}\langle n_1\rangle^{2s}\langle \tau-\tau_1+(n-n_1)^2\rangle^{2b}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \lesssim 1. \tag{3.9}$$

In the region $\mathcal{B}$, $N_1 > 100$, and either $N_1 \gg N_0$ or $N_1 \ll N_0$. In any case, it is not difficult to see that

$$\frac{\langle n\rangle^{2k}}{\langle n-n_1\rangle^{2k}\langle n_1\rangle^{2s}} \lesssim 1.$$

In fact, this is an easy consequence of $s \geqslant 0$ and $N_2 \gtrsim N_i$ if $N_i \gg N_j$, for $\{i, j\} = \{0, 1\}$. So, we obtain the bound

$$\begin{aligned} &\left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle} \sum_{n_1} \int \frac{\chi_{\mathcal{B}}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle n_1\rangle^{2s}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \\ &\quad \lesssim \sup_{n,\tau} \sum_{n_1} \int \frac{d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}} \lesssim \sum_{n_1} \frac{\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}} \lesssim 1, \end{aligned} \tag{3.10}$$

where, as before, we have used Lemmas 3.1 and 3.2.

In the region $\mathcal{C}_0$, it is convenient to consider the following bound

$$\begin{aligned} &\left\| \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle} \sum_{n_1} \int \frac{\chi_{\mathcal{C}_0}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle n_1\rangle^{2s}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}\langle n-n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \\ &\quad \lesssim \left\| \frac{1}{\langle \tau+n^2\rangle} \sum_{n_1} \int \frac{\langle n_1\rangle^{2k-2s}\chi_{\mathcal{C}_0}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}} \right\|_{L^\infty_\tau L^\infty_n}, \end{aligned}$$

which is an immediate corollary of $\langle n\rangle \leqslant \langle n-n_1\rangle\langle n_1\rangle$. Integrating with respect to $\tau_1$ and using Lemma 3.1 gives, as before,

$$\begin{aligned} &\frac{1}{\langle \tau+n^2\rangle} \sum_{n_1} \int \frac{\langle n_1\rangle^{2k-2s}\chi_{\mathcal{C}_0}\, d\tau_1}{\langle \tau_1-n_1^3\rangle^{1-}\langle \tau-\tau_1+(n-n_1)^2\rangle^{1-}} \\ &\quad \lesssim \frac{1}{\langle \tau+n^2\rangle} \sum_{n_1} \frac{\langle n_1\rangle^{2k-2s}\chi_{\mathcal{C}_0}\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}}. \end{aligned}$$

Since, by the dispersion relation (3.8), $L_0 = L_{\max} \gtrsim N_1^3$ in the region $\mathcal{C}_0$, we have

$$\frac{1}{\langle \tau+n^2\rangle} \sum_{n_1} \frac{\langle n_1\rangle^{2k-2s}\chi_{\mathcal{C}_0}\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}} \lesssim \frac{L_{\max}^{(2k-2s)/3}}{L_{\max}} \sum_{n_1} \frac{\log(1+\langle p(n_1)\rangle)}{\langle p(n_1)\rangle^{1-}}.$$

Hence, $k - s \leqslant 3/2$ and Lemma 3.2 together allow us to conclude

$$\left\| \frac{\langle n\rangle^{2k}}{\langle \tau + n^2\rangle} \sum_{n_1} \int \frac{\chi_{\mathcal{C}_0}\, d\tau_1}{\langle \tau_1 - n_1^3\rangle^{1-} \langle n_1\rangle^{2s} \langle \tau - \tau_1 + (n-n_1)^2\rangle^{1-} \langle n - n_1\rangle^{2k}} \right\|_{L^\infty_\tau L^\infty_n} \lesssim 1. \quad (3.11)$$

By definition of $\mathcal{R}_0$, the bounds (3.9)–(3.11) concludes the proof of Claim 3.1. □

**Proof of Claim 3.2.** Using that $\langle n\rangle \leqslant \langle n_1\rangle \langle n - n_1\rangle$, integrating in the variable $\tau$ and applying Lemma 3.1 (with $\theta = 1/2$ and $\tilde{\theta} = 1/2-$), we get

$$\left\| \frac{1}{\langle n_1\rangle^{2s} \langle \tau_1 - n_1^3\rangle} \sum_{n} \int \frac{\langle n\rangle^{2k} \chi_{\mathcal{R}_1}\, d\tau}{\langle \tau + n^2\rangle \langle \tau - \tau_1 + (n-n_1)^2\rangle^{1-} \langle n - n_1\rangle^{2k}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}}$$
$$\lesssim \left\| \frac{\langle n_1\rangle^{2k-2s}}{\langle \tau_1 - n_1^3\rangle} \sum_{n} \frac{\chi_{\mathcal{R}_1} \log(1 + \langle q(n)\rangle)}{\langle q(n)\rangle^{1-}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}},$$

where $q(x) := \tau_1 - n_1^2 + 2n_1 x$. Note that in the region $\mathcal{R}_1$, $N_1 > 100$ and $N_0 \sim N_1$, $|\lambda_1| \sim L_1 = L_{\max}$ and, by the dispersion relation (3.8), $L_{\max} \gtrsim N_1^3$; this permits us to apply Lemma 3.3 to conclude

$$\frac{\langle n_1\rangle^{2k-2s}}{\langle \tau_1 - n_1^3\rangle^{2b}} \sum_{n} \frac{\chi_{\mathcal{R}_1} \log(1 + \langle q(n)\rangle)}{\langle q(n)\rangle^{1-}} \lesssim \frac{L_{\max}^{(2k-2s)/3}}{L_{\max}}.$$

Thus, if we remember that $k - s \leqslant 3/2$, we get

$$\frac{\langle n_1\rangle^{2k-2s}}{\langle \tau_1 - n_1^3\rangle^{2b}} \sum_{n} \frac{\chi_{\mathcal{R}_1} \log(1 + \langle q(n)\rangle)}{\langle q(n)\rangle^{1-}} \lesssim 1.$$

This completes the proof of Claim 3.2. □

**Proof of Claim 3.3.** Using that $\langle n_1 - n_2\rangle \leqslant \langle n_1\rangle \langle n_2\rangle$, integrating in the $\tau_1$ and applying Lemma 3.1 with $\theta = 1/2-$ and $\tilde{\theta} = 1/2$,

$$\left\| \frac{1}{\langle n_2\rangle^{2k} \langle \tau_2 - n_2^2\rangle^{2b}} \sum_{n_1} \int \frac{\langle n_1 - n_2\rangle^{2k} \chi_{\widetilde{\mathcal{R}}_2}\, d\tau_1}{\langle \tau_1 - \tau_2 + (n_1 - n_2)^2\rangle^{2a} \langle \tau_1 - n_1^3\rangle^{2b} \langle n_1\rangle^{2s}} \right\|_{L^\infty_{\tau_2} L^\infty_{n_2}}$$
$$\lesssim \left\| \frac{1}{\langle \tau_2 - n_2^2\rangle^{2b}} \sum_{n_1} \frac{\langle n_1\rangle^{2k-2s} \chi_{\widetilde{\mathcal{R}}_2} \log(1 + \langle r(n_1)\rangle)}{r(n_1)} \right\|_{L^\infty_{\tau_2} L^\infty_{n_2}},$$

where $r(x) := x^3 + x^2 - 2n_2 x - (\tau_2 - n_2^2)$. Note that the change of variables $\tau = \tau_1 - \tau_2$ and $n = n_1 - n_2$ transforms the region $\mathcal{R}_2$ into a set $\widetilde{\mathcal{R}}_2$ such that

$$\widetilde{\mathcal{R}}_2 \subseteq \{(n_1, n_2, \tau_1, \tau_2) \in \mathbb{Z}^2 \times \mathbb{R}^2;\ N_1 > 100 \text{ and } L_2 = L_{\max}\}.$$

In particular, the dispersion relation (3.8) implies that $|\lambda_2| \sim L_2 = L_{\max} \gtrsim N_1^3$ in the region $\widetilde{\mathcal{R}}_2$. So, an application of Lemma 3.2 and the hypothesis $k - s \leqslant 3/2$ yields

$$\frac{1}{\langle \tau_2 - n_2^2 \rangle^{2b}} \sum_{n_1} \frac{\langle n_1 \rangle^{2k-2s} \chi_{\widetilde{\mathcal{R}}_2} \log(1 + \langle r(n_1) \rangle)}{r(n_1)} \lesssim \frac{L_{\max}^{(2k-2s)/3}}{L_{\max}} \lesssim 1.$$

This concludes the proof of Claim 3.3. □

It remains now only to prove the second estimate (3.2), i.e.,

$$\left\| \frac{\langle n \rangle^k}{\langle \tau + n^2 \rangle} \widehat{uv}(n, \tau) \right\|_{L_n^2 L_\tau^1} \lesssim \|u\|_{X^{k,1/2-}} \cdot \|v\|_{Y^{s,1/2}} + \|u\|_{X^{k,1/2}} \cdot \|v\|_{Y^{s,1/2-}}.$$

We can rewrite the left-hand side as

$$\left\| \int_{n=n_1+n_2} \langle n \rangle^k \int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau + n^2 \rangle} \hat{u}(n_1, \tau_1) \hat{v}(n_2, \tau_2) \right\|_{L_n^2 L_\tau^1}.$$

To begin with, we split the domain of integration into three regions. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$, where

$$\begin{aligned} \mathcal{L}_1 &:= \big\{(n, \tau, n_2, \tau_2)\colon\ |n_2| \leqslant 100\big\}, \\ \mathcal{L}_2 &:= \big\{(n, \tau, n_2, \tau_2)\colon\ |n_2| > 100 \text{ and } |n| \ll |n_2|\big\}, \\ \mathcal{L}_3 &:= \big\{(n, \tau, n_2, \tau_2)\colon\ |n_2| > 100 \text{ and } |n| \gg |n_2|\big\}, \end{aligned}$$

$\mathcal{M} := \{(n, \tau, n_2, \tau_2)\colon\ |n_2| > 100,\ |n| \sim |n_2| \text{ and either } |\tau_1 + n_1^2| = L_{\max} \text{ or } |\tau_2 + n_2^3| = L_{\max}\}$ and $\mathcal{N} := \{(n, \tau, n_2, \tau_2)\colon\ |n_2| > 100,\ |n| \sim |n_2| \text{ and } |\tau + n^2| = L_{\max}\}$. Clearly, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ completely decomposes our domain of integrations, so that, in order to prove (3.2), it suffices to get the bounds

$$\begin{aligned} &\left\| \int_{n=n_1+n_2} \frac{\langle n \rangle^k}{\langle n_1 \rangle^k \langle n_2 \rangle^s} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{L}}}{\langle \tau + n^2 \rangle \langle \tau_1 + n_1^2 \rangle^{1/2-} \langle \tau_2 - n_2^3 \rangle^{1/2-}} \hat{u}(n_1, \tau_1) \hat{v}(n_2, \tau_2) \right\|_{L_n^2 L_\tau^1} \\ &\quad \lesssim \|u\|_{X^{0,0}} \|v\|_{Y^{0,0}}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &\left\| \int_{n=n_1+n_2} \frac{\langle n \rangle^k}{\langle n_1 \rangle^k \langle n_2 \rangle^s} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{M}}}{\langle \tau + n^2 \rangle} \hat{u}(n_1, \tau_1) \hat{v}(n_2, \tau_2) \right\|_{L_n^2 L_\tau^1} \\ &\quad \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,1/2}} + \|u\|_{X^{0,1/2}} \|v\|_{Y^{0,1/2-}}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\left\| \int_{n=n_1+n_2} \frac{\langle n \rangle^k}{\langle n_1 \rangle^k \langle n_2 \rangle^s} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{N}}}{\langle \tau + n^2 \rangle} \hat{u}(n_1, \tau_1) \hat{v}(n_2, \tau_2) \right\|_{L_n^2 L_\tau^1} \\ &\quad \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,1/2}} + \|u\|_{X^{0,1/2}} \|v\|_{Y^{0,1/2-}}. \end{aligned} \tag{3.14}$$

To proceed further, we need to recall the following Bourgain–Strichartz inequalities.

**Lemma 3.5.** (Bourgain [5]) $X^{0,3/8}([0,1]), Y^{0,1/3}([0,1]) \subset L^4(\mathbb{T}\times[0,1])$. *More precisely,*

$$\|\psi(t)f\|_{L^4_{xt}} \lesssim \|f\|_{X^{0,3/8}} \quad \textit{and} \quad \|\psi(t)g\|_{L^4_{xt}} \lesssim \|g\|_{Y^{0,1/3}}.$$

To prove the first bound (3.12), we start with the simple observation that

$$\frac{\langle n\rangle^k}{\langle n_1\rangle^k\langle n_2\rangle^s} \lesssim 1,$$

if either $|n_2| \leqslant 100$, or $|n_2| > 100$ and $|n| \ll |n_2|$, or $|n_2| > 100$ and $|n| \gg |n_2|$. This follows from the fact that $\langle n\rangle \leqslant \langle n_1\rangle\langle n_2\rangle$ and $s \geqslant 0$. Hence,

$$\left\| \int_{n=n_1+n_2} \frac{\langle n\rangle^k}{\langle n_1\rangle^k\langle n_2\rangle^s} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{L}}}{\langle \tau+n^2\rangle\langle \tau_1+n_1^2\rangle^{1/2-}\langle \tau_2-n_2^3\rangle^{1/2-}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^1_\tau}$$
$$\lesssim \left\| \int_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau+n^2\rangle\langle \tau_1+n_1^2\rangle^{1/2-}\langle \tau_2-n_2^3\rangle^{1/2-}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^1_\tau}.$$

Therefore, this reduces our goal to prove that

$$\left\| \int_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau+n^2\rangle\langle \tau_1+n_1^2\rangle^{1/2-}\langle \tau_2-n_2^3\rangle^{1/2-}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^1_\tau} \lesssim \|u\|_{X^{0,0}}\|v\|_{Y^{0,0}}.$$

This can be rewritten as

$$\left\| \frac{1}{\langle \tau+n^2\rangle^{5/8}\langle \tau+n^2\rangle^{3/8}} \int_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^1_\tau} \lesssim \|u\|_{X^{0,1/2-}}\|v\|_{Y^{0,1/2-}}.$$

Since $2(-5/8) < -1$, the Cauchy–Schwarz inequality in $\tau$ reduces this bound to showing

$$\left\| \frac{1}{\langle \tau+n^2\rangle^{3/8}} \int_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^2_\tau} \lesssim \|u\|_{X^{0,1/2-}}\|v\|_{Y^{0,1/2-}}.$$

However, this bound is an easy consequence of duality, $L^4_{xt}L^2_{xt}L^4_{xt}$ Hölder and the Bourgain–Strichartz inequalities $X^{0,3/8}, Y^{0,1/3} \subset L^4$ in Lemma 3.5.

The second bound (3.13) can be proved in an analogous fashion, using the dispersion relation

$$\left(\tau+n^2\right) - \left(\tau_2-n_2^3\right) + \left(\tau_1+n_1^2\right) = n_2^3 - n_2^2 - 2nn_2, \tag{3.15}$$

which implies that, in the region $\mathcal{M}$, either $|\tau_1+n_1^2| \gtrsim |n_2|^3$ or $|\tau_2-n_2^3| \gtrsim |n_2|^3$. Thus, using that $k-s \leqslant 3/2$ and making the corresponding cancellation, we see that it suffices to prove that

$$\left\| \frac{1}{\langle \tau+n^2\rangle^{5/8}\langle \tau+n^2\rangle^{3/8}} \int_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\right\|_{L^2_nL^1_\tau} \lesssim \|u\|_{X^{0,0}}\|v\|_{Y^{0,1/2-}}$$

and

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{5/8}\langle \tau + n^2\rangle^{3/8}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,0}}.$$

Again, we use Cauchy–Schwarz to reduce these estimates to

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{3/8}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^2_\tau} \lesssim \|u\|_{X^{0,0}} \|v\|_{Y^{0,1/2-}} \quad \text{and}$$

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{3/8}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^2_\tau} \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,0}},$$

which follows from duality, Hölder and Bourgain–Strichartz, as above.

Finally, the third bound (3.14) requires a subdivision into two cases. When $|\tau_1 + n_1^2| \gtrsim |n_2|^{2-}$ (respectively, $|\tau_2 - n_2^3| \gtrsim |n_2|^{2-}$), we use $\langle \tau_1 + n_1^2\rangle^{1/8}$ leaving $\langle \tau_1 + n_1^2\rangle^{3/8}$ in the denominator and $|n_2|^{k-s-}$ in the numerator (respectively, a similar argument with $(\tau_2 - n_2^3)$ instead of $(\tau_1 + n_1^2)$, using $\langle \tau_2 - n_2^3\rangle^{1/6}$ and leaving $\langle \tau_2 - n_2^3\rangle^{1/3}$). After another cancellation using $|\tau + n^2| \gtrsim |n_2|^3$, we need to prove

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{1/2+}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{0,3/8}} \|v\|_{Y^{0,1/2-}} \quad \text{and}$$

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{1/2+}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,1/3}}.$$

These bounds follow again from Cauchy–Schwarz in $\tau$, duality, Hölder and Bourgain–Strichartz. So it remains only the case $|\tau_1 + n_1^2|, |\tau_2 - n_2^3| \ll |n_2|^{2-}$. In this case, the dispersion relation says that, in the region $\mathcal{N}$,

$$\tau + n^2 = n_2^3 - n_2^2 - 2nn_2 - O\big(|n_2|^{2-}\big).$$

On the other hand, the cancellation using $|\tau + n^2| \gtrsim |n_2|^3$ and $k - s \leqslant 3/2$ reduces the proof to the bound

$$\left\| \frac{1}{\langle \tau + n^2\rangle^{1/2}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\chi_{\Omega(n)}\big(\tau + n^2\big) \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{0,1/2-}} \|v\|_{Y^{0,1/2-}},$$

where $\Omega(n) = \{\eta \in \mathbb{R}\colon\ \eta = r^3 - r^2 - 2nr + O(|r|^{2-}), \text{ for some } r \in \mathbb{Z}, |r| \sim |n| > 100\}$. Applying Cauchy–Schwarz in $\tau$, we can estimate the left-hand side by

$$\left\| \left( \int \big\langle \tau + n^2\big\rangle^{-1} \chi_{\Omega(n)}\big(\tau + n^2\big) \right)^{1/2} \left\| \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_\tau} \right\|_{L^2_n}.$$

Therefore, the point is to show

$$\sup_n \left( \int \langle \tau + n^2 \rangle^{-1} \chi_{\Omega(n)}(\tau + n^2)\, d\tau \right) \lesssim 1. \tag{3.16}$$

We need the following lemma.

**Lemma 3.6.** *There exists some $\delta > 0$ such that, for any fixed $n \in \mathbb{Z}$, $|n| \gg 1$ and for all $M \geqslant 1$ dyadic, we have*

$$\left| \left\{ \mu \in \mathbb{R}\colon |\mu| \sim M,\ \mu = r^3 - r^2 - 2nr + O\left(|r|^{2-}\right),\ \textit{for some } r \in \mathbb{Z},\ |r| \sim |n| \right\} \right| \lesssim M^{1-\delta}.$$

**Proof.** Note that the dyadic block $\{|\mu| \sim M\}$ contains at most $O(M/N^2) + 1$ integer numbers of the form $r^3 - r^2 - 2nr$ with $|r| \sim |n|$, $r \in \mathbb{Z}$, where $N \sim |n|$. Indeed, this follows from the fact that the distance between two consecutive numbers of this form is $\sim N^2$. Thus, the set of $\mu$ verifying $\mu = r^3 - r^2 - 2nr + O(|r|^{2-})$ is the union of $O(M/N^2) + 1$ intervals of size $O(N^{2-})$. Since the relation $\mu = r^3 - r^2 - 2nr + O(|r|^{2-})$ with $|\mu| \sim M$ and $|r| \sim |n| \sim N \gg 1$ implies that $M \sim N^3$, we get

$$\begin{aligned} &\left| \left\{ \mu \in \mathbb{R}\colon |\mu| \sim M,\ \mu = r^3 - r^2 - 2nr + O\left(|r|^{2-}\right),\ \text{for some } r \in \mathbb{Z},\ |r| \sim |n| \right\} \right| \\ &\quad \lesssim N^{2-} \cdot \frac{M}{N^2} \lesssim M^{1-}. \end{aligned}$$

This completes the proof of Lemma 3.6. □

Using Lemma 3.6, it is not difficult to conclude the proof of (3.16): by changing variables, we have to estimate

$$\sup_n \int \langle \mu \rangle^{-1} \chi_{\Omega(n)}(\mu)\, d\mu.$$

By decomposing the domain of integration into dyadic blocks $\{|\mu| \sim M\}$, Lemma 3.6 gives

$$\int \langle \mu \rangle^{-1} \chi_{\Omega(n)}(\mu)\, d\mu \leqslant 1 + \sum_{M \geqslant 1} \int_{|\mu| \sim M} \langle \mu \rangle^{-1} \chi_{\Omega(n)}(\mu)\, d\mu \lesssim 1 + \sum_{M \geqslant 1;\ M \text{ dyadic}} M^{-1} M^{1-\delta} \lesssim 1.$$

This proves estimate (3.2), thus completing the proof of Lemma 3.4. □

### *3.3. Proof of Proposition 1.2: bilinear estimates for the coupling term $\partial_x(|u|^2)$*

By Lemma 2.2, it suffices to prove the bilinear estimate:

**Lemma 3.7.** $\|\partial_x(u_1 \overline{u_2})\|_{W^s} \lesssim \|u_1\|_{X^{k,1/2-}} \|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}} \|u_2\|_{X^{k,1/2-}}$ *whenever* $1 + s \leqslant 4k$ *and* $k - s \geqslant -1/2$.

**Proof.** From the definition of $W^s$, we have to prove that

$$\left\|\partial_x(u_1\overline{u_2})\right\|_{Y^{s,-1/2}} \lesssim \|u_1\|_{X^{k,1/2-}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,1/2-}} \quad \text{and} \tag{3.17}$$

$$\left\|\frac{\langle n\rangle^s}{\langle\tau-n^3\rangle}\widehat{\partial_x(u_1\overline{u_2})}\right\|_{L^2_nL^1_\tau} \lesssim \|u_1\|_{X^{k,1/2-}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,1/2-}}. \tag{3.18}$$

We begin with the proof of (3.17). First, we reduce the bilinear estimate to some multiplier estimates as follows. By the definition of Bourgain's spaces,

$$\begin{aligned}\left\|\partial_x(u_1\overline{u_2})\right\|_{Y^{s,-a}} &= \left\|\langle\tau-n^3\rangle^{-a}\langle n\rangle^s\widehat{\partial_x(u_1\overline{u_2})}\right\|_{L^2_\tau L^2_n}\\ &= \left\|n\langle\tau-n^3\rangle^{-a}\langle n\rangle^s\widehat{u_1}*\widehat{\overline{u_2}}(n,\tau)\right\|_{L^2_\tau L^2_n}.\end{aligned}$$

Let

$$f(n,\tau)=\langle n\rangle^k\langle\tau+n^2\rangle^b\widehat{u_1}(n,\tau) \quad\text{and}\quad g(n,\tau)=\langle n\rangle^k\langle-\tau+n^2\rangle^c\overline{\widehat{u_2}(-n,-\tau)}.$$

By duality,

$$\begin{aligned}&\left\|\partial_x(u_1\overline{u_2})\right\|_{Y^{s,-a}}\\
&= \sup_{\|\varphi\|_{L^2_\tau l^2_n}\leqslant 1}\sum_{n\in\mathbb{Z}}\int d\tau\sum_{n_1\in\mathbb{Z}}\int d\tau_1\,\frac{|n|\langle n\rangle^s}{\langle\tau-n^3\rangle^a}\widehat{u_1}(n-n_1,\tau-\tau_1)\overline{\widehat{u_2}(-n_1,-\tau_1)}\cdot\overline{\varphi(n,\tau)}\\
&= \sup_{\|\varphi\|_{L^2_\tau l^2_n}\leqslant 1}\sum_{n\in\mathbb{Z}}\int d\tau\sum_{n_1\in\mathbb{Z}}\int d\tau_1\,\frac{|n|\langle n\rangle^s\langle\tau-n^3\rangle^{-a}f(n-n_1,\tau-\tau_1)g(n_1,\tau_1)\overline{\varphi(n,\tau)}}{\langle n-n_1\rangle^k\langle(\tau-\tau_1)+(n-n_1)^2\rangle^b\langle n_1\rangle^k\langle-\tau_1+n_1^2\rangle^c}\\
&= \sum\int\sum\int_{(n,n_1,\tau,\tau_1)\in\mathcal{V}_0} + \sum\int\sum\int_{(n,n_1,\tau,\tau_1)\in\mathcal{V}_1} + \sum\int\sum\int_{(n,n_1,\tau,\tau_1)\in\mathcal{V}_2}\\
&\equiv V_0+V_1+V_2,\end{aligned}\tag{3.19}$$

whenever $\mathbb{Z}^2\times\mathbb{R}^2=\mathcal{V}_0\cup\mathcal{V}_1\cup\mathcal{V}_2$.

As before, we have three general ways to estimate the quantity

$$V=\sup_{\|\varphi\|_{L^2_\tau l^2_n}\leqslant 1}\sum_{n\in\mathbb{Z}}\int d\tau\sum_{n_1\in\mathbb{Z}}\int d\tau_1\,\frac{|n|\langle n\rangle^s\langle\tau-n^3\rangle^{-a}f(n-n_1,\tau-\tau_1)g(n_1,\tau_1)\overline{\varphi(n,\tau)}}{\langle n-n_1\rangle^k\langle(\tau-\tau_1)+(n-n_1)^2\rangle^b\langle n_1\rangle^k\langle-\tau_1+n_1^2\rangle^c}\chi_{\mathcal{V}}. \tag{3.20}$$

First, we integrate over $\tau_1$ and $n_1$ and then use Cauchy–Schwarz and Hölder inequalities to obtain

$$\begin{aligned}|V|^2 &\leqslant \|\varphi\|^2_{L^2_{n,\tau}}\left\|\frac{|n|\langle n\rangle^s}{\langle\tau-n^3\rangle^a}\sum_{n_1}\int d\tau_1\,\frac{g(n_1,\tau_1)f(n-n_1,\tau-\tau_1)\chi_{\mathcal{V}}}{\langle n-n_1\rangle^k\langle(\tau-\tau_1)+(n-n_1)^2\rangle^b\langle n_1\rangle^k\langle-\tau_1+n_1^2\rangle^c}\right\|_{L^2_\tau L^2_n}\\
&\leqslant \|u_1\|^2_{X^{k,b}}\|u_2\|^2_{X^{k,c}}\\
&\quad\times\left\|\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle^{2a}}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{V}}}{\langle-\tau_1+n_1^2\rangle^{2c}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{2b}}\right\|_{L^\infty_\tau L^\infty_n}.\end{aligned}\tag{3.21}$$

Second, we put $\tilde{f}(n,\tau) = f(-n,-\tau)$, integrate over $n$ and $\tau$ and then use the same steps above to get

$$\begin{aligned} |V|^2 &\leqslant \|g\|^2_{L^2_{\tau_1} L^2_{n_1}} \\ &\quad \times \left\| \frac{1}{\langle n_1\rangle^k \langle -\tau_1 + n_1^2\rangle^c} \sum_n \int d\tau \, \frac{|n|\langle n\rangle^s}{\langle \tau - n^3\rangle^a} \frac{\tilde{f}(n_1 - n, \tau_1 - \tau)\overline{\varphi(n,\tau)}\chi_{\mathcal{V}}}{\langle (\tau - \tau_1) + (n - n_1)^2\rangle^b \langle n_1 - n\rangle^k} \right\|^2_{L^2_{\tau_1} L^2_{n_1}} \\ &\leqslant \|u_1\|^2_{X^{k,b}} \|u_2\|^2_{X^{k,c}} \\ &\quad \times \left\| \frac{1}{\langle n_1\rangle^{2k} \langle -\tau_1 + n_1^2\rangle^{2c}} \sum_n \int d\tau \, \frac{|n|^2\langle n\rangle^{2s}\chi_{\mathcal{V}}}{\langle \tau - n^3\rangle^{2a} \langle (\tau - \tau_1) + (n - n_1)^2\rangle^{2b} \langle n_1 - n\rangle^{2k}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}}. \end{aligned} \tag{3.22}$$

Finally, using the change of variables $\tau_1 = \tau + \tau_2$ and $n_1 = n + n_2$, we transform $\mathcal{V}$ into the region

$$\widetilde{\mathcal{V}} = \{(n, n_2, \tau, \tau_2)\colon\ (n, n+n_2, \tau, \tau+\tau_2) \in \mathcal{V}\}$$

and, hence, integrating over $\tau$ and $n$, we can estimate

$$\begin{aligned} |V|^2 &\leqslant \|\tilde{f}\|^2_{L^2_{\tau_2} L^2_{n_2}} \left\| \frac{1}{\langle n_2\rangle^k \langle -\tau_2 + n_2^2\rangle^b} \sum_{n\in\mathbb{Z}} \int d\tau \, \frac{|n|\langle n\rangle^s g(n+n_2, \tau+\tau_2)\overline{\varphi(n,\tau)}\chi_{\widetilde{\mathcal{V}}}}{\langle \tau - n^3\rangle^a \langle -(\tau+\tau_2) + (n+n_2)^2\rangle^c} \right\|^2_{L^2_{\tau_2} L^2_{n_2}} \\ &\leqslant \|u_1\|^2_{X^{k,b}} \|u_2\|^2_{X^{k,c}} \\ &\quad \times \left\| \frac{1}{\langle n_2\rangle^{2k} \langle -\tau_2 + n_2^2\rangle^{2b}} \sum_{n\in\mathbb{Z}} \frac{|n|^2\langle n\rangle^{2s}}{\langle n+n_2\rangle^{2k}} \int d\tau \, \frac{\chi_{\widetilde{\mathcal{V}}}}{\langle \tau - n^3\rangle^{2a} \langle -(\tau+\tau_2) + (n+n_2)^2\rangle^{2c}} \right\|_{L^\infty_{\tau_2} L^\infty_{n_2}}. \end{aligned} \tag{3.23}$$

The next step is to use estimates (3.21)–(3.23) for expression (3.20) to reduce the bilinear estimate $\|\partial_x(u_1\overline{u_2})\|_{Y^{s,-1/2}} \lesssim \|u_1\|_{X^{k,1/2-}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,1/2-}}$ to $L^\infty$ bounds for certain multipliers localized in some well-chosen regions $\mathcal{V}_0$, $\mathcal{V}_1$ and $\mathcal{V}_2$. We consider $n_0 := n$, $n_1$ and $n_2 := n_1 - n$ the frequencies of our waves and $\lambda_0 := \tau - n^3$, $\lambda_1 := -\tau_1 + n_1^2$, $\lambda_2 := -\tau_2 + n_2^2 := (\tau - \tau_1) + (n - n_1)^2$ the modulations of our waves; again, $L_j = |\lambda_j|$ are variables measuring the magnitude of the modulations, $j = 0, 1, 2$. We define $L_{\max} \geqslant L_{\text{med}} \geqslant L_{\min}$ to be the maximum, median and minimum of $L_0, L_1, L_2$. In order to define the regions $\mathcal{V}_i$, we split $\mathbb{Z}^2 \times \mathbb{R}^2$ into three regions $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$,

$$\begin{aligned} \mathcal{O} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2\colon\ |n| \leqslant 100\}, \\ \mathcal{P} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2\colon\ |n| \geqslant 100 \text{ and } |n_1| \gtrsim |n|^2\}, \\ \mathcal{Q} &= \{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2\colon\ |n| \geqslant 100 \text{ and } |n_1| \ll |n|^2\}. \end{aligned}$$

Now we separate $\mathcal{Q}$ into three parts

$$\mathcal{Q}_0 = \{(n, n_1, \tau, \tau_1) \in \mathcal{C}\colon\ L_0 = L_{\max}\},$$
$$\mathcal{Q}_1 = \{(n, n_1, \tau, \tau_1) \in \mathcal{C}\colon\ L_1 = L_{\max}\},$$
$$\mathcal{Q}_2 = \{(n, n_1, \tau, \tau_1) \in \mathcal{C}\colon\ L_2 = L_{\max}\}.$$

At this point, we put

$$\mathcal{V}_0 = \mathcal{O} \cup \mathcal{P} \cup \mathcal{Q}_0, \qquad \mathcal{V}_1 = \mathcal{Q}_1, \qquad \mathcal{V}_2 = \mathcal{Q}_2.$$

We have the following multiplier estimates:

**Claim 3.4.** *If* $1 + s \leqslant 4k$ *and* $k - s \geqslant -1/2$,

$$\left\| \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau - n^3 \rangle} \sum_{n_1} \frac{1}{\langle n_1 \rangle^{2k} \langle n - n_1 \rangle^{2k}} \int d\tau_1 \, \frac{\chi_{\mathcal{V}_0}}{\langle -\tau_1 + n_1^2 \rangle^{1-} \langle (\tau - \tau_1) + (n - n_1)^2 \rangle^{1-}} \right\|_{L^\infty_\tau L^\infty_n} \lesssim 1.$$

**Claim 3.5.** *If* $1 + s \leqslant 4k$ *and* $k - s \geqslant -1/2$,

$$\left\| \frac{1}{\langle n_1 \rangle^{2k} \langle -\tau_1 + n_1^2 \rangle} \sum_{n} \int d\tau \, \frac{|n|^2 \langle n \rangle^{2s} \chi_{\mathcal{V}_1}}{\langle \tau - n^3 \rangle \langle (\tau - \tau_1) + (n - n_1)^2 \rangle^{1-} \langle n_1 - n \rangle^{2k}} \right\|_{L^\infty_{\tau_1} L^\infty_{n_1}} \lesssim 1.$$

**Claim 3.6.** *If* $1 + s \leqslant 4k$ *and* $k - s \geqslant -1/2$,

$$\left\| \frac{1}{\langle n_2 \rangle^{2k} \langle -\tau_2 + n_2^2 \rangle} \sum_{n \in \mathbb{Z}} \frac{|n|^2 \langle n \rangle^{2s}}{\langle n + n_2 \rangle^{2k}} \int d\tau \, \frac{\chi_{\widetilde{\mathcal{V}_2}}}{\langle \tau - n^3 \rangle \langle -(\tau + \tau_2) + (n + n_2)^2 \rangle^{1-}} \right\|_{L^\infty_{\tau_2} L^\infty_{n_2}} \lesssim 1,$$

*where* $\widetilde{\mathcal{V}_2}$ *is the image of* $\mathcal{V}_2$ *under the change of variables* $n_2 := n_1 - n$ *and* $\tau_2 := \tau_1 - \tau$.

Again, it is easy to show that these facts implies the desired bilinear estimate (3.17). Indeed, by Eqs. (3.21)–(3.23), we see that, for $a = 1/2$ and well-chosen $b$, $c$, these claims means that, whenever $1 + s \leqslant 4k$ and $k - s \geqslant -1/2$, $|V_0| \lesssim \|u_1\|_{X^{k,1/2-}} \|u_2\|_{X^{k,1/2-}}$, $|V_1| \lesssim \|u_1\|_{X^{k,1/2-}} \|u_2\|_{X^{k,1/2}}$ and $|V_2| \lesssim \|u_1\|_{X^{k,1/2}} \|u_2\|_{X^{k,1/2-}}$. Putting these informations into Eq. (3.19), we obtain the bilinear estimate (3.1). Hence, we have only to prove these claims. For later use, we recall that our dispersion relation is

$$\lambda_0 + \lambda_1 - \lambda_2 = -n^3 - n^2 + 2n_1 n. \tag{3.24}$$

**Proof of Claim 3.4.** In the region $\mathcal{O}$, using that $|n| \leqslant 100$,

$$\sup_{n,\tau} \frac{|n|^2 \langle n \rangle^{2s}}{\langle \tau - n^3 \rangle} \sum_{n_1} \frac{1}{\langle n_1 \rangle^{2k} \langle n - n_1 \rangle^{2k}} \int d\tau_1 \, \frac{\chi_{\mathcal{O}}}{\langle -\tau_1 + n_1^2 \rangle^{1-} \langle (\tau - \tau_1) + (n - n_1)^2 \rangle^{1-}}$$
$$\lesssim \frac{1}{\langle \tau - n^3 \rangle} \sum_{n_1} \frac{1}{\langle n_1 \rangle^{2k} \langle n - n_1 \rangle^{2k}} \frac{1}{\langle \lambda_1 - \lambda_2 \rangle^{1-}},$$

by Lemma 3.1. By the dispersion relation (3.24) and the fact $\langle x + y \rangle \leqslant \langle x \rangle \langle y \rangle$, we obtain the bound

$$\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{O}}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$\lesssim\sup_{n\neq0}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}\langle-n^3-n^2+2nn_1\rangle^{1-}}\lesssim 1, \tag{3.25}$$

if $k>0$.

In the region $\mathcal{P}$, we consider the cases $n_1=(n^2+n)/2$ and $|n_1-(n^2+n)/2|\geqslant 1$. Using that $|n|\lesssim|n_1|^{1/2}$, $4k\geqslant 1+s$, the dispersion relation (3.24) and the fact that $\langle xy\rangle\gtrsim\langle x\rangle\langle y\rangle$ whenever $|x|,|y|\geqslant 1$, we see that

$$\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{P}}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$\lesssim C+\sup_{n,\tau}|n|^2\langle n\rangle^{2s}\sum_{|n_1-(n^2+n)/2|\geqslant1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}\langle n\rangle^{1-}\langle n_1-(n^2+n)/2\rangle^{1-}}.$$

Thus,

$$\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{P}}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$\lesssim C+\sup_{n,\tau}|n|^{1+}\langle n\rangle^{2s}\sum_{|n_1-(n^2+n)/2|\geqslant1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}\langle n_1-(n^2+n)/2\rangle^{1-}}$$
$$\lesssim C+\sum_{|n_1-(n^2+n)/2|\geqslant1}\frac{1}{\langle n_1\rangle^{1/2-}\langle n_1-(n^2+n)/2\rangle^{1-}}\lesssim 1. \tag{3.26}$$

In the region $\mathcal{Q}_0$, using that $L_0\gtrsim|n|^3$ and $k-s\geqslant-1/2$, we get

$$\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{n_1}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{Q}_0}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$=\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{|n_1|\gtrsim|n|}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{Q}_0}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$+\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle\tau-n^3\rangle}\sum_{|n_1|\ll|n|}\frac{1}{\langle n_1\rangle^{2k}\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{Q}_0}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$\lesssim\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle n\rangle^3\langle n\rangle^{2k}}\sum_{|n_1|\gtrsim|n|}\frac{1}{\langle n-n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{Q}_0}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$+\sup_{n,\tau}\frac{|n|^2\langle n\rangle^{2s}}{\langle n\rangle^3\langle n\rangle^{2k}}\sum_{|n_1|\ll|n|}\frac{1}{\langle n_1\rangle^{2k}}\int d\tau_1\,\frac{\chi_{\mathcal{Q}_0}}{\langle-\tau_1+n_1^2\rangle^{1-}\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}\lesssim 1, \tag{3.27}$$

if $k>0$. □

**Proof of Claim 3.5.** In the region $\mathcal{Q}_1$, using that $L_1 = L_{\max} \gtrsim |n|^3$ (by the dispersion relation (3.24) and $|n_1| \ll |n|^2$), $\langle n\rangle \leqslant \langle n_1\rangle\langle n-n_1\rangle$ and $k-s \geqslant -1/2$, it is not difficult to see that

$$\sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle -\tau_1+n_1^2\rangle} \sum_n \frac{|n|^2\langle n\rangle^{2s}}{\langle n_1-n\rangle^{2k}} \int d\tau \frac{\chi_{\mathcal{Q}_1}}{\langle \tau-n^3\rangle\langle(\tau-\tau_1)+(n-n_1)^2\rangle^{1-}}$$
$$\lesssim \sup_{n_1,\tau_1} \sum_{n\in\mathbb{Z}} \frac{1}{\langle -\tau_1+(n-n_1)^2+n^3\rangle^{1-}} \lesssim 1. \qquad \square \tag{3.28}$$

**Proof of Claim 3.6.** In the region $\mathcal{Q}_2$, using that $L_2 = L_{\max} \gtrsim |n|^3$ (by the dispersion relation (3.24) and $|n_1| \ll |n|^2$), $\langle n\rangle \leqslant \langle n_2\rangle\langle n+n_2\rangle$ and $k-s \geqslant -1/2$, it follows that

$$\sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2k}\langle -\tau_2+n_2^2\rangle} \sum_{n\in\mathbb{Z}} \frac{|n|^2\langle n\rangle^{2s}}{\langle n+n_2\rangle^{2k}} \int d\tau \frac{\chi_{\widetilde{\mathcal{Q}}_2}}{\langle \tau-n^3\rangle\langle-(\tau+\tau_2)+(n+n_2)^2\rangle^{1-}}$$
$$\lesssim \sup_{n_2,\tau_2} \sum_{n\in\mathbb{Z}} \frac{1}{\langle \tau_2-(n+n_2)^2+n^3\rangle^{\theta}} \lesssim 1. \qquad \square \tag{3.29}$$

Once (3.17) is proved, we start the proof of estimate (3.18), that is,

$$\left\| \frac{\langle n\rangle^s}{\langle \tau-n^3\rangle} \partial_x\widehat{(u_1\overline{u_2})} \right\|_{L^2_n L^1_\tau} \lesssim \|u_1\|_{X^{k,1/2-}}\|u_2\|_{X^{k,1/2}} + \|u_1\|_{X^{k,1/2}}\|u_2\|_{X^{k,1/2-}}.$$

We can rewrite the left-hand side as

$$\left\| \int_{n=n_1+n_2} |n|\langle n\rangle^s \int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau-n^3\rangle} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}.$$

To begin with, we split the domain of integration into three regions. Let $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$\mathcal{S}_1 := \big\{(n,\tau,n_2,\tau_2)\colon\ |n| \leqslant 100\big\},$$
$$\mathcal{S}_2 := \big\{(n,\tau,n_2,\tau_2)\colon\ |n| > 100 \text{ and } |n_2| \gtrsim |n|^2\big\},$$

$\mathcal{T} := \{(n,\tau,n_2,\tau_2)\colon\ |n_2| > 100,\ |n_2| \ll |n|^2 \text{ and either } |\tau_1+n_1^2| = L_{\max} \text{ or } |-\tau_2+n_2^2| = L_{\max}\}$ and $\mathcal{U} := \{(n,\tau,n_2,\tau_2)\colon\ |n_2| > 100,\ |n| \sim |n_2| \text{ and } |\tau-n^3| = L_{\max}\}$. Clearly, $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ completely decomposes our domain of integrations, so that, in order to prove (3.18), it suffices to get the bounds:

$$\left\| \int_{n=n_1+n_2} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^k\langle n_2\rangle^k} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{S}}}{\langle \tau+n^2\rangle\langle \tau_1+n_1^2\rangle^{\frac12-}\langle -\tau_2+n_2^2\rangle^{\frac12-}} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u_1\|_{X^{0,0}}\|u_2\|_{X^{0,0}}, \tag{3.30}$$

$$\left\| \int_{n=n_1+n_2} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^k\langle n_2\rangle^k} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{T}}}{\langle \tau-n^3\rangle} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u_1\|_{X^{0,1/2-}}\|u_2\|_{X^{0,1/2}} + \|u_1\|_{X^{0,1/2}}\|u_2\|_{X^{0,1/2-}}, \tag{3.31}$$

$$\left\| \int_{n=n_1+n_2} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^k\langle n_2\rangle^k} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{U}}}{\langle \tau-n^3\rangle} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u_1\|_{X^{0,1/2-}}\|u_2\|_{X^{0,1/2}} + \|u_1\|_{X^{0,1/2}}\|u_2\|_{X^{0,1/2-}}. \tag{3.32}$$

To prove (3.30), we note that

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^k\langle n_2\rangle^k} \lesssim 1,$$

if either $|n| \leqslant 100$, or $|n| > 100$ and $|n_2| \gtrsim |n|^2$, since $\langle n\rangle \leqslant \langle n_1\rangle\langle n_2\rangle$ and $1+s \leqslant 4k$. Hence,

$$\left\| \int_{n=n_1+n_2} \frac{|n|\langle n\rangle^s}{\langle n_1\rangle^k\langle n_2\rangle^k} \int_{\tau=\tau_1+\tau_2} \frac{\chi_{\mathcal{S}}}{\langle \tau-n^3\rangle\langle \tau_1+n_1^2\rangle^{\frac12-}\langle -\tau_2+n_2^2\rangle^{\frac12-}} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \left\| \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau-n^3\rangle\langle \tau_1+n_1^2\rangle^{\frac12-}\langle -\tau_2+n_2^2\rangle^{\frac12-}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2) \right\|_{L^2_n L^1_\tau}.$$

Therefore, this reduces our goal to prove that

$$\left\| \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{1}{\langle \tau-n^3\rangle\langle \tau_1+n_1^2\rangle^{\frac12-}\langle -\tau_2+n_2^2\rangle^{\frac12-}} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u_1\|_{X^{0,0}}\|u_2\|_{X^{0,0}}.$$

This can be rewritten as

$$\left\| \frac{1}{\langle \tau-n^3\rangle^{2/3}\langle \tau-n^3\rangle^{1/3}} \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u_1\|_{X^{0,1/2-}}\|u_2\|_{X^{0,1/2-}}.$$

Since $2(-2/3) < -1$, the Cauchy–Schwarz inequality in $\tau$ reduces this bound to showing

$$\left\| \frac{1}{\langle \tau-n^3\rangle^{1/3}} \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(n_2,\tau_2)} \right\|_{L^2_n L^2_\tau} \lesssim \|u_1\|_{X^{0,1/2-}}\|u_2\|_{X^{0,1/2-}},$$

which is an easy consequence of duality, $L^4_{xt}L^2_{xt}L^4_{xt}$ Hölder and the Bourgain–Strichartz inequalities $X^{0,3/8}$, $Y^{0,1/3} \subset L^4$ in Lemma 3.5.

The second bound (3.31) can be proved in an analogous fashion, using the dispersion relation

$$\left(\tau-n^3\right) - \left(-\tau_2+n_2^2\right) + \left(\tau_1+n_1^2\right) = -n^3+n^2+2nn_2, \tag{3.33}$$

which implies that, in the region $\mathcal{M}$, either $|\tau_1 + n_1^2| \gtrsim |n|^3$ or $|-\tau_2 + n_2^2| \gtrsim |n|^3$. Thus, using that $s - k \leqslant 1/2$ and making the corresponding cancellation, we see that it suffices to prove that

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{2/3} \langle \tau - n^3 \rangle^{1/3}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^1} \lesssim \|u_1\|_{X^{0,0}} \|u_2\|_{X^{0,1/2-}}$$

and

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{2/3} \langle \tau - n^3 \rangle^{1/3}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^1} \lesssim \|u_1\|_{X^{0,1/2-}} \|u_2\|_{X^{0,0}}.$$

Again, we use Cauchy–Schwarz to reduce these estimates to

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{1/3}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^2} \lesssim \|u_1\|_{X^{0,0}} \|u_2\|_{X^{0,1/2-}}$$

and

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{1/3}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^2} \lesssim \|u_1\|_{X^{0,1/2-}} \|u_2\|_{X^{0,0}},$$

which follows from duality, Hölder and Bourgain–Strichartz, as above.

Finally, the third bound (3.32) requires a subdivision into two cases. When $|\tau_1 + n_1^2| \gtrsim |n|^{1-}$ (respectively, $|-\tau_2 + n_2^2| \gtrsim |n|^{1-}$), we use $\langle \tau_1 + n_1^2 \rangle^{1/8}$ leaving $\langle \tau_1 + n_1^2 \rangle^{3/8}$ in the denominator and $|n|^{1+s-k-}$ in the numerator (respectively, the same argument with $(-\tau_2 + n_2^2)$ instead of $(\tau_1 + n_1^2)$). After another cancellation using $|\tau - n^3| \gtrsim |n|^3$, we need to prove

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{1/2+}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^1} \lesssim \|u_1\|_{X^{0,3/8}} \|u_2\|_{X^{0,1/2-}}$$

and

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{1/2+}} \int\limits_{n=n_1+n_2} \int\limits_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1, \tau_1) \overline{\widehat{u_2}(-n_2, -\tau_2)} \right\|_{L_n^2 L_\tau^1} \lesssim \|u_1\|_{X^{0,1/2-}} \|u_2\|_{X^{0,3/8}}.$$

These bounds follow again from Cauchy–Schwarz in $\tau$, duality, Hölder and Bourgain–Strichartz. So it remains only the case $|\tau_1 + n_1^2|$, $|\tau_2 - n_2^3| \ll |n|^{1-}$. In this case, the dispersion relation says that, in the region $\mathcal{N}$,

$$\tau - n^3 = -n^3 + n^2 + 2nn_2 - O\big(|n|^{1-}\big).$$

On the other hand, the cancellation using $|\tau - n^3| \gtrsim |n|^3$ and $s - k \leqslant 1/2$ reduces the proof to the bound

$$\left\| \frac{1}{\langle \tau - n^3 \rangle^{1/2}} \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)}\chi_{\widetilde{\Omega}(n)}\big(\tau - n^3\big) \right\|_{L^2_n L^1_\tau} \lesssim \|u_1\|_{X^{0,1/2-}} \|u_2\|_{X^{0,1/2-}},$$

where $\widetilde{\Omega}(n) = \{\eta \in \mathbb{R}\colon\ \eta = n^3 - n^2 - 2nr + O(|n|^{1-})$, for some $r \in \mathbb{Z}$, $|r| \ll |n|^2\}$ if $|n| > 100$ and $\widetilde{\Omega}(n) = \emptyset$, otherwise. Applying Cauchy–Schwarz in $\tau$, we can estimate the left-hand side by

$$\left\| \left( \int \big\langle \tau - n^3 \big\rangle^{-1} \chi_{\widetilde{\Omega}(n)}\big(\tau - n^3\big) \right)^{1/2} \left\| \int_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{u_1}(n_1,\tau_1)\overline{\widehat{u_2}(-n_2,-\tau_2)} \right\|_{L^2_\tau} \right\|_{L^2_n}.$$

Therefore, the point is to show

$$\sup_n \left( \int \big\langle \tau - n^3 \big\rangle^{-1} \chi_{\widetilde{\Omega}(n)}\big(\tau - n^3\big)\, d\tau \right) \lesssim 1. \tag{3.34}$$

We need the following lemma:

**Lemma 3.8.** *There exists some $\delta > 0$ such that, for any fixed $n \in \mathbb{Z}$, $|n| \gg 1$ and for all $M \geqslant 1$ dyadic, we have*

$$\left| \left\{ \mu \in \mathbb{R}\colon\ |\mu| \sim M,\ \mu = n^3 - n^2 - 2nr + O\big(|n|^{1-}\big),\ \textit{for some } r \in \mathbb{Z},\ |r| \ll |n|^2 \right\} \right| \lesssim M^{1-\delta}.$$

**Proof.** Note that the dyadic block $\{|\mu| \sim M\}$ contains at most $O(M/N) + 1$ integer numbers of the form $n^3 - n^2 - 2nr$ with $r \in \mathbb{Z}$, where $N \sim |n|$. Indeed, this follows from the fact that the distance between two consecutive numbers of this form is $\sim N$. Thus, the set of $\mu$ verifying $\mu = r^3 - r^2 - 2nr + O(|r|^{2-})$ is the union of $O(M/N) + 1$ intervals of size $O(N^{2-})$. Since the relation $\mu = n^3 - n^2 - 2nr + O(|n|^{1-})$ with $|\mu| \sim M$ and $|r| \ll |n|^2 \sim N^2 \gg 1$ implies that $M \sim N^3$, we get

$$\left| \left\{ \mu \in \mathbb{R}\colon\ |\mu| \sim M,\ \mu = n^3 - n^2 - 2nr + O\big(|n|^{1-}\big),\ \text{for some } r \in \mathbb{Z},\ |r| \ll |n|^2 \right\} \right| \lesssim N^{1-} \cdot \frac{M}{N} \lesssim M^{1-}.$$

This completes the proof of Lemma 3.8. □

It is now easy to conclude the proof of (3.16): by changing variables, we have to estimate

$$\sup_n \int \langle \mu \rangle^{-1} \chi_{\widetilde{\Omega}(n)}(\mu)\, d\mu.$$

By decomposing the domain of integration into dyadic blocks $\{|\mu| \sim M\}$, Lemma 3.8 gives

$$\int \langle \mu \rangle^{-1} \chi_{\widetilde{\Omega}(n)}(\mu)\, d\mu \leqslant 1 + \sum_{M \geqslant 1} \int_{|\mu| \sim M} \langle \mu \rangle^{-1} \chi_{\widetilde{\Omega}(n)(\mu)}\, d\mu \lesssim 1 + \sum_{M \geqslant 1;\ M \text{ dyadic}} M^{-1} M^{1-\delta} \lesssim 1.$$

This proves estimate (3.18), thus completing the proof of Lemma 3.7. □

## 4. Local well-posedness for rough initial data

This section contains the proof of Theorem 1.1 concerning the local well-posedness of the NLS–KdV. First of all, we observe that the NLS–KdV (1.1) is equivalent to the integral equation

$$u(t) = U(t)u_0 - i\int_0^t U(t-t')\{\alpha u(t')v(t') + \beta|u|^2u(t')\}\,dt',$$

$$v(t) = V(t)v_0 + \int_0^t V(t-t')\left\{\gamma\partial_x\left(|u|^2\right)(t') - \frac{1}{2}\partial_x\left(v^2\right)(t')\right\}dt'.$$

Since we are seeking for local-in-time solutions for (1.1), it suffices to find a fixed point $u$ for the map $\Phi = (\Phi_1, \Phi_2): \widetilde{X}^k([0,T]) \times \widetilde{Y}^s([0,T]) \to \widetilde{X}^k([0,T]) \times \widetilde{Y}^s([0,T])$,

$$\Phi_1(u,v) = \psi_1(t)U(t)u_0 - i\psi_T(t)\int_0^t U(t-t')\{\alpha u(t')v(t') + \beta|u|^2u(t')\}\,dt',$$

$$\Phi_2(u,v) = \psi_1(t)V(t)v_0 + \psi_T(t)\int_0^t V(t-t')\left\{\gamma\partial_x\left(|u|^2\right)(t') - \frac{1}{2}\partial_x\left(v^2\right)(t')\right\}dt'.$$

From now on, our efforts are to show that $\Phi$ is a contraction of (a large ball of) the space $\widetilde{X}^k([0,T]) \times \widetilde{Y}^s([0,T])$ for sufficiently small $T > 0$. To accomplish this goal, we need the following well-known linear and multilinear estimates related to the cubic NLS and the KdV equations:

**Lemma 4.1** *(Linear estimates). It holds*

- $\|\psi_1(t)U(t)u_0\|_{X^k} \lesssim \|u_0\|_{H^k}$ *and* $\|\psi_T(t)\int_0^t U(t-t')F(t')\,dt'\|_{X^k} \lesssim \|F\|_{Z^k}$;
- $\|\psi_1(t)V(t)v_0\|_{Y^s} \lesssim \|v_0\|_{H^s}$ *and* $\|\psi_T(t)\int_0^t V(t-t')G(t')\,dt'\|_{Y^s} \lesssim \|G\|_{W^s}$.

**Lemma 4.2** *(Trilinear estimate for the cubic term $|u|^2u$). For $k \geqslant 0$, we have*

$$\left\|\psi(t)uv\overline{w}\right\|_{Z^k} \lesssim \|u\|_{X^{k,3/8}}\|v\|_{X^{k,3/8}}\|w\|_{X^{k,3/8}}.$$

**Lemma 4.3** *(Bilinear estimate for $\partial_x(v^2)$). For $s \geqslant -1/2$, we have*

$$\left\|\psi(t)\partial_x(v_1v_2)\right\|_{W^s} \lesssim \|v_1\|_{Y^{s,1/2}}\|v_2\|_{Y^{s,1/2-}} + \|v_1\|_{Y^{s,1/2-}}\|v_2\|_{Y^{s,1/2}},$$

*if $v_1 = v_1(x,t)$ and $v_2 = v_2(x,t)$ are $x$-periodic functions having zero $x$-mean for all $t$ (i.e., $\int_{\mathbb{T}} v_j(x,t)\,dx = 0$ for all $t$ and $j = 1,2$).*

**Remark 4.1.** The zero-mean assumption in Lemma 4.3 above is crucial for some of the analysis of the multiplier associated to this bilinear estimate. However, in the proof of our local well-posedness result, this hypothesis is not restrictive by a standard argument based on the conservation of the mean of $v$ under the flow (1.1). See Remark 4.2 below.

We present the proofs of these lemmas in Appendix A of this paper because some of these estimates are not stated as above in the literature, although they are contained in the works [5,6] for instance. See Appendix A below for more details. Returning to the proof of Theorem 1.1, in order to apply Lemma 4.3, we make the following observation:

**Remark 4.2.** The spatial mean $\int_{\mathbb{T}} v(t,x)\,dx$ is preserved during evolution (1.1). Thus, we can assume that the initial data $v_0$ has zero-mean, since otherwise we make the change $w = v - \int_{\mathbb{T}} v_0\,dx$ at the expense of two harmless linear terms (namely, $u\int_{\mathbb{T}} v_0\,dx$ and $\partial_x v \int_{\mathbb{T}} v_0$).

After this reduction, we are ready to finish the proof of Theorem 1.1. Accordingly with the linear estimates (Lemma 4.1), trilinear estimate for the cubic term $|u|^2u$ (Lemma 4.2), bilinear estimate for $\partial_x(v^2)$ (Lemma 4.3) and the bilinear estimates for the coupling terms (Propositions 1.1 and 1.2), we obtain

$$\begin{aligned}\big\|\Phi_1(u,v)\big\|_{\widetilde{X}^k([0,T])} &\leqslant C_0\|u_0\|_{H^k} + C_1\big\{\|uv\|_{Z^k} + \|u\|^3_{X^{k,3/8}([0,T])}\big\}\\ &\leqslant C_0\|u_0\|_{H^k} + C_1\|u\|_{X^{k,1/2-}([0,T])}\|v\|_{Y^{s,1/2}([0,T])}\\ &\quad + C_1\|u\|_{X^{k,1/2}([0,T])}\|v\|_{Y^{s,1/2-}([0,T])} + C_1\|u\|^3_{X^{k,3/8}([0,T])} \quad \text{and}\end{aligned}$$

$$\begin{aligned}\big\|\Phi_2(u,v)\big\|_{\widetilde{Y}^s([0,T])} &\leqslant C_0\|v_0\|_{H^s} + C_1\big\{\big\|\partial_x\big(v^2\big)\big\|_{W^s} + \big\|\partial_x\big(|u|^2\big)\big\|_{W^s}\big\}\\ &\leqslant C_0\|v_0\|_{H^k} + C_1\big\{\|v\|_{Y^{s,1/2}}\|v\|_{Y^{s,1/2-}([0,T])} + \|u\|_{X^{k,1/2}}\|u\|_{X^{k,1/2-}([0,T])}\big\},\end{aligned}$$

if $s \geqslant 0$, $-1/2 \leqslant k-s \leqslant 3/2$ and $1+s \leqslant 4k$. At this point we invoke the following elementary lemma concerning the stability of Bourgain's spaces with respect to time localization:

**Lemma 4.4.** *Let* $X^{s,b}_{\tau=h(\xi)} := \{f\colon \langle\tau - h(\xi)\rangle^b\langle\xi\rangle^s|\hat{f}(\tau,\xi)| \in L^2\}$. *Then,*

$$\big\|\psi(t)f\big\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{\psi,b} \|f\|_{X^{s,b}_{\tau=h(\xi)}}$$

*for any* $s,b\in\mathbb{R}$ *and, furthermore, if* $-1/2 < b' \leqslant b < 1/2$, *then for any* $0 < T < 1$ *we have*

$$\big\|\psi_T(t)f\big\|_{X^{s,b'}_{\tau=h(\xi)}} \lesssim_{\psi,b',b} T^{b-b'}\|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

**Proof.** First of all, note that $\langle\tau-\tau_0-h(\xi)\rangle^b \lesssim_b \langle\tau_0\rangle^{|b|}\langle\tau - h(\xi)\rangle^b$, from which we obtain

$$\big\|e^{it\tau_0}f\big\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_b \langle\tau_0\rangle^{|b|}\|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Using that $\psi(t) = \int\hat{\psi}(\tau_0)e^{it\tau_0}\,d\tau_0$, we conclude

$$\big\|\psi(t)f\big\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_b \left(\int\big|\hat{\psi}(\tau_0)\big|\langle\tau_0\rangle^{|b|}\right)\|f\|_{X^{s,b}_{\tau=h(\xi)}}.$$

Since $\psi$ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s=0$ and, by composition it suffices to treat the cases $0 \leqslant b' \leqslant b$ or $b' \leqslant b \leqslant 0$. By duality, we may take $0 \leqslant b' \leqslant b$. Finally, by interpolation with the trivial case $b'=b$, we may consider $b'=0$. This reduces matters to show that

$$\left\| \psi_T(t) f \right\|_{L^2} \lesssim_{\psi,b} T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

for $0 < b < 1/2$. Partitioning the frequency spaces into the cases $\langle \tau - h(\xi)\rangle \geqslant 1/T$ and $\langle \tau - h(\xi)\rangle \leqslant 1/T$, we see that in the former case we will have

$$\|f\|_{X^{0,0}_{\tau=h(\xi)}} \leqslant T^b \|f\|_{X^{0,b}_{\tau=h(\xi)}}$$

and the desired estimate follows because the multiplication by $\psi$ is a bounded operation in Bourgain's spaces. In the latter case, by Plancherel and Cauchy–Schwarz

$$\begin{aligned} \left\| f(t) \right\|_{L^2_x} &\lesssim \left\| \widehat{f(t)}(\xi) \right\|_{L^2_\xi} \lesssim \left\| \int_{\langle \tau - h(\xi)\rangle \leqslant 1/T} \left| \hat f(\tau,\xi)\right| d\tau \right\|_{L^2_\xi} \\ &\lesssim_b T^{b-1/2} \left\| \int \left( \langle \tau - h(\xi)\rangle^{2b} \left| \hat f(\tau,\xi)\right|^2 d\tau \right)^{1/2} \right\|_{L^2_\xi} = T^{b-1/2} \|f\|_{X^{s,b}_{\tau=h(\xi)}}. \end{aligned}$$

Integrating this against $\psi_T$ concludes the proof of the lemma. □

Now, a direct application of this lemma yields

$$\left\| \Phi_1(u,v) \right\|_{\widetilde{X}^k([0,T])} \leqslant C_0 \|u_0\|_{H^k} + C_1 T^{0+} \left\{ \|u\|_{X^{k,1/2}([0,T])} \|v\|_{Y^{s,1/2}([0,T])} + \|u\|^3_{X^{k,1/2}([0,T])} \right\}$$

and

$$\left\| \Phi_2(u,v) \right\|_{\widetilde{Y}^s([0,T])} \leqslant C_0 \|v_0\|_{H^s} + C_1 T^{0+} \left\{ \|v\|^2_{Y^{s,1/2}([0,T])} + \|u\|^2_{X^{k,1/2}([0,T])} \right\},$$

if $s \geqslant 0$, $-1/2 \leqslant k-s \leqslant 3/2$ and $1+s \leqslant 4k$. Hence, if $T>0$ is sufficiently small (depending on $\|u_0\|_{H^k}$ and $\|v_0\|_{H^s}$), we see that for every sufficiently large $R>0$, $\Phi$ sends the ball of radius $R$ of the space $\widetilde{X}^k([0,T]) \times \widetilde{Y}^s([0,T])$ into itself. Similarly, we have that

$$\begin{aligned} &\left\| \Phi_1(u,v) - \Phi_1(\tilde u, \tilde v) \right\|_{\widetilde{X}^k} \\ &\quad \lesssim T^{0+} \left\{ \|u\|_{X^{k,1/2}} + \|u\|^2_{X^{k,1/2}} + \|v\|_{Y^{s,1/2}} \right\} \left\{ \|u - \tilde u\|_{X^{k,1/2}} + \|v - \tilde v\|_{Y^{s,1/2}} \right\} \end{aligned}$$

and

$$\begin{aligned} &\left\| \Phi_2(u,v) - \Phi_2(\tilde u, \tilde v) \right\|_{\widetilde{Y}^s([0,T])} \\ &\quad \lesssim T^{0+} \left\{ \|u\|_{X^{k,1/2}} + \|v\|_{Y^{s,1/2}} \right\} \left\{ \|u - \tilde u\|_{X^{k,1/2}} + \|v - \tilde v\|_{Y^{s,1/2}} \right\}, \end{aligned}$$

if $s \geqslant 0$, $-1/2 \leqslant k-s \leqslant 3/2$ and $1+s \leqslant 4k$. So, up to taking $T>0$ smaller, we get that $\Phi$ is a contraction. This concludes the proof of Theorem 1.1.

## 5. Global well-posedness in the energy space $H^1 \times H^1$

This section is devoted to the proof of Theorem 1.2. First of all, we recall the following conserved functionals for the NLS–KdV system.

**Lemma 5.1.** *The evolution* (1.1) *preserves the quantities*

- $M(t) := \int_{\mathbb{T}} |u(t)|^2\, dx,$
- $Q(t) := \int_{\mathbb{T}} \{\alpha v(t)^2 + 2\gamma \Im(u(t)\overline{\partial_x u(t)})\}\, dx$ *and*
- $E(t) := \int_{\mathbb{T}} \{\alpha\gamma v(t)|u(t)|^2 - \frac{\alpha}{6} v(t)^3 + \frac{\beta\gamma}{2}|u(t)|^4 + \frac{\alpha}{2}|\partial_x v(t)|^2 + \gamma|\partial_x u(t)|^2\}\, dx.$

*In other words,* $M(t) = M(0)$, $Q(t) = Q(0)$ *and* $E(t) = E(0)$.

In order to do not interrupt the proof of the global well-posedness result, we postpone the proof of this lemma to Appendix A.

Let $\alpha\gamma > 0$ and $t > 0$. From the previous lemma, we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, and

$$\|v(t)\|_{L^2}^2 \leqslant \frac{1}{|\alpha|}\bigl\{|Q_0| + 2|\gamma|\|u_0\|_{L^2}\|\partial_x u(t)\|_{L^2}\bigr\}.$$

Put $\mu = \min\{|\gamma|, |\alpha|/2\}$. Then, using again the previous lemma, Gagliardo–Nirenberg and Young inequalities, we deduce

$$\begin{aligned}
\|\partial_x u(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2 &\leqslant \frac{1}{\mu}\bigl(|\gamma|\|\partial_x u(t)\|_{L^2}^2 + |\alpha|\|\partial_x v(t)\|_{L^2}^2\bigr) \\
&\leqslant C\bigl(|E(0)| + \|v(t)\|_{L^2}\|u(t)\|_{L^4}^2 + \|v(t)\|_{L^3}^3 + \|u(t)\|_{L^4}^4\bigr) \\
&\leqslant C\bigl(|E(0)| + \|v(t)\|_{L^2}^2 + \|v(t)\|_{L^3}^3 + \|u(t)\|_{L^4}^4\bigr) \\
&\leqslant C\bigl(|E(0)| + |Q(0)| + \|u_0\|_{L^2}\|\partial_x u(t)\|_{L^2} + \|v(t)\|_{L^3}^3 + \|u(t)\|_{L^4}^4\bigr) \\
&\leqslant C\bigl\{|E(0)| + |Q(0)| + |Q(0)|^{5/3} + M(0)^5 + M(0)^3 + M(0)\bigr\} \\
&\quad + \frac{1}{2}\bigl\{\|\partial_x u(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2\bigr\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|\partial_x u(t)\|_{L^2}^2 + \|\partial_x v(t)\|_{L^2}^2 \\
&\quad \leqslant C\bigl\{|E(0)| + |Q(0)| + |Q(0)|^{5/3} + M(0)^5 + M(0)^3 + M(0)\bigr\}. \qquad (5.1)
\end{aligned}$$

We can estimate the right-hand side of (5.1) using the conservation laws in Lemma 5.1 and Sobolev's lemma to get

$$\|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 \leqslant \Psi\bigl(\|u_0\|_{H^1}, \|v_0\|_{H^1}\bigr), \qquad (5.2)$$

where $\Psi$ is a function depending only on $\|u_0\|_{H^1}$ and $\|v_0\|_{H^1}$. We observe that the constants depend only on the parameters $\alpha$, $\beta$ and $\gamma$. Since the right-hand side in (5.2) only depends of

$\|u_0\|_{H^1}$ and $\|v_0\|_{H^1}$, we can repeat the argument of local existence of solution at time $T$ arriving to a solution for any positive time. This completes the proof of Theorem 1.2.

## 6. Final remarks

We conclude this paper with some comments and questions related to our results in Theorems 1.1, 1.2.

Concerning the local well-posedness result in Theorem 1.1, the gap between our endpoint $H^{1/4} \times L^2$ and the "natural" $L^2 \times H^{-1/2}$ endpoint[5] suggests the ill-posedness question:

**Question 6.1.** Is the periodic NLS–KdV system (1.1) ill-posed for initial data $(u_0, v_0) \in H^k \times H^s$ with $0 \leqslant k < 1/4$, $1 + s \leqslant 4k$ and $-1/2 \leqslant k - s \leqslant 3/2$?

On the other hand, one should be able to improve the global well-posedness result in Theorem 1.2 using the *I-method* of Colliander, Keel, Staffilani, Takaoka and Tao [6]. In the continuous case, the global well-posedness result in the energy space of Corcho and Linares [7] was refined by Pecher [13] via the I-method. This motivates the following question in the periodic context:

**Question 6.2.** Is the periodic NLS–KdV system (1.1) globally well-posed for initial data $(u_0, v_0) \in H^{1-} \times H^{1-}$?

We plan to address this issue in a forthcoming paper by using our bilinear estimates for the coupling terms $uv$ and $\partial_x(|u|^2)$ and the I-method.

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## Appendix A

This appendix collects some well-known results concerning linear and multilinear estimates related to the periodic cubic NLS and the periodic KdV, and also includes a brief comment about three conserved functionals for the NLS–KdV discovered by M. Tsutsumi.

### *A.1. Linear estimates*

We begin with the proof of the linear estimates in Lemma 4.1. The basic strategy of the argument is contained in the work [6] of Colliander, Keel, Staffilani, Takaoka and Tao. First,

[5] As we said before, from the sharp well-posedness theory for the NLS and the KdV equations, the well-posedness endpoint for the periodic NLS equation is $L^2$ and for the periodic KdV is $H^{-1/2}$.

we observe that $\widehat{\psi U(u_0)}(n,\tau)=\widehat{u_0}(n)\hat{\psi}(\tau+n^2)$ and $\widehat{\psi V(v_0)}(n,\tau)=\widehat{v_0}(n)\hat{\psi}(\tau-n^3)$. Thus, it follows that

$$\left\|\psi(t)U(t)u_0\right\|_{Z^k}\lesssim\|u_0\|_{H^k}\quad\text{and}\quad\left\|\psi(t)V(t)v_0\right\|_{W^s}\lesssim\|v_0\|_{H^s}.\tag{A.1}$$

Hence, it remains only to show that

$$\left\|\psi_T(t)\int_0^t U(t-t')F(t')\,dt'\right\|_{X^k}\lesssim\|F\|_{Z^k}\quad\text{and}\quad\left\|\psi_T(t)\int_0^t V(t-t')G(t')\,dt'\right\|_{Y^s}\lesssim\|G\|_{W^s}.$$

Up to a smooth cutoff, we can assume that both $F$ and $G$ are supported on $\mathbb{T}\times[-3,3]$. Let $a(t)=\operatorname{sgn}(t)\eta(t)$, where $\eta(t)$ is a smooth bump function supported on $[-10,10]$ which equals 1 on $[-5,5]$. The identity

$$\chi_{[0,t]}(t')=\frac{1}{2}\big(a(t')-a(t-t')\big),$$

for $t\in[-2,2]$ and $t'\in[-3,3]$ permits to rewrite $\psi_T(t)\int_0^t U(t-t')F(t')\,dt'$ (respectively, $\psi_T(t)\int_0^t V(t-t')G(t')\,dt'$) as a linear combination of

$$\psi_T(t)U(t)\int_{\mathbb{R}} a(t')U(-t')F(t')\,dt'\qquad\left(\text{respectively, }\psi_T(t)V(t)\int_{\mathbb{R}} a(t')V(-t')G(t')\,dt'\right)\tag{A.2}$$

and

$$\psi_T(t)\int_{\mathbb{R}} a(t-t')U(t-t')F(t')\,dt'\qquad\left(\text{respectively, }\psi_T(t)\int_{\mathbb{R}} a(t-t')V(t-t')G(t')\,dt'\right).\tag{A.3}$$

For (A.2), we note that by (A.1), it suffices to prove that

$$\left\|\int_{\mathbb{R}} a(t')U(-t')F(t')\,dt'\right\|_{H^k}\lesssim\|F\|_{Z^k}$$
$$\left(\text{respectively, }\left\|\int_{\mathbb{R}} a(t')V(-t')G(t')\,dt'\right\|_{H^s}\lesssim\|G\|_{W^s}\right).$$

Since the Fourier transform of $\int_{\mathbb{R}} a(t')U(-t')F(t')\,dt'$ (respectively, $\int_{\mathbb{R}} a(t')V(-t')G(t')\,dt'$) at $n$ is $\int\hat{a}(\tau+n^2)\widehat{F}(n,\tau)\,d\tau$ (respectively, $\int\hat{a}(\tau-n^3)\widehat{G}(n,\tau)\,d\tau$) and $|\hat{a}(\tau)|=O(\langle\tau\rangle^{-1})$, the desired estimate follows. For (A.3), we discard the cutoff $\psi_T(t)$ and note that the Fourier transform of $\int_{\mathbb{R}} a(t-t')U(t-t')F(t')\,dt'$ (respectively, $\int_{\mathbb{R}} a(t-t')V(t-t')G(t')\,dt'$) evaluated at $(n,\tau)$ is $\hat{a}(\tau+n^2)\widehat{F}(n,\tau)$ (respectively, $\hat{a}(\tau-n^3)\widehat{G}(n,\tau)$). Therefore, the decay estimate $|\hat{a}(\tau)|=O(\langle\tau\rangle^{-1})$ give us the claimed estimate. This proves Lemma 4.1.

*A.2. Trilinear estimates for* $(|u|^2u)$

Next, we prove the trilinear estimate in Lemma 4.2. The argument is essentially contained in the work [5] of Bourgain.[6]

By definition of $Z^k$, the hypothesis $k \geqslant 0$ says that it suffices to show that

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2-n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \overline{\phi(n,\tau)} \frac{\langle n\rangle^k}{\langle \tau+n^2\rangle^{1/2}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\overline{\widehat{w}(n_3,\tau_3)}$$
$$\lesssim \|u\|_{X^{k,3/8}}\|v\|_{X^{k,3/8}}\|w\|_{X^{k,3/8}} \quad \text{and}$$

$$\left\| \frac{\langle n\rangle^k}{\langle \tau+n^2\rangle} \widehat{uv\overline{w}}(n,\tau) \right\|_{L^2_n L^1_\tau} \lesssim \|u\|_{X^{k,3/8}}\|v\|_{X^{k,3/8}}\|w\|_{X^{k,3/8}}.$$

Observe that $\langle n\rangle^k \lesssim \max\{\langle n_1\rangle^k, \langle n_2\rangle^k, \langle n_3\rangle^k\}$. By symmetry, we can assume that $\langle n\rangle^k \lesssim \langle n_1\rangle^k$. This reduces matters to show that

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2-n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{\overline{\phi(n,\tau)}}{\langle \tau+n^2\rangle^{1/2}} \big\langle \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\overline{\widehat{w}(n_3,\tau_3)}\big\rangle$$
$$\lesssim \|u\|_{X^{0,3/8}}\|v\|_{X^{0,3/8}}\|w\|_{X^{0,3/8}} \quad \text{and} \tag{A.4}$$
$$\left\| \sum_{n=n_1+n_2-n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{1}{\langle \tau+n^2\rangle} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\overline{\widehat{w}(n_3,\tau_3)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u\|_{X^{0,3/8}}\|v\|_{X^{0,3/8}}\|w\|_{X^{0,3/8}}. \tag{A.5}$$

First, it is not difficult to see that duality, $L^4_{x,t}L^4_{x,t}L^4_{x,t}L^4_{x,t}$ Hölder inequality and the Bourgain–Strichartz estimate in Lemma 3.5 (i.e., $X^{0,3/8} \subset L^4$) implies (A.4). Next, consider the contribution of (A.5). By Cauchy–Schwarz in $\tau$, since $2(-5/8) < -1$, we need only to prove that

$$\left\| \sum_{n=n_1+n_2-n_3} \int_{\tau=\tau_1+\tau_2+\tau_3} \frac{1}{\langle \tau+n^2\rangle^{3/8}} \hat{u}(n_1,\tau_1)\hat{v}(n_2,\tau_2)\overline{\widehat{w}(n_3,\tau_3)} \right\|_{L^2_n L^1_\tau}$$
$$\lesssim \|u\|_{X^{0,3/8}}\|v\|_{X^{0,3/8}}\|w\|_{X^{0,3/8}},$$

which follows again by duality, $L^4_{x,t}L^4_{x,t}L^4_{x,t}L^4_{x,t}$ Hölder inequality and the Bourgain–Strichartz estimate. This concludes the proof of Lemma 4.2.

[6] The "novelty" here is to estimate the contribution of the weighted $L^2_n L^1_\tau$ portion of the $Z^k$ norm, although this is not hard, as we are going to see.

*A.3. Bilinear estimates for $\partial_x(v^2)$*

Now, we present the proof of the bilinear estimate in Lemma 4.3. Since this bilinear estimate was used only in the case $s \geqslant 0$, we will restrict ourselves to this specific context (although the proof of the bilinear estimate for $-1/2 \leqslant s \leqslant 0$ is similar). Again, the argument is due to Bourgain [5] (except for the bound on the weighted $L^2_n L^1_\tau$ portion of the $W^s$ norm, which is due to Colliander, Keel, Staffilani, Takaoka and Tao [6]). By definition of $W^s$, it suffices to prove that

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s}{\langle \tau - n^3\rangle^{1/2}} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2)\overline{\phi(n,\tau)}$$

$$\lesssim \|v_1\|_{Y^{s,1/2}} \|v_2\|_{Y^{s,1/2-}} + \|v_1\|_{Y^{s,1/2-}} \|v_2\|_{Y^{s,1/2}} \quad \text{and} \tag{A.6}$$

$$\left\| \frac{|n|\langle n\rangle^s}{\langle \tau - n^3\rangle} \widehat{v_1 v_2}(n,\tau) \right\|_{L^2_n L^1_\tau} \lesssim \|v_1\|_{Y^{s,1/2}} \|v_2\|_{Y^{s,1/2-}} + \|v_1\|_{Y^{s,1/2-}} \|v_2\|_{Y^{s,1/2}}. \tag{A.7}$$

Note that our hypothesis of zero mean implies that $nn_1n_2 \neq 0$. Since

$$\tau - n^3 = \left(\tau_1 - n_1^3\right) + \left(\tau_2 - n_2^3\right) - 3nn_1n_2,$$

we obtain that

$$\max\left\{\left\langle \tau - n^3\right\rangle, \left\langle \tau_1 - n_1^3\right\rangle, \left\langle \tau_2 - n_2^3\right\rangle\right\} \gtrsim |nn_1n_2| \gtrsim |n|^2.$$

Also, observe that $s \geqslant 0$ implies that $\langle n\rangle^s \lesssim \langle n_1\rangle^s \langle n_2\rangle^s$.

First, we deal with (A.6). To do so, we analyse two cases:

- $\langle \tau - n^3\rangle = \max\{\langle \tau - n^3\rangle, \langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle\}$: in this case, the estimate (A.6) follows from

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}\,\widehat{v_2}\,\overline{\phi(n,\tau)} \lesssim \|v_1\|_{Y^{0,1/3}} \|v_2\|_{Y^{0,1/3}},$$

  which is an easy consequence of duality, $L^4_{x,t} L^4_{x,t} L^2_{x,t}$ Hölder inequality and Bourgain–Strichartz estimate in Lemma 3.5 ($Y^{0,1/3} \subset L^4$).
- $\langle \tau_j - n_j^3\rangle = \max\{\langle \tau - n^3\rangle, \langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle\}$ for $j \in \{1,2\}$: in this case, estimate (A.6) follows from

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \frac{\overline{\phi(n,\tau)}}{\langle \tau - n^3\rangle^{1/2}} \lesssim \|v_1\|_{Y^{0,0}} \|v_2\|_{Y^{0,1/2-}} \quad \text{and}$$

$$\sup_{\|\phi\|_{L^2_{n,\tau}} \leqslant 1} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \frac{\overline{\phi(n,\tau)}}{\langle \tau - n^3\rangle^{1/2}} \lesssim \|v_1\|_{Y^{0,1/2-}} \|v_2\|_{Y^{0,0}},$$

  which are valid by duality, Hölder and the Bourgain–Strichartz estimate.

Second, we consider (A.7). Again, we distinguish two cases:

- $\langle \tau_j - n_j^3\rangle = \max\{\langle \tau - n^3\rangle, \langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle\}$ for $j \in \{1,2\}$: after doing the natural cancellations, we see that (A.7) is a corollary of

$$\left\| \langle \tau - n^3\rangle^{-2/3} \langle \tau - n^3\rangle^{-1/3} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^1} \lesssim \|v_1\|_{Y^{0,0}} \|v_2\|_{Y^{0,1/3}}$$

and

$$\left\| \langle \tau - n^3\rangle^{-2/3} \langle \tau - n^3\rangle^{-1/3} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^1} \lesssim \|v_1\|_{Y^{0,1/3}} \|v_2\|_{Y^{0,0}}.$$

Applying Cauchy–Schwarz in $\tau$, since $2(-2/3) < -1$, it suffices to prove

$$\left\| \langle \tau - n^3\rangle^{-1/3} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^2} \lesssim \|v_1\|_{Y^{0,0}} \|v_2\|_{Y^{0,1/3}} \quad \text{and}$$

$$\left\| \langle \tau - n^3\rangle^{-1/3} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^2} \lesssim \|v_1\|_{Y^{0,1/3}} \|v_2\|_{Y^{0,0}}.$$

Rewriting the left-hand sides by duality, using Hölder inequality and Bourgain–Strichartz estimate $Y^{0,1/3} \subset L^4$ we finish off this case.

- $\langle \tau - n^3\rangle = \max\{\langle \tau - n^3\rangle, \langle \tau_1 - n_1^3\rangle, \langle \tau_2 - n_2^3\rangle\}$: we subdivide this case into two situations. If $\langle \tau_j - n_j^3\rangle \gtrsim |nn_1n_2|^{1/100} \gtrsim |n|^{1/50}$ for some $j \in \{1,2\}$, we cancel $\langle \tau_j - n_j^3\rangle^{1/6}$ leaving $\langle \tau_j - n_j^3\rangle^{1/3}$ so that we need to show

$$\left\| \langle \tau - n^3\rangle^{-1/2-} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^1} \lesssim \|v_1\|_{Y^{0,0}} \|v_2\|_{Y^{0,1/3}} \quad \text{and}$$

$$\left\| \langle \tau - n^3\rangle^{-1/2-} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2) \right\|_{L_n^2 L_\tau^1} \lesssim \|v_1\|_{Y^{0,1/3}} \|v_2\|_{Y^{0,0}}.$$

This is an easy consequence of Cauchy–Schwarz in $\tau$, Hölder inequality and Bourgain–Strichartz. If $\langle \tau_j - n_j^3\rangle \ll |nn_1n_2|^{1/100}$ for $j = 1, 2$, we observe that

$$\tau - n^3 = -3nn_1n_2 + O\big(\langle nn_1n_2\rangle^{1/100}\big).$$

After some cancellations, we need to prove that

$$\left\| \langle \tau - n^3\rangle^{-1/2} \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1,\tau_1)\widehat{v_2}(n_2,\tau_2)\chi_{\Omega(n)}\big(\tau - n^3\big) \right\|_{L_n^2 L_\tau^1}$$
$$\lesssim \|v_1\|_{Y^{0,1/3}} \|v_2\|_{Y^{0,1/3}},$$

where $\Omega(n) := \{\eta \in \mathbb{R}\colon\ \eta = -3nn_1n_2 + O(\langle nn_1n_2\rangle^{1/100})$ for $n_1, n_2 \in \mathbb{Z}$ with $n = n_1 + n_2\}$. By Cauchy–Schwarz in $\tau$, we bound the left-hand side by

$$\left\| \left( \int \langle \tau - n^3 \rangle^{-1} \chi_{\Omega(n)}\left(\tau - n^3\right) d\tau \right)^{1/2} \left\| \sum_{n=n_1+n_2} \int_{\tau=\tau_1+\tau_2} \widehat{v_1}(n_1, \tau_1)\widehat{v_2}(n_2, \tau_2) \right\|_{L^2_n} \right\|_{L^2_\tau}.$$

Therefore, it remains only to prove that

$$\left( \int \langle \tau - n^3 \rangle^{-1} \chi_{\Omega(n)}\left(\tau - n^3\right) d\tau \right)^{1/2} \lesssim 1.$$

To estimate the integral on the left-hand side, we need the following lemma about the distribution of points in $\Omega(n)$ in a fixed dyadic block:

**Lemma A.1.** *Fix $n \in \mathbb{Z} - \{0\}$. For $n_1, n_2 \in \mathbb{Z} - \{0\}$, we have for all dyadic $M \geqslant 1$*

$$\left|\left\{\mu \in \mathbb{R}\colon\ |\mu| \sim M,\ \mu = -3nn_1n_2 + O\left(\langle nn_1n_2\rangle^{1/100}\right)\right\}\right| \lesssim M^{1-\delta},$$

*for some $\delta > 0$.*

**Proof.** By symmetry, we may assume $|n_1| \geqslant |n_2|$. Consider first the situation $|n| \geqslant |n_1|$. Since $\mu = -3nn_1n_2 + O(\langle nn_1n_2\rangle^{1/100})$, we get $|n| \lesssim |\mu| \lesssim |n|^3$ because $n_1, n_2 \in \mathbb{Z} - \{0\}$ and $|nn_1n_2| \lesssim |n|^3$. Suppose $\mu \sim M$ and $|n| \sim N$. For some $1 \leqslant p \leqslant 3$, we have $M \sim N^p$. Thus, the expression of $\mu$ implies that $|n_1n_2| \sim M^{1-1/p}$. Observe that there are at most $M^{1-1/p}$ multiples of $M^{1/p}$ in the dyadic block $\{|\mu| \sim M\}$. Therefore, the set of $\mu$ with the form $-3nn_1n_2 + O(\langle nn_1n_2\rangle^{1/100})$ is the union of $M^{1-1/p}$ intervals of size $M^{1/100}$, each of them containing an integer multiple of $n$. Then,

$$\left|\left\{\mu \in \mathbb{R}\colon\ |\mu| \sim M,\ \mu = -3nn_1n_2 + O\left(\langle nn_1n_2\rangle^{1/100}\right)\right\}\right| \leqslant M^{1-1/p} M^{1/100} \lesssim M^{3/4},$$

since $1 \leqslant p \leqslant 3$.

In the situation $|n| \leqslant |n_1|$, we have $|n_1| \lesssim |\mu| \lesssim |n_1|^3$. So, if $|n_1| \sim N_1$, we obtain $M \sim N_1^p$ for some $1 \leqslant p \leqslant 3$. Thus, we can repeat the previous argument. □

Using this lemma, it is not hard to prove that

$$\int \langle \tau - n^3 \rangle^{-1} \chi_{\Omega(n)}\left(\tau - n^3\right) d\tau \lesssim 1.$$

Indeed, we change the variables to rewrite the left-hand side as

$$\int \langle \mu \rangle^{-1} \chi_{\Omega(n)}(\mu)\, d\mu.$$

Decomposing the domain of integration and using the previous lemma, we have

$$\int \langle\mu\rangle^{-1}\chi_{\Omega(n)}(\mu)\,d\mu = \int_{|\mu|\leqslant 1} \langle\mu\rangle^{-1}\chi_{\Omega(n)}(\mu)\,d\mu + \sum_{M\geqslant 1 \text{ dyadic}} \int_{|\mu|\sim M} \langle\mu\rangle^{-1}\chi_{\Omega(n)}(\mu)\,d\mu$$
$$\lesssim 1 + \sum_{M\geqslant 1 \text{ dyadic}} M^{-1}M^{1-\delta} \lesssim 1.$$

This finishes the proof of Lemma 4.3.

### *A.4. Three conserved quantities for the NLS–KdV flow*

In the sequel, we show that the quantities

- $M(t) := \int_{\mathbb{T}} |u(t)|^2\,dx$,
- $Q(t) := \int_{\mathbb{T}} \{\alpha v(t)^2 + 2\gamma \Im(u(t)\overline{\partial_x u(t)})\}\,dx$ and
- $E(t) := \int_{\mathbb{T}} \{\alpha\gamma v(t)|u(t)|^2 - \frac{\alpha}{6}v(t)^3 + \frac{\beta\gamma}{2}|u(t)|^4 + \frac{\alpha}{2}|\partial_x v(t)|^2 + \gamma|\partial_x u(t)|^2\}\,dx$

are conserved by the NLS–KdV flow, as discovered by M. Tsutsumi [15]. By the local well-posedness result in Theorem 1.1, we may assume that $u$ and $v$ are smooth in both $x$ and $t$ variables. First, we consider $M(t)$. Differentiating with respect to $t$, we have

$$\partial_t M(t) = \int_{\mathbb{T}} \partial_t u \cdot \bar{u} + \int_{\mathbb{T}} u \cdot \overline{\partial_t u}.$$

Since Eq. (1.1) implies

$$\partial_t u = i\partial_x^2 u - i\alpha uv - i\beta|u|^2 u,$$

we see that, by integration by parts,

$$\int_{\mathbb{T}} \partial_t u \cdot \bar{u} = i\int \partial_x^2 u \cdot \bar{u} - i\int \alpha u\bar{u}v - i\int \beta|u|^4$$
$$= -\int u \cdot \overline{i\partial_x^2 u} - \int u(\overline{-i\alpha uv}) - \int u\big(\overline{-i\beta|u|^2 u}\big) = -\int_{\mathbb{T}} u \cdot \overline{\partial_t u}.$$

Hence, $\partial_t M(t) = 0$, i.e., $M(t)$ is a conserved quantity. Second, we analyse $Q(t)$. Differentiating with respect to $t$ and using that $v$ is a real-valued function,

$$\partial_t Q(t) = 2\alpha\int_{\mathbb{T}} \partial_t v \cdot v + 2\gamma\int_{\mathbb{T}} \Im(\partial_t u\overline{\partial_x u}) + 2\gamma\int_{\mathbb{T}} \Im(u\overline{\partial_x\partial_t u}).$$

Applying (1.1) and using integration by parts, we obtain

$$2\alpha\int_{\mathbb{T}}\partial_t v\cdot v = 2\alpha\gamma\int_{\mathbb{T}}\left\{-\partial_x^3 v-\frac{1}{2}\partial_x\left(v^2\right)+\gamma\partial_x\left(|u|^2\right)\right\}\cdot v = 2\alpha\gamma\int_{\mathbb{T}}\partial_x\left(|u|^2\right)\cdot v,$$

$$\int_{\mathbb{T}}\partial_t u\partial_x\bar{u} = i\int_{\mathbb{T}}\partial_x^2 u\partial_x\bar{u}-i\alpha\int_{\mathbb{T}}uv\partial_x\bar{u}-i\beta\int_{\mathbb{T}}|u|^2u\partial_x\bar{u}$$

and

$$\begin{aligned}\int_{\mathbb{T}}u\partial_x\partial_t\bar{u} &= \int_{\mathbb{T}}u\cdot\partial_x\left\{-i\partial_x^2 u+i\alpha\bar{u}v+i\beta|u|^2\bar{u}\right\}\\ &= i\alpha\int_{\mathbb{T}}uv\partial_x\bar{u}+i\alpha\int_{\mathbb{T}}|u|^2\partial_x v+i\beta\int_{\mathbb{T}}|u|^2u\partial_x\bar{u}+i\beta\int_{\mathbb{T}}|u|^2\partial_x\left(|u|^2\right)\\ &= i\alpha\int_{\mathbb{T}}uv\partial_x\bar{u}+i\alpha\int_{\mathbb{T}}\partial_x\left(|u|^2\right)v+i\beta\int_{\mathbb{T}}|u|^2u\partial_x\bar{u}.\end{aligned}$$

In particular,

$$\int_{\mathbb{T}}\partial_t u\partial_x\bar{u}+\int_{\mathbb{T}}u\partial_x\partial_t\bar{u} = i\int_{\mathbb{T}}\partial_x^2 u\partial_x\bar{u}+i\alpha\int_{\mathbb{T}}\partial_x\left(|u|^2\right)v.$$

Since $i\int_{\mathbb{T}}\partial_x^2 u\partial_x\bar{u}=\overline{i\int_{\mathbb{T}}\partial_x\bar{u}\partial_x^2 u}$, we get

$$\Im\left(\int_{\mathbb{T}}\partial_t u\partial_x\bar{u}+\int_{\mathbb{T}}u\partial_x\partial_t\bar{u}\right)=\alpha\int_{\mathbb{T}}\partial_x\left(|u|^2\right)v.$$

Hence, putting these informations together, we obtain $\partial_t Q(t)=0$. Third, we compute $\partial_t E(t)$. Writing $E(t)=I-II+III+IV+V$, where $I:=\alpha\gamma\int_{\mathbb{T}}|u|^2v$, $II=\frac{\alpha}{6}\int_{\mathbb{T}}v^3$, $III:=\frac{\beta\gamma}{2}\int_{\mathbb{T}}|u|^4$, $IV:=\frac{\alpha}{2}\int_{\mathbb{T}}|\partial_x v|^2$ and $V:=\gamma\int_{\mathbb{T}}|\partial_x u|^2$. Using (1.1) and integrating by parts,

$$\partial_t I = -\alpha\gamma\int_{\mathbb{T}}|u|^2\left\{\partial_x^3 v+\frac{1}{2}\partial_x\left(v^2\right)\right\}+v\left\{i\partial_x^2 u\cdot\bar{u}-i\partial_x^2\bar{u}\cdot u\right\},$$

$$-\partial_t II = -\frac{\alpha}{2}\int_{\mathbb{T}}\left\{-v^2\partial_x^3 v+\gamma v^2\partial_x\left(|u|^2\right)\right\}=\frac{\alpha}{2}\int_{\mathbb{T}}v^2\partial_x^3 v-\alpha\gamma\int_{\mathbb{T}}|u|^2\frac{1}{2}\partial_x\left(v^2\right),$$

$$\partial_t III = \beta\gamma\int_{\mathbb{T}}\left\{\partial_t u|u|^2\bar{u}+\partial_t\bar{u}|u|^2u\right\}=\beta\gamma\int_{\mathbb{T}}\left\{i\bar{u}|u|^2\partial_x^2 u-iu|u|^2\partial_x^2\bar{u}\right\},$$

$$\partial_t IV = \alpha\int_{\mathbb{T}}\partial_x v\cdot\partial_x\partial_t v = -\frac{\alpha}{2}\int_{\mathbb{T}}v^2\partial_x^3 v+\alpha\gamma\int_{\mathbb{T}}|u|^2\partial_x^3 v,$$

$$\begin{aligned}\partial_t V &= \gamma \int_{\mathbb{T}} \{\partial_x \partial_t u \cdot \partial_x \bar{u} + \partial_x u \cdot \partial_x \partial_t \bar{u}\} \\ &= \gamma \int_{\mathbb{T}} \{i\partial_x^2 u - i\alpha u v + i\beta |u|^2 u\} \partial_x^2 \bar{u} + \{-i\partial_x^2 \bar{u} + i\alpha \bar{u} v - i\beta |u|^2 \bar{u}\} \partial_x^2 u \\ &= -\alpha\gamma \int_{\mathbb{T}} v \cdot \{i\partial_x^2 \bar{u} \cdot u - i\partial_x^2 \bar{u} \cdot u\} - \beta\gamma \int_{\mathbb{T}} \{i\bar{u}|u|^2 \partial_x^2 u - iu|u|^2 \partial_x^2 \bar{u}\}.\end{aligned}$$

From these expressions, it is not hard to conclude that $\partial_t Q(t) = 0$.

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