Proof of the invariance of the Birkhoff measure of some billiards

by "two young brave warriors"

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Abstract

We show a simple argument to prove the invariance of the *Birkhoff measure* in billiards with piecewise affine boundary.

1 Introduction

During a conversation with prof. Marcelo Viana, he pointed out to us that there is a simple argument which shows the invariance of the Birkhoff measure (at least when the billiard is *poligonal*). Since the usual textbooks of ergodic theory (e.g., Mañé [M]) do not prove this well-known result, this motivates the existence of this short note about that argument.

Recall that, if U is a bounded open set of $\mathbb{R}^2$ whose boundary ∂U is the finite union of piecewise smooth curves, we can define a dynamical system as follows. Consider a particle inside U with constant velocity and rectilienar motion. Moreover, suppose that the collisions of this particle with ∂U are always perfectly elastic (i.e., the angles of incidency and reflection are the same). The moving particle can be studied with the help of the transformation $T : K \to K$, where $K = S^1 \times \partial U$ defined by: if $(\theta, p) \in K$, we consider the particle collinding with ∂U at the point p making an angle of θ with the tangent vector to ∂U at p. See the figure 1 below.

This information allow us to determine the next point q of collision of the particle with ∂U and the next angle η of incidency. We put $T(\theta, p) = (\eta, q)$.

Exercise 1. Prove that T is a measurable transformation. Hint: Prove that T^{-1} (or equivalently T) maps retangles to measurable sets of K, where we call a set $A \subset K$ a retangle if $A = I \times J$, where I and J are arcs of S^1 and

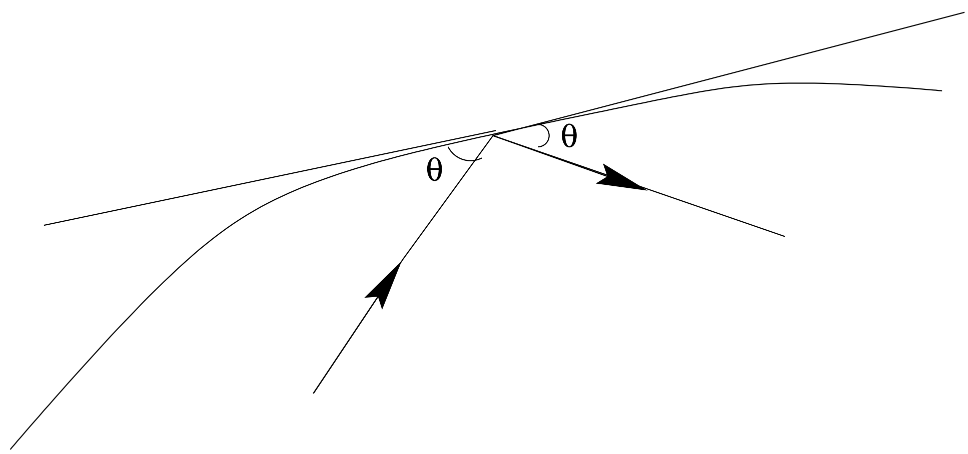

Figure 1: Definition of the system associated to the billiard

∂U, resp. This finishes the exercise since the retangles generates the Borel σ-algebra.

Let dr be the Lebesgue measure of ∂U (i.e., for every arc I of ∂U, $dr(I)$ is the lenght of I) and $d\theta$ be the Lebesgue measure of S^1.

Definition 1.1. The *Birkhoff measure* on K is:

$$\mu(A) := \int_A \sin\theta \; dr \; d\theta.$$

Theorem 1.2 (Birkhoff). *The Birkhoff measure μ is T-invariant, i.e., $\mu(T^{-1}(A)) = \mu(A)$ for any measurable $A \subset K$.*

2 Proof of Birkhoff's theorem

During this section, we assume that ∂U is piecewise affine.

We have that the incidency and reflection angles are always the same. In particular, if $A = \{\theta_0\} \times J_0 \subset K$, then, by parallelism of the possible trajectories of the particle with initial data in A, we obtain that $T(A) = \{\theta_1\} \times J_1$. See figure 2, where, in this figure, dr_i denotes the lenght of the arcs J_i, $i = 0, 1$.

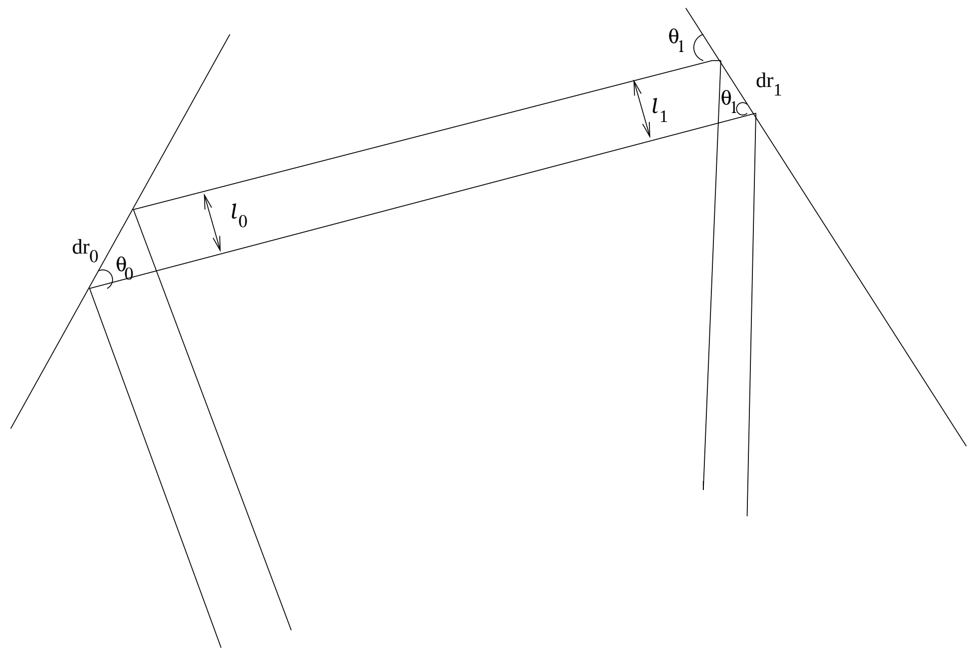

Figure 2: The case of constant angle θ_0

Moreover, again by parallelism, we obtain that the quantities l_0, l_1 in figure 2 are *equal*,

$$l_0 = l_1. \tag{1}$$

But, by definition (see figure2),

$$l_i = \sin\theta_i dr_i \text{ for } i = 0, 1. \tag{2}$$

So, we proved the following claim:

Claim 2.1. *If $A \subset K$ satisfies $A = \{\theta_0\} \times J_0$, then $T(A) = \{\theta_1\} \times J_1$. Moreover, if dr_i are the lenghts of J_i $(i = 0, 1)$, then* $\sin\theta_0 dr_0 = \sin\theta_1 dr_1$.

Define $\pi^1 : K = S^1 \times \partial U \to S^1$, $\pi^2 : K = S^1 \times \partial U \to \partial U$ the canonical projections. Now, if $A = I_0 \times \{r_0\}$, where $r_0 \in \partial U$ and I_0 an arc of S^1, then we have $\pi^1(T(A)) = I_1$ and $\pi^2(T(A)) = J_1$, for some intervals $I_1 \subset S^1, J_1 \subset \partial U$. Putting $\partial I_i = \{x_i, y_i\}$, $\partial J_i = \{p_i, q_i\}$ if we put $d\theta_i :=$ lenght of (I_i), by definition,

$$d\theta_0 = \text{ the angle between the trajectories of } (x_0, r_0) \text{ and } (y_0, r_0). \quad (3)$$

$$d\theta_1 = \text{ the angle between the trajectories of } (x_1, p_1) \text{ and } (y_1, q_1). \quad (4)$$

See the figure 3 below.

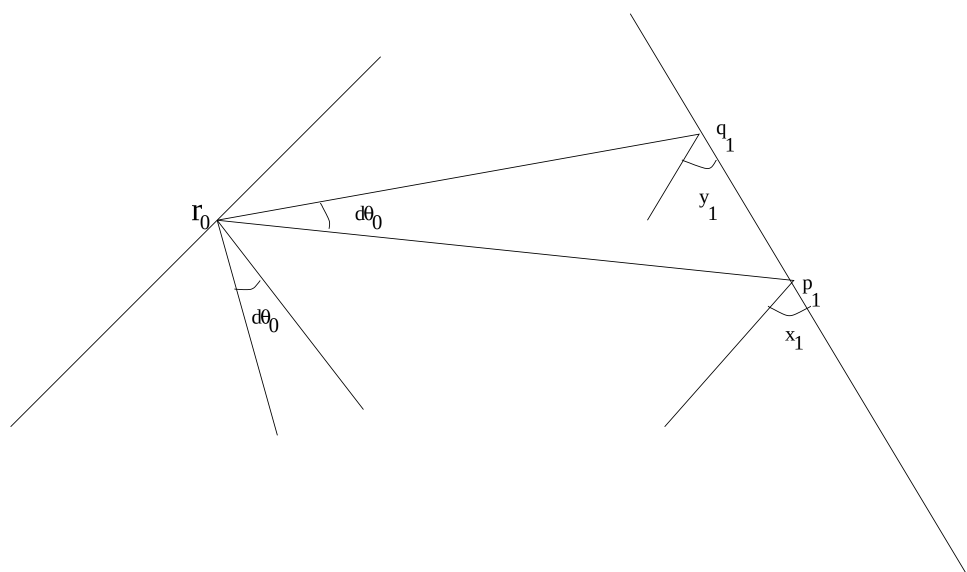

Figure 3: The case of constant position r_0

Take the parallel translation of the ray associated with (x_1, p_1) to the point $\pi^2(T(y_0, r_0)) := q_1$ of ∂U, as in figure 4 below.

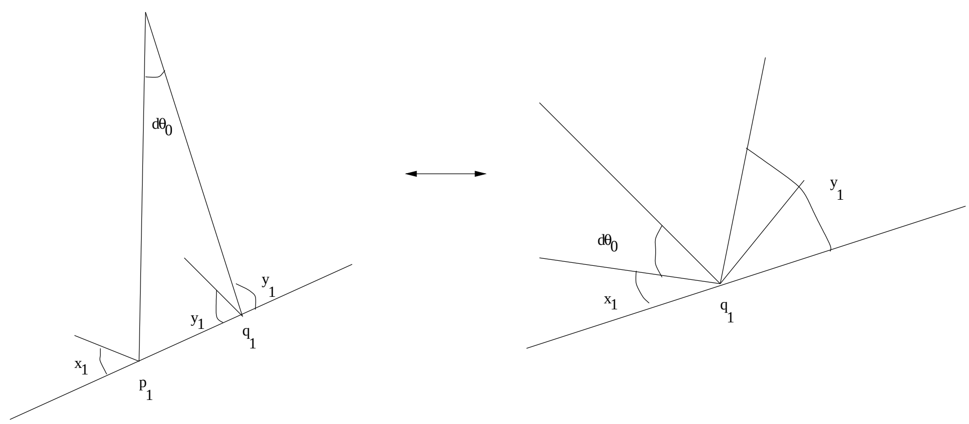

Figure 4: Parallel transport of the trajectories

We infer from the figure 4 (since the incidency and reflection angles are always the same) that

$$\text{angle between } (x_0, r_0) \text{ and } (y_0, r_0) = \text{angle between } (x_1, p_1) \text{ and } (y_1, q_1). \tag{5}$$

The equations (3), (4) and (5) together prove the claim:

Claim 2.2. *If $A \subset K$ satisfies $A = I_0 \times \{r_0\}$, then $\pi^1(T(A)) = I_1$. Moreover, $d\theta_0 = d\theta_1$, where $d\theta_i$ is the lenght of the arcs I_i, $i = 0, 1$.*

To conclude the proof, we left the following exercise:

Exercise 2. Prove that the proof follows from the claims 2.1, 2.2. Hint: Note that μ is a product measure and apply Fubini's theorem.

To end this note, we propose:

Exercise 3. Try to prove Birkhoff's theorem in the general case using the ideias of the proof in the piecewise affine case. Hint: If you can not solve this exercise, see the proof in [CM].

References

[CM] N. Chernov and R. Markarian, Introduction to the Ergodic Theory of Chaotic Billiards, *Instituto de Matematica y Ciencias Afines, Lima, Peru*, 2001. (see also *www.math.uab.edu/chernov/papers/pubs.html*)

[M] R. Mañé, Ergodic theory and differentiable dynamics, *Springer-Verlag*, 1987.