Forni's proof of Veech's theorem

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1 Context

In these notes we present a short proof (due to Forni [F]) of a theorem of Veech [V] which says:

Theorem 1.1. *The Teichmüller geodesic flow is non-uniformly hyperbolic.*

Before describing the history of this problem, we will introduce some notation (and define the Teichmüller geodesic flow).

- M is a Riemann surface of genus g;
- T_g = corresponding Teichmüller space;
- Q_g = non-zero holomorphic quadratic differentials $\simeq$ cotangent bundle of T_g minus the zero section.

Given $\kappa = (\kappa_1, \ldots, \kappa_\sigma)$, κ_i an even positive integer and $\sum_i \kappa_i = 4g-4$, we define $Q_\kappa \subset Q_g$ as the set of quadratic differentials q such that q is the square of an holomorphic differential and whose zeroes have orders $(\kappa_1, \ldots, \kappa_\sigma)$.

The complex analytic submanifold Q_κ de Q_g is called a *stratum* of Q_g.

Remark 1.2. The condition $\sum_i \kappa_i = 4g-4$ implies that the number of strata is finite.

Defining $\Gamma_g = \mathrm{Diff}^+(M)/\mathrm{Diff}^+_0(M)$ (called the *mapping class group*), then $\mathcal{M}_g := Q_g/\Gamma_g$ is the *moduli space* and $\mathcal{M}_\kappa := Q_\kappa/\Gamma_g$ are the strata of $\mathcal{M}_g$.

For a quadratic differential q, the set of zeroes of q is denoted by Σ_q.

$\mathcal{M}_\kappa$ admits a natural complex structure given by the *period map*: locally, if $q \in Q_\kappa$, we can define the square root $q^{1/2}$ of q, a holomorphic differential with zeroes in Σ_q. The cohomology class of $q^{1/2}$ on $H^1(M_q, \Sigma_q, \mathbb{C})$ is given by integrating along relative cycles of (M_q, Σ_q).

Fix the Lebesgue measure in $H^1(M_q, \Sigma_q, \mathbb{C})$ to be (uniquely) normalized such that the complex $(2g-1+\sigma)$-torus obtained as a quocient over integer lattice $\mathbb{C} \otimes_{\mathbb{Z}} H^1(M_q, \Sigma_q, \mathbb{C})$ is equal to 1. Then, the push-forward of the Lebesgue measure over $\mathcal{M}_\kappa$ (by projection) is denoted by μ_κ.

In addition, the function $A : Q_g \to \mathbb{R}^+$ associating to each q the total area of M_q (with respect to the area form ω_q of the metric R_q induced by q; see the next section for the precise definition of ω_q and R_q).

Note that the group $GL_+(2,\mathbb{R})$ acts on Q_κ by linear transformations on the pairs of real-valued 1-forms $(\Re(q^{1/2}), \Im(q^{1/2}))$. Moreover, this action commutes with Γ_g, and hence it induces an action on $\mathcal{M}_\kappa$. In the affine coordinates given by the *period map*, this action is an action of the group $GL_+(2,\mathbb{R})$ over the vector space

$$H^1(M_q, \Sigma_q, \mathbb{C}) \equiv \mathbb{C} \otimes_{\mathbb{R}} H^1(M_q, \Sigma_q, \mathbb{R}) \equiv \mathbb{R}^2 \otimes_{\mathbb{R}} H^1(M_q, \Sigma_q, \mathbb{R}) \qquad (1)$$

through the first factor of the tensorial product. Such an action commutes with Γ_g and, so it induces an action on $\mathcal{M}_\kappa$. It follows from the definitions that $SL(2,\mathbb{R})$ preserves μ_κ and A. In particular, $Q_\kappa^{(1)} := A^{-1}(1) \cap Q_\kappa$ has a natural measure $\mu_\kappa^{(1)} := \mu_\kappa / dA$. The action of the diagonal matrices $G_t = \mathrm{diag}(e^t, e^{-t})$ generates a measure-preserving flow. By the previous arguments, this flow, in fact, acts on $\mathcal{M}_\kappa^{(1)}$.

Definition 1.3. The flow G_t of $\mathcal{M}_\kappa^{(1)}$ is called *Teichmüller geodesic flow.*

Concerning the properties of this flow, we have:

Theorem 1.4 (Veech [V]). *$\mu_\kappa^{(1)}$ is a finite measure of $\mathcal{M}_\kappa^{(1)}$ and $G_t = diag(e^t, e^{-t})$ is ergodic in $\mathcal{M}_\kappa^{(1)}$ with respect to $\mu_\kappa^{(1)}$.*

On the other hand, the non-trivial part of the information about the Lyapounov exponents of this flow is contained in a natural cocycle (defined by Kontsevich and Zorich) over Teichmüller's flow.

The *Kontsevich-Zorich cocycle* is constructed as follows: let $H_g^1(M, \mathbb{C})$ be the holomorphic bundle over $Q_g^{(1)}$ with fiber $H^1(M_q, \mathbb{C})$. Since $Q_g^{(1)}$ is simply connected, $H_g^1(M, \mathbb{C})$ can be trivialized. Moreover, such a trivialization may be defined by parallel transportation with respect to the *Gauss-Manin* connection (which, in this case, is given by the property that the parallel sections are the locally constant holomorphic sections). That is, $H^1_g(M,\mathbb{C}) \equiv Q^{(1)}_g \times H^1(M,\mathbb{C})$. However, the Teichmüller flow G_t on $Q^{(1)}_g$ lifts trivially to a product bundle as the identity in the second factor. So, it lifts to a flow on $H^1_g(M,\mathbb{C})$. Passing to the quocient wit respect to the group Γ_g, this flow is defined on $\mathcal{H}^1_g(M,\mathbb{C}) := H^1_g(M,\mathbb{C})/\Gamma_g$. At this point we define:

Definition 1.5. The restriction of the previous flow (defined on $\mathcal{H}^1_g(M,\mathbb{C})$) to the (real) subbundle $\mathcal{H}^1_g(M,\mathbb{R}) := H^1_g(M,\mathbb{R})/\Gamma_g$ is called the *Kontsevich-Zorich cocycle*, which is denoted by G^{KZ}_t.

Remark 1.6. This cocycle leaves the strata $\mathcal{H}^1_\kappa(M,\mathbb{R})$ (defined by the restriction to $\mathcal{M}_\kappa$ of the bundle $\mathcal{H}^1_g(M,\mathbb{R})$).

Since $H^1(M,\mathbb{R})$ has a natural symplectic structure given by the exterior product of de Rham cohomology classes, the Kontsevich-Zorich cocycle is *symplectic*. In this case, the Lyapounov exponents are symmetric with respect to the origin.

Based on computational experiments, Kontsevich and Zorich [K] conjectured that *the Lyapounov spectrum is simple*:

KONTSEVICH-ZORICH CONJECTURE. The $2g$ Lyapounov exponents of the Kontsevich-Zorich cocycle satisfies:

$$1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g > 0 > \lambda_{g+1} = -\lambda_g > \cdots > \lambda_{2g} = -\lambda_1 = -1. \quad (2)$$

Remark 1.7. Veech [V] showed that the Teichmüller geodesic flow is *nonuniformly hyperbolic*. However, since the bundle $\mathcal{H}^1_g(M,\mathbb{R})$ does not coincide with the whole tangent space of $\mathcal{M}^{(1)}_\kappa$, Veech's theorem does not imply the Kontsevich-Zorich conjecture. By (1), the Lyapounov exponents of the Teichmüller flow (in terms of the exponents of the Kontsevich-Zorich cocycle) are (see [K]):

$$\begin{aligned} 2 \geq 1+\lambda_2 \ldots 1+\lambda_g \geq 1 \\ \geq 1-\lambda_g \cdots \geq 1-\lambda_2 \geq 0 \geq -1+\lambda_2 \geq \cdots \geq -1+\lambda_g \\ \geq -1 \geq -1-\lambda_{g-1} \geq \cdots \geq -1-\lambda_2 \geq -2. \end{aligned} \quad (3)$$

where the exponents 1 e -1 above have, at least, multiplicity $\sigma_\kappa - 1$.

The non uniform hyperbolicity of the *Teichmüller flow* (theorem 1.1) means:

Theorem A. $\lambda_2 < 1 (= \lambda_1)$ *(i.e., the first Lyapounov exponent is simple).*

The purpose of this notes is to present a simplification (due to Forni [F]) of the original proof of Veech [V]. Note that in the same work, Forni applies the methods of the proof given below to prove also the *non uniform hyperbolicity* of the Kontsevich-Zorich cocycle (i.e., $\lambda_g > 0$).

To end the introduction, these notes are organized as follows. In the next section we exhibit the (Hodge) coordinates which allows us to estimate the second exponent and to prove the theorem A, modulo some more or less general lemmas about ODE's in Banach spaces, which will be proved in the appendix.

2 A variational formula

During this section we study a variational formula used by Forni to prove the theorem A. To do this, we need to introduce:

2.1 Hodge's representation as good coordinates

Let $q \in \mathcal{M}_\kappa^{(1)}$. Every cohomology class $c \in H^1(M_q, \mathbb{R})$ can be represented by a harmonic differential, since $c = [\Re(h^+)]$, where h^+ is a holomorphic differential of M_q. This allows us to introduce in $H^1(M_q, \mathbb{R})$ the Hodge norm. This remark will be used to construct good coordinates to calculate the Lyapounov exponents of the Kontsevich-Zorich cocycle.

Define $R_q := |q|^{1/2}$ the flat (smooth) metric induced by q in M, which only degenerate on Σ_q, and ω_q the corresponding area form. That is, if $p \in M_q - \Sigma_q$, fixing a canonical holomorphic coordinate $z := x + iy$ such that $q = dz^2$, then

$$R_q = (dx^2 + dy^2)^{1/2} \quad \text{e} \quad \omega_q = dx \wedge dy.$$

If $p \in \Sigma_q$ is a zero of order k, then in canonical coordinates holds $q = z^k dz^2$, so

$$R_q = |z|^{k/2}(dx^2 + dy^2)^{1/2} \quad \text{e} \quad \omega_q = |z|^k dx \wedge dy.$$

We denote by $\mathcal{F}_q := \{\Im(q^{1/2}) = 0\}$ the *horizontal foliation* of q and $\mathcal{F}_{-q} := \{\Re(q^{1/2}) = 0\}$ the *vertical foliation* of q.

Remark 2.1. These foliations are *measured foliations* in the sense of Thurston. Their transversal measures are $|\Im(q^{1/2})|$ and $|\Re(q^{1/2})|$ resp.

There exists a unique *positively oriented* R_q-orthonormal frame $\{S, T\}$ of $TM|_{M_q - \Sigma_q}$ such that S (resp. T) is tangent to the horizantal (resp. vertical) foliation of q. Note that S, T are smooth on $M_q - \Sigma_q$, but not on Σ_q.

Define $\eta_T := \Re(q^{1/2}) = -i_T\omega_q$, $\eta_S := \Im(q^{1/2}) = i_S\omega_q$, where if ω is a differential form and X is a vector field, then $i_X\omega$ is the inner product of X by ω. Clearly $\omega_q = \eta_T \wedge \eta_S$ and, since η_S, η_T are closed 1-forms, S, T preserves the area form ω_q.

Definition 2.2.

- $L^2_q(M) := L^2(M, \omega_q)$;
- $H^1(M)$ is the Sobolev space of weakly differentiable functions with L^2-weak derivatives;
- The *Cauchy-Riemann* operators are $\partial^\pm_q := S \pm iT$, which are closed on $H^1(M) \subset L^2(M)$;
- R^+_q (resp. R^-_q) is the image of ∂^+_q (resp. ∂^-_q);
- $\mathcal{M}^-_q$ (resp. $\mathcal{M}^+_q$) is the space of anti-meromorphic (resp. meromorphic) L^2_q functions with poles on Σ_q.

Observe that the images $R^\pm_q$ of the Cauchy-Riemann operators are closed and have finite codimension. In fact, the adjoints of the Cauchy-Riemann operators satisfies $(\partial^\pm_q)^* = -\partial^\mp_q$. Hence by the Hilbert space theory, $R^\pm_q$ is orthogonal to $\mathcal{M}^\mp_q$, since $\mathcal{M}^\mp_q$ are the kernels of the operators $\partial^\pm_q$. So, we have the orthogonal decompositions

$$L^2_q(M) = R^-_q \oplus \mathcal{M}^+_q = R^+_q \oplus \mathcal{M}^-_q \tag{4}$$

and the associated orthogonal projections $\pi^\pm_q : L^2_q(M) \to \mathcal{M}^\pm_q$.

The linear (real) transformations $c^\pm_q : \mathcal{M}^\pm_q \to H^1(M_q, \mathbb{R})$

$$c^+_q(m^+) := [\Re(m^+ q^{1/2})], \quad c^-_q(m^-) := [\Re(m^- \overline{q}^{1/2})] \tag{5}$$

are isomorphisms (between vector spaces) such that

$$\|c^\pm_q(m^\pm)\|^2_q := \int_M c^\pm_q(m^\pm) \wedge * c^\pm_q(m^\pm) = |m^\pm|^2_0, \tag{6}$$

where $\|.\|_q$ is the Hodge norm in the cohomology $H^1(M_q, \mathbb{R})$, the Hodge $*$ operator is given by $*\eta_T = \eta_S, *\eta_S = -\eta_T$ and $|.|_0$ is the usual $L^2_q(M)$-norm.

We can summarize this assertions with the proposition:

Proposition 2.3. *The spaces $\mathcal{M}_q^{\pm}$ with the $L^2_q(M)$-inner product are* iso-morphic *to $H^1(M_q, \mathbb{R})$ with the Hodge inner product defined by the metric R_q and the area form ω_q.*

Proof. We need only to prove that $c_q^{\pm}$ are isomorphisms. But this follows from the fact that, if $h^{\pm}$ is a holomorphic, resp. anti-holomorphic, differential then $h^+/q^{\frac{1}{2}}$, resp. $h^-/\overline{q}^{\frac{1}{2}}$, is a meromorphic function, resp. anti-meromorphic, which lies in $L^2_q(M)$ and with poles on Σ_q. □

This means that the Hodge representation of any cohomology class as the real part of a holomorphic differential gives a natural identification with the spaces of meromorphic and anti-meromorphic functions.

For later use, we note that the *symplectic form* in $H^1(M_q, \mathbb{R})$ defined by the exterior product can be written in $\mathcal{M}_q^{\pm}$ as:

$$c_q^{\pm}(m_1^{\pm}) \wedge c_q^{\pm}(m_2^{\pm}) = \Im(m_1^{\pm}, m_2^{\pm})_q, \tag{7}$$

where $(.,.)_q$ is the $L^2_q(M)$ inner product.

Now we compute the Kontsevich-Zorich cocycle using these isomorphims.

By the decomposition of $L^2_q(M)$ in equation (4), any $u \in L^2_q(M)$ can be written as $u = \partial_q^+ v + \pi_q^-(u)$, $v \in H^1(M)$. The operator $U_q : L^2_q(M) \to L^2_q(M)$ defined by

$$U_q(u) := \partial_q^- v - \overline{\pi_q^-(u)} \quad \text{if} \;\; u = \partial_q^- v + \pi_q^-(u), \tag{8}$$

is an $\mathbb{R}$-linear isometry. Indeed, by the orthogonality of the decomposition (4), we have

$$|U_q(u)|_0^2 = |\partial_q^- v|_0^2 + |\pi_q^-(u)|_0^2 = |\partial_q^+ v|_0^2 + |\pi_q^-(u)|_0^2 = |u|_0^2. \tag{9}$$

Let $q_t := G_t(q)$ be the orbit of a quadratic differential $q \in Q_\kappa^{(1)}$ by the Teichmüller flow. Using the definition, q_t is determined by the equations

$$\eta_T(t) := \Re(q_t^{1/2}) = e^t \Re(q^{1/2}) = e^t \eta_T, \quad \eta_S(t) := \Im(q_t^{1/2}) = e^{-t} \Im(q^{1/2}) = e^{-t} \eta_S \tag{10}$$

Remark 2.4. An *important* fact in what follows is that, the previous equation says that the area form ω_q of the metric R_q is *invariant* by the Teichmüller geodesic flow.

Let $\{S_t, T_t\}$ be the orthonormal frame and $\partial_t^{\pm} := S_t \pm iT_t$ be the Cauchy-Riemann operators determined by q_t. Then

$$\partial_t^{\pm} = S_t \pm iT_t = e^{-t}S \pm ie^{t}T. \tag{11}$$

Let $\mathcal{M}_t^{\pm} := N(\partial_t^{\pm}) \subset L_q^2(M)$ be the subspaces of the meromorphic, resp. anti-meromorphic functions of $M_t := M_{q_t}$.

The Kontsevich-Zorich cocycle G_t^{KZ} along the orbit $q_t = G_t(q)$ of the Teichmüller flow is given by the following:

Lemma 2.5. *The ordinary differential equation*

$$u' = U_{q_t}(u) \tag{12}$$

is well-defined in $L_q^2(M)$ and satisfies the properties:

1. *The Cauchy problem for (12) is well-posed (i.e., it holds existence for every time and uniqueness);*
2. *If $u_t \in L_q^2(M)$ is a solution of (12) with initial condition $u_0 \in \mathcal{M}_q^+$, then $u_t \in \mathcal{M}_t^+$ for all $t \in \mathbb{R}$;*
3. *Let $m_t^+ \in \mathcal{M}_t^+$ be the unique solution of (12) with initial condition $m_0^+ = m^+ \in \mathcal{M}_q^+$. Then, for all $t \in \mathbb{R}$ holds*

$$G_t^{KZ}(c_q^+(m^+)) = c_{q_t}^+(m_t^+). \tag{13}$$

Since the proof of this lemma is a consequence of general facts about ODEs in Hilbert spaces, it will be postponed to the appendix.

Returning to the Lyapounov exponents of the Kontsevich-Zorich cocycle, it is not sufficient to know only the cocycle along the orbits, since we need also to understand how the norm of tangent vector varies.

Hence, to prove the theorem A, the first variation of Hodge norm plays an important role. With this in mind, let $c \in H^1(M_q, \mathbb{R})$ and $c_t := G_t^{KZ}(c)$, say $c_t = [\Re(m_t^+ q_t^{1/2})], m_t^+ \in \mathcal{M}_t^+$. We have

Lemma 2.6. *The Hodge norm evolves under the Kontsevich-Zorich cocycle by the formula*

$$\frac{d}{dt}|m_t^+|_0^2 = -2\Re B_q(m_t^+) := -2\Re \int_M (m_t^+)^2 \omega_q, \tag{14}$$

where $B_q(m^+) := \int_M (m^+)^2 \omega_q$.

The proof of this lemma is contained in the appendix.

Proceeding with our considerations, we wish to give an upper bound for the second Lyapounov exponent of the Kontsevich-Zorich cocycle. To do this, we define the function $\Lambda^+ : \mathcal{M}_\kappa^{(1)}/SO(2,\mathbb{R}) \to \mathbb{R}$

$$\Lambda^+(q) := \max\left\{ \frac{|B_q(m^+)|}{|m^+|_0^2} : m^+ \in \mathcal{M}_q^+ - \{0\}, \int_M m^+ \omega_q = 0 \right\}.$$

Now we are ready to prove the theorem A.

2.2 Proof of the theorem A

We will show the following:

Corollary 2.7. *Let* μ *be a* G_t*-ergodic measure in* $\mathcal{M}_\kappa^{(1)}$. *Then it holds the following upper bound for the second Lyapounov exponent of* μ*:*

$$\lambda_2^\mu \leq \int_{\mathcal{M}_\kappa^{(1)}} \Lambda^+ d\mu < \lambda_1^\mu = 1. \tag{15}$$

Remark 2.8. At this point it is clear that the inequality (15) implies the theorem A.

Proof. For $q \in \mathcal{M}_\kappa^{(1)}$, define the subbundles

- $E_1(q) := \mathbb{R} \cdot [\Im(q^{1/2})]$,
- $E_{-1}(q) := \mathbb{R} \cdot [\Re(q^{1/2})]$,
- $E_0(q) := \{c \in H^1(M_q, \mathbb{R}) : c \wedge [\Re(q^{1/2})] = c \wedge [\Im(q^{1/2})] = 0\}$.

This gives a decomposition $H^1_\kappa(M,\mathbb{R}) = E_1 \oplus E_0 \oplus E_{-1}$ of $H^1_\kappa(M,\mathbb{R})$ into three continuous invariant (with respect to the Kontsevich-Zorich cocycle) subbundles. Note that E_1 and E_{-1} are 1-dimensional, so it follows that for any G_t-ergodic probability measure μ, it holds $\lambda_1^\mu = 1, \lambda_{2g}^\mu = -\lambda_1^\mu = -1$.

On the other hand, a cohomology class $c = [\Re(m^+ q^{1/2})] \in E_0(q)$ if and only if $\int_M m^+ \omega_q = 0$, and the subbundle E_0 is invariant by the action of the group $SO(2,\mathbb{R})$ over $\mathcal{M}_\kappa^{(1)}$. Let $c \in E_0(q)$, $q_t := G_t(q)$ and $c_t := G_t^{KZ}(c)$. In particular, $c_t = [\Re(m_t^+ q_t^{1/2})]$, where $m_t^+ \in \mathcal{M}_t^+$ has zero integral.

Then by the lemma 2.6,

$$\frac{d}{dt} \log |m_t^+|_0 = -\frac{\Re B_q(m_t^+)}{|m_t^+|_0^2} \leq \Lambda^+(q_t).$$

Integrating

$$\frac{1}{\tau} \log \frac{|m_\tau^+|_0}{|m^+|_0} \leq \frac{1}{\tau} \int_0^\tau \Lambda^+(q_t) dt. \tag{16}$$

By the Oseledets' theorem, since μ is *ergodic*, taking the limit in (16) when $\tau \to \infty$, we obtain the first inequality of (15).

To obtain the second inequality, we will show that $\Lambda^+(q) < 1$ *for all* $q \in \mathcal{M}_\kappa^{(1)}$. Indeed, by Cauchy-Schwartz

$$|B_q(m^+)| = |(m^+, \overline{m^+})_q| \leq |m^+|_0^2, \tag{17}$$

with equality if and only if there exists $\lambda \in \mathbb{C}, |\lambda| = 1$ such that $\overline{m^+} = \lambda m^+$. However, in this case m^+ would be meromorphic and anti-meromorphic, and so constant. Since m^+ has zero integral, we conclude that the equality in (17) implies $m^+ = 0$. It follows that for $m^+ \neq 0$, the inequality (17) is strict. Becuase the unit sphere of $\mathcal{M}_q^+ \subset L_q^2(M)$ is compact, we have $\Lambda^+(q) < 1$.

This concludes the proof. □

3 Appendix: ODEs in Hilbert spaces

In this appendix we prove the lemmas 2.5 and 2.6.

Proof of the lemma 2.5. The ODE (12) is well defined since the space $L^2_q(M)$ is invariant along the orbits of the Teichmüller flow. Furthermore, the function $F(t,u) := U_{q_t}(u)$ is uniformly Lipschitz in the second variable since U_q is an isometry. Moreover F is smooth on $\mathbb{R} \times L^2_q(M)$. Indeed, the Cauchy-Riemann operators $\partial_t^{\pm}$, defined on $L^2_q(M)$ with common domain $H^1(M)$, depends smoothly on $t \in \mathbb{R}$ (this follows from the explicit expression of these operators in terms of the orthornormal frame $\{S, T\}$). So, by the classical theory of ODEs in Banach spaces, we have existence and uniqueness.

Because every solution satisfies $|u_t'|_0 = |u_t|_0$ (U_q is an isometry),

$$|u_t|_0 \leq |u_0|_0 + \int_0^t |u_s|_0 ds, \tag{18}$$

hence by Gronwall $|u_t|_0$ does not blow up in finite time. In particular, local solutions can be extended for all time. This proves item 1 of lemma 2.5.

By the equations (10) and (11),

$$\frac{d}{dt} q_t^{1/2} = \overline{q_t}^{1/2}, \; \frac{d}{dt} \partial^{\pm} = -\partial^{mp},$$

so, if u_t is a solution of the ODE (12), we obtain

$$\frac{d}{dt}(\partial_t^+ u_t) = -\partial_t^-(u_t) + \partial_t^+(u_t') = 0.$$

This proves that the flow generated by the ODE (12) preserves meromorphic functions, that is, the item 2 of the lemma 2.5.

Finally, let $\pi_t^{\pm} : L^2_q(M) \to \mathcal{M}_t^{\pm} \subset L^2_q(M)$ be the orthogonal projections. If $m_t^+ \in \mathcal{M}_t^+$ is a solution of (12), there exists $v_t \in H^1(M)$ (unique up to additive constants) such that

$$m_t^+ = \partial_t^+ v_t + \pi_t^-(m_t^+), \quad \frac{d}{dt} m_t^+ = \partial_t^- v_t - \overline{\pi_t^-(m_t^+)}. \tag{19}$$

If v_t is choosen with zero integral for all $t \in \mathbb{R}$, then $t \to v_t \in H^1(M)$ is a smooth function. By equation (19),

$$\frac{d}{dt} \Re(m_t^+ q_t^{1/2}) = \Re\big((\frac{d}{dt} m_t^+ + \overline{m_t^+}) q_t^{1/2}\big) = 2\Re(dv_t) \equiv 0 \in H^1(M, \mathbb{R}).$$

The definition of the Kontsevich-Zorich cocycle G_t^{KZ} implies that the equation (13) follows. In fact, the cocycle acts as the identity in cohomology classes. This proves item 3 of the lemma 2.5. □

Proof of the lemma 2.6. By (6) and the invariance of the L_q^2 inner product under the Teichmüller flow, $\|c_t\| = |m_t^+|_0$. Using (19) we obtain

$$\frac{d}{dt}|m_t^+|_0^2 = 2\Re(m_t^+, \frac{d}{dt}m_t^+)_q = -\Re(m_t^+, \overline{\pi_t^-(m_t^+)})_q \tag{20}$$

$$= -2\Re(m_t^+, \overline{m_t^+})_q = -2\Re\int_M (m_t^+)^2\omega_q. \tag{21}$$

This concludes the proof. □

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