Mathematische Zeitschrift manuscript No.
(will be inserted by the editor)

On the sensitivity of sectional-Anosov flows

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Received: date / Accepted: date

Abstract Consider a sectional-Anosov flow X on a compact 3-manifold for which the maximal invariant and nonwandering sets coincide. We prove that every vector field close to X is sensitive with respect to initial conditions.

Keywords Anosov Flow · Sectional-Anosov Flow · Sensitive.

Mathematics Subject Classification (2000) (Primary) 37D30 · (Secondary) 37D45.

1 Introduction

Let X be a C^1 vector field with flow X_t on a compact connected Riemannian manifold M. If the boundary ∂M of M is not empty we assume that X is transverse to ∂M pointing inward (in such a case X_t is defined in ∂M for $t \geq 0$ only). Denote by $d(\cdot, \cdot)$ the metric in M induced by the Riemannian structure. We say that X is *sensitive to initial conditions* if there is $\delta > 0$ such that for every $x \in M$ and every neighborhood U of x there are $y \in U$ and $t \geq 0$ satisfying $d(X_t(x), X_t(y)) \geq \delta$.

The basic examples of vector fields which are sensitive to initial conditions are the Anosov ones. They motivate the question as to whether such a property holds true for more general dynamical system as, for instance, the partially hyperbolic ones. Nevertheless, partially hyperbolic vector fields on compact manifolds with boundary may be insensitive to initial conditions as, for instance, the product of the zero vector field in the torus by a strong contraction. These counterexamples suggest one more question, namely, which conditions suffice for a partially hyperbolic vector field to be sensitive to initial conditions. The one we shall consider here is that of being *sectionally expanding*, i.e., area expanding along every two-dimensional subspace of the central subbundle. Indeed, we study the *sectional-Anosov flows* [17] and, more precisely, sectional-Anosov

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Partially supported by CNPq, FAPERJ and PRONEX/DYN-SYS. from Brazil.

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flows X on compact three-dimensional manifolds for which the maximal invariant and nonwandering sets coincide. We shall prove that sensitiveness with respect to initial conditions holds for every vector field C^1 close to X. Let us state this result in a precise way.

Hereafter all vector fields X on M will be of class C^1. Define the *maximal invariant set* of X by

$$M(X) = \bigcap_{t \geq 0} X_t(M).$$

A *nonwandering point* of X is a point $x \in M$ such that for all neighborhood U of x and all $T > 0$ there is $t > T$ such that $X_t(U) \cap U \neq \emptyset$. The *nonwandering set* of X is the set $\Omega(X)$ formed by the nonwandering points of X. Evidently $\Omega(X) \subset M(X)$ but not conversely. A subset $\Lambda \subset M(X)$ is called *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in I\!R$. Clearly $\Omega(X)$ is a compact invariant set of X.

Define the *omega-limit set* of $x \in M$ by

$$\omega(x) = \left\{ y \in M : y = \lim_{k \to \infty} X_{t_k}(x) \text{ for some sequence } t_k \to \infty \right\}.$$

An invariant set Λ is called *transitive* if $\Lambda = \omega(x)$ for some $x \in \Lambda$. We say that X is transitive if $M(X)$ is a transitive set of X.

Denote by $m(\cdot)$ and $Det(\cdot)$ the minimal norm and jacobian operations respectively. A compact invariant set Λ is *hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ and positive constants K, λ such that E^X_Λ is the subbundle generated by X,

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad m(DX_t(x)/E^u_x) \geq K^{-1}e^{\lambda t}, \quad \forall x \in \Lambda, \forall t \geq 0.$$

(Notice that a hyperbolic set may contain singularities, all of then isolated.) A periodic orbit or singularity is hyperbolic if it does as a compact invariant set.

We say that Λ is *partially hyperbolic* if there are a continuous invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$ and positive constants K, λ such that

$$\|DX_t(x)/E^s_x\| \leq Ke^{-\lambda t} \quad \text{and} \quad \frac{\|DX_t(x)/E^s_x\|}{m(DX_t(x)/E^c_x)} \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda, \forall t \geq 0.$$

If, additionally, $|Det(DX_t(x)/L_x)| \geq K^{-1}e^{\lambda t}$ for all $x \in \Lambda, t \geq 0$ and all two-dimensional subspace L_x of E^c_x we say that the central subbundle E^c_Λ is *sectionally expanding.*

Definition 1 A *sectional-hyperbolic set* is a partially hyperbolic set with hyperbolic singularities and sectionally expanding central subbundle [15].

The sectional-hyperbolic sets on 3-manifolds are precisely the singular-hyperbolic sets defined in [22]. Partially hyperbolic sets for which the central subbundle is volume (but not necessarily sectional) expanding were considered in [25] with the name *pseudohyperbolic set.* The following definition was introduced in [17].

Definition 2 A C^1 vector field is a *sectional-Anosov flow* if its maximal invariant set is a sectional-hyperbolic set.

Now we present our main result in which we fix the standard C^1 topology in the space of C^1 vector fields of M.

Theorem 1 *Let X be a sectional-Anosov flow on a compact three-dimensional manifold M. If $\Omega(X) = M(X)$, then every vector field close to X is sensitive to initial conditions.*

Notice that, unlike Anosov flows, the property $\Omega(X) = M(X)$ for sectional-Anosov flows is not stable by small perturbations in general ([23]).

A concept related to that of sensitiveness with respect to initial conditions is the following notion of expansiveness [13]: We say that a vector field X of M is *K*-expansive* if for every $\epsilon > 0$ there is $\delta > 0$ such that, for any surjective increasing continuous functions $h : I\!R \to I\!R$, if $x, y \in M(X)$ satisfy $d(X_t(x), X_{h(t)}(y)) \leq \delta$ for all $t \in I\!R$, then $X_{h(t_0)}(y) \in X_{[t_0\epsilon,t_0+\epsilon]}(x)$, for some $t_0 \in I\!R$.

Nevertheless K*-expansiveness do not imply sensitiveness with respect to initial conditions as the former property involves the maximal invariant set $M(X)$ only while, the latter one, involves the whole ambient manifold M. A concrete counterexample is a vector field on a 3-ball, inwardly transverse to the boundary, whose maximal invariant set is an attracting singularity.

However, Theorem A p. 2434 in [3] implies that every *transitive* sectional-Anosov flow X on a compact 3-manifold M is K*-expansive. Therefore, the restricted flow $X_t/M(X)$ is weakly sensitive in the sense that there is $\delta > 0$ such that for every $x \in M(X)$ and every neighborhood U of x in M there are $y \in M(X) \cap U$ and $t \in I\!R$ such that $d(X_t(x), X_t(y)) \geq \delta$ (see the definition of sensitiveness in p. 18 of [2] and the argument in p. 20 of [2]). Finally we observe that, unlike Theorem 1, the results in [3] do not consider any perturbation of X.

This paper is organized as follows. In Section 2 we prove some elementary properties. In particular, Proposition 1 asserts that a flow with hyperbolic saddle-type singularities on a compact manifold is sensitive to initial conditions as soon as every point can be approximated by points for which the omega-limit set is a singularity. Next we introduce what we shall call *Property (P)*, recall the definition of singular partition and give a simple criterion for the existence of these partitions. We start Section 3 by proving that every flow as in the statement of Theorem 1 has Property (P). Finally we prove using singular partitions that, for every sectional-Anosov flow with Property (P) on a compact 3-manifold, every point can be approximated by points for which the omega-limit set is a singularity. Then, Proposition 1 applies.

2 Preliminary facts

We start with an elementary sufficient condition for a flow to have sensitivity to initial conditions. Recall that a hyperbolic singularity or periodic orbit is *saddle-type* if none of its hyperbolic subbundles vanish. Denote by $W^s(\cdot)$ and $W^u(\cdot)$ the usual stable and unstable manifold operations respectively [12]. Denote also by $Sing(X)$ the set of singularities of X. All the vector fields considered in this paper will be of class C^1.

The following gives an useful sufficient condition for sensitiveness with respect to initial conditions.

Proposition 1 *Let X be a vector field on a compact manifold M all of whose singularities are hyperbolic of saddle-type. If every point in M is approximated by points for which the omega-limit set is a singularity, then X is sensitive to initial conditions.*

Proof We have that $Sing(X)$ is finite since M is compact. Then, there is $\delta > 0$ such that the set of δ-balls with center in $Sing(X)$ is pairwise disjoint. Let $x \in M$ and U be a neighborhood of x. We have two possibilities, namely, either $x \in W^s(\sigma)$ for some $\sigma \in Sing(X)$ or not.

In the first case we can select $y \in U$ outside the stable manifolds of the singularities since the union of such manifolds has empty interior (recall that every singularity is saddle-type). Since the positive orbit of x converges to σ, and that of y does not, we eventually find $t > 0$ such that $d(X_t(x), X_t(y)) \geq \delta$.

In the second case we can select $\sigma \in Sin(X)$ and $y \in W^s(\sigma) \cap U$ since the union of the stable manifolds of the singularities is dense by assumption. Again we argue that since the positive orbit of y converges to σ, and that of x does not, we eventually find $t > 0$ such that $d(X_t(x), X_t(y)) \geq \delta$. This proves the result. □

This proposition will be used together with the following definition. By a closed orbit we mean an orbit which is either singular or periodic.

Definition 3 A vector field X with hyperbolic closed orbits has *Property (P)* if for every periodic orbit O there is a singularity σ such that $W^u(O) \cap W^s(\sigma) \neq \emptyset$.

The following elementary lemma relates Property (P) with approximation by points whose omega-limit set is a singularity.

Lemma 1 *Every point in the closure of the periodic orbits of a vector field with the Property (P) is accumulated by points for which the omega-limit set is a singularity.*

Next we recall the definition of singular partition generalizing that of global cross section [10]. Consider a vector field X in a manifold M. A *cross section* of X is a codimension one submanifold Σ transverse to X. The interior and the boundary of Σ as a submanifold are denoted by $Int(\Sigma)$ and $\partial\Sigma$ respectively. If $\mathcal{R} = \{S_1, \cdots, S_k\}$ is a collection of cross sections we still denote by $\mathcal{R}$ the union of its elements. We denote

$$\partial\mathcal{R} = \bigcup_{i=1}^{k} \partial S_i \quad \text{and} \quad Int(\mathcal{R}) = \bigcup_{i=1}^{k} Int(S_i).$$

The *diameter* of $\mathcal{R}$ will be the sum of the diameters of its elements.

Definition 4 A *singular-partition* of an invariant set Λ of X is a finite disjoint collection $\mathcal{R}$ of cross sections satisfying $\Lambda \cap \partial\mathcal{R} = \emptyset$ and $\Lambda \cap Sing(X) = \{y \in \Lambda : X_t(y) \notin \mathcal{R}, \forall t \in I\!R\}$.

In the sequel we present a general existence result for singular partition.

Proposition 2 *Let Λ be a compact invariant set all of whose singularities are hyperbolic of X. Suppose that for every $\delta > 0$ and every $z \in \Lambda \setminus Sin(X)$ there is a cross section Σ_z of diameter at most δ such that $z \in Int(\Sigma_z)$ and $\Lambda \cap \partial\Sigma_z = \emptyset$. Then, Λ has singular partitions of arbitrarily small diameter.*

Proof Since every singularity in Λ is hyperbolic we can choose $\beta > 0$ such that

$$\Lambda \cap Sing(X) = \bigcap_{t \in I\!R} X_t \left(\bigcup_{\sigma \in \Lambda \cap Sing(X)} B_\beta(\sigma) \right). \tag{1}$$

For such a β we define

$$H = \varLambda \setminus \left(\bigcup_{\sigma \in \varLambda \cap Sing(X)} B_\beta(\sigma) \right).$$

We can assume that $H \neq \emptyset$ for, otherwise, (1) would imply $\varLambda = \varLambda \cap Sing(X)$ in whose case we are done. Clearly $H \subset \varLambda$ and $H \cap Sing(X) = \emptyset$ so $\varSigma_z$ as in the statement exists for all $z \in H$. For all such z we define

$$V_z = \bigcup_{t \in (-1,1)} X_t(Int(\varSigma_z)).$$

Obviously $z \in V_z$ and then $\{V_z : z \in H\}$ is an open covering of H which is clearly compact. So, there is a finite subset $\{z_1, \cdots, z_r\} \in H$ such that

$$H \subset \bigcup_{i=1} V_{z_i}.$$

By moving the cross sections $\varSigma_{z_1}, \cdots, \varSigma_{z_r}$ along the flow we can assume that the collection

$$\mathcal{R} = \{\varSigma_{z_1}, \cdots, \varSigma_{z_r}\}$$

is pairwise disjoint. Moreover, since $\varLambda \cap \partial \varSigma_z = \emptyset$ we have

$$\varLambda \cap \partial \mathcal{R} = \emptyset.$$

If $z \in \varLambda \setminus Sing(X)$, then (1) implies that there is $t \in \mathbb{R}$ such that

$$X_t(z) \notin \bigcup_{\sigma \in \varLambda \cap Sing(X)} B_\beta(\sigma).$$

But $X_t(z) \in \varLambda$ since z does therefore $X_t(x) \in H$ by definition. Hence $X_t(z) \in V_{z_i}$ for some i and then the orbit of z intersects $\varSigma_{z_i}$ by the definition of V_{z_i}. This proves

$$\varLambda \cap Sing(X) = \{z \in \varLambda : X_t(z) \notin \mathcal{R}\}$$

from which the result follows.

3 Proof of Theorem 1

Recall that an *attractor* of a vector field X is a transitive set A for which there is a compact neighborhood U satisfying

$$A = \bigcap_{t \geq 0} Y_t(U).$$

Now we state a theorem originally proved in [20] for sectional-Anosov flows with transitive maximal invariant set. We include its proof the Appendix for the sake of completeness. In what follows M will denote a compact manifold of dimension 3.

Theorem 2 *If X is a sectional-Anosov flow with singularities of M satisfying $\varOmega(X) = M(X)$, then every attractor of every vector field C^1 close to X has a singularity.*

With this theorem we obtain the following.

Corollary 1 *If X is a sectional-Anosov flow* with singularities *of M satisfying $\Omega(X) = M(X)$, then every vector field C^1 close to X has the Property (P).*

Proof By the sectional-Anosov connecting lemma [5] it suffices to show $Cl(W^u_Y(O)) \cap Sin(Y) \neq \emptyset$ for every flow Y close to X and every periodic orbit O of Y.

Suppose by contradiction that there is Y close to X with a periodic orbit O such that $Cl(W^u_Y(O)) \cap Sin(Y) = \emptyset$. It follows that $Cl(W^u_Y(O))$ is a hyperbolic set by [15]. Since $W^u_Y(O)$ is a two-dimensional submanifold we can easily prove that $Cl(W^u_Y(O))$ is an attracting set of Y. This attracting set necessarily contains a hyperbolic attractor. However, such attractors do not exist by Theorem 2. □

Now we introduce a notation to be used in the sequel. Let X be a sectional-Anosov flow of M. It follows from the Invariant Manifold Theory [12] that the stable subbundle $E^s_{M(X)}$ of X can be extended continuously to a stable subbundle E^s_U in a full neighborhood U of $M(X)$. Since the flow of X contracts M into $M(X)$ we can assume that such a neighborhood is M itself. On the other hand, it turns out that the direct sum $E^s_M \oplus E^X_M$ is tangent to a stable singular foliation W^s. We can use it to define a continuous one-dimensional foliation $\mathcal{F}^s = \{\mathcal{F}^s(x, D) : x \in D\}$, in each cross section D, by intersecting the leaves of W^s with D. We then say that D is a *foliated rectangle* if it is diffeomorphic to $[0, 1] \times [0, 1]$ and the leaves of $\mathcal{F}^s$ have the form $* \times [0, 1]$ up to identification. In such a case there are a vertical boundary $\partial^v D$ formed by leaves of $\mathcal{F}^s$ and an horizontal boundary $\partial^h D$ transverse to $\mathcal{F}^s$.

Theorem 3 *If X is a sectional-Anosov flow with the Property (P) on M and $x \in M$ is not approximated by points for which the omega-limit set is a singularity, then $\omega(x)$ has a singular partition of arbitrarily small diameter.*

Proof By Proposition 2 we have to prove that for all $z \in \omega(x) \setminus Sing(X)$ there is a cross section of small diameter Σ_z such that $z \in Int(\Sigma_z)$ and $\omega(x) \cap \partial\Sigma_z = \emptyset$.

We claim that $\omega(x) \cap W^{ss}(z)$ has empty interior in $W^{ss}(z)$ for all $z \in \omega(x)$. Indeed, suppose by contradiction that it is not so. Then, $\omega(x)$ contains a local strong stable manifold $W^{ss}_\epsilon(y)$ through some $y \in \omega(x)$. Now we have two cases, namely, either $\omega(x)$ has singularities or not. If not, then $\omega(x)$ is hyperbolic [21] so we would have that $x \in \omega(x)$ by using the unstable manifolds through $W^{ss}_\epsilon(y)$. It would follow from the improved sectional-Anosov closing lemma in [16] that x is approximated by periodic points or by points for which the omega-limit set is a singularity. Since the former points are accumulated by points for which the omega-limit set is a singularity by Lemma 1 we conclude that x itself is approximated by points for which the omega-limit set is a singularity. This is clearly a contradiction. If $\omega(x)$ has a singularity, then we get a contradiction by taking the backward orbit of $W^{ss}_\epsilon(y)$ as done in [19]. These contradictions prove the claim.

By the claim we can fix for all $z \in \omega(x)$ a foliated rectangle of small diameter R^0_z such that $z \in Int(R^0_z)$ and $\omega(x) \cap \partial^h R^0_z = \emptyset$. If the positive orbit of x intersects $\mathcal{F}^s(z, R^0_z)$ infinitely many times we would have that $\omega(x)$ is a periodic orbit in whose case the result is trivial. Therefore, we can assume that the positive orbit of x does not intersect $\mathcal{F}^s(z, R^0_z)$. Then, it intersects either only one or the two connected components of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$.

If the positive orbit intersects only one component we select some point x' of the positive orbit inside that component, a point z' in the other component and define Σ_z

as the subrectangle of R^0_z bounded by $\mathcal{F}^s(x', R^0_z)$ and $\mathcal{F}^s(z', R^0_z)$. Since the positive orbit does not passes through the connected component of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$ containing z' we have that $\omega(q) \cap \mathcal{F}^s(z', R^0_z) = \emptyset$. Now suppose for a while that there is $h \in \omega(x) \cap \mathcal{F}^s(x', R^0_z)$. Since $\omega(x) \subset \Omega(X)$ the improved sectional-Anosov closing lemma [16] and Lemma 1 as before would imply that h is accumulated by points for which the omega-limit set is a singularity. Since the stable manifolds have uniformly large size we have that x' is also accumulated by points for which the omega-limit set is a singularity. This would imply the same for x contradiction. Therefore $\omega(x) \cap \mathcal{F}^s(x', R^0_z) = \emptyset$. Since $\partial^h \Sigma_z \subset \partial^h R^0_z$ and $\partial^v \Sigma_z = \mathcal{F}^s(z', R^0_z) \cup \mathcal{F}^s(x', R^0_z)$ we have that Σ_z has the required properties.

In the case when the positive orbit intersects both components of $R^0_z \setminus \mathcal{F}^s(z, R^0_z)$ we choose two points x', x'' of that orbit, in each connected component, and define Σ_z as the rectangle of R^0_z bounded by $\mathcal{F}^s(x', R^0_Z)$ and $\mathcal{F}^s(x'', R^0_Z)$. By repeating the above argument we see that $\omega(x) \cap (\mathcal{F}^s(x', R^0_Z) \cup \mathcal{F}^s(x'', R^0_Z)) = \emptyset$ so Σ_z satisfies the required properties. This completes the proof. ☐

The following theorem improving [9] says that the singularities of a sectional-Anosov flow with Property (P) play the same role of the critical points in the Henón-like maps (see the comment after Theorem B in [7] p. 377).

Theorem 4 *Let X be a sectional-Anosov flow with the Property (P) of M. Then, every point of M is approximated by points for which the omega-limit set is a singularity.*

Proof Suppose by contradiction that there are a sectional-Anosov flow X with the Property (P) on a compact 3-manifold M and $x \in M$ which is not approximated by points for which the omega-limit set is a singularity. In particular, $\omega(x)$ *is not a singularity.* Since X has the Property (P) we can apply Theorem 3 in order to find a singular partition $\mathcal{S} = \{S_1, \cdots, S_r\}$ of $\omega(x)$ close to it.

On the one hand, since x is not approximated by points for which the omega-limit set is a singularity, we can fix an open interval I around (and close to) x tangent to E^c and orthogonal to E^X such that:

I does not intersect the stable manifold of any singularity.

On the other hand, since $\omega(x)$ is not a singularity, we can find $S \in \mathcal{R}$, a sequence $x_n \in S$ of points in the positive orbit of x and a sequence of intervals $J_n \subset S$ in the positive orbit of I with $x_n \in J_n$ such that if J^+_n and J^-_n are the connected components of $J_n \setminus \{x_n\}$ then both sequences $\{\text{Length}(J^+_n) : n = 1, 2, 3, \cdots\}$ and $\{\text{Length}(J^-_n) : n = 1, 2, 3, \cdots\}$ are bounded away from 0 (where $Length(\cdot)$ here denotes the length operation).

Take a limit point $w \in S$ of x_n. Then $w \in \omega(x) \cap Int(S)$ since $\mathcal{S}$ is a singular partition. Because I is tangent to E^c the interval sequence J_n converges to an interval J tangent to $E^c_w \cap T_w S$ in the C^1 topology. We have that J is not trivial since $\{\text{Length}(J^+_j) : j = 1, 2, 3, \cdots\}$ and $\{\text{Length}(J^-_j) : j = 1, 2, 3, \cdots\}$ are bounded away from 0. It follows from these lower bounds and $x_j \to w$ that J_n intersects $W^s(w)$ for some n large. Now, since w is nonwandering we have from the improved sectional-Anosov closing lemma [16] and Lemma 1 that w is approximated by points for which the omega-limit set is a singularity. It then follows from the continuous dependence in compact parts of the stable manifolds that there is an intersection point between J_n and the stable manifold of a singularity (see Figure 1). Since the stable manifolds of the singularities are flow-invariant we get:

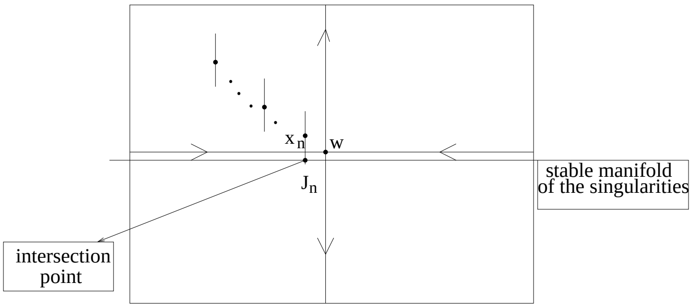

Fig. 1 Proof of Theorem 4

I intersects the stable manifold of a singularity.

This contradicts the choice of I and the proof follows. □

Proof of Theorem 1: Let X be a sectional-Anosov flow of a compact three-dimensional manifold M such that $\Omega(X) = M(X)$. If X has no singularities, then $M(X)$ is a hyperbolic attractor so the sensitivity to initial conditions for nearby vector fields is well known. Hence, we can suppose that X has a singularity. In particular, every vector field C^1 close to X has a singularity too. By Corollary 1 we have that every vector field C^1 close to X also has Property (P). Since all these vector fields are also sectional-Anosov we have from Theorem 4 that all of them satisfy that every point is approximated by points for which the omega-limit set is a singularity. Since the singularities of a sectional-Anosov flow are all hyperbolic of saddle-type we can apply Proposition 1 in order to conclude that all these vector fields are sensitive to initial conditions. This ends the proof. □

Appendix

In this Appendix we shall prove Theorem 2. The proof is similar to that in [20] but using the lemma below instead of the transitive hypothesis used there.

Lemma 2 (Corollary 1 in [16]) *Let X be a sectional-Anosov flow with singularities of M satisfying $\Omega(X) = M(X)$. Then, every nonwandering point of X is approximated by points for which the omega-limit set is a singularity.*

Hereafter we use the notations $W^s(\cdot)$, $W^{ss}(\cdot)$, $W^u(\cdot)$ and $W^{uu}(\cdot)$ for the standard invariant manifold operations [12]. We write $W^s_X(\cdot)$, $W^{ss}_X(\cdot)$, etc to indicate dependence on flows X. The notations $Int(\cdot)$, $Cl(\cdot)$ and $B_\delta(\cdot)$ stand for the interior, closure and δ-ball operations respectively. A singularity σ is called *Lorenz-like* if it has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$ up to some order.

The proof of Theorem 2 is by contradiction, namely, we assume that there is a sectional-Anosov flow X of M satisfying $\Omega(X) = M(X)$ which is the limit of a sequence of flows X^n each one exhibiting an attractor without singularities A^n. Clearly we can assume that each X^n is sectional-Anosov, and so, A^n is a hyperbolic attractor of X^n for all n (c.f. [22]).

We claim that

$$Sing(X) \cap Cl\left(\bigcup_{n\in I\!N} A^n\right) \neq \emptyset.$$

Otherwise there is $\delta > 0$ such that

$$B_\delta(Sing(X)) \cap \left(\bigcup_{n\in I\!N} A^n\right) = \emptyset. \tag{2}$$

Define

$$H = \bigcap_{t\in I\!R} X_t\left(M \setminus B_{\frac{\delta}{2}}(Sing(X))\right).$$

Obviously $Sing(X) \cap H = \emptyset$ and then H is a hyperbolic set. Denote by $E^s \oplus E^X \oplus E^u$ the corresponding hyperbolic splitting.

By the stability of hyperbolic sets we can fix compact neighborhoods V, W

$$H \subset Int(V) \subset V \subset Int(W) \subset W$$

of H and $\epsilon > 0$ such that if Y is a vector field that is C^1 close to X and H_Y is a compact invariant set of Y in W then:

(H1) H_Y is hyperbolic and its hyperbolic splitting $E^{s,Y} \oplus E^Y \oplus E^{u,Y}$ satisfies

$$dim(E^u) = dim(E^{u,Y}), \quad dim(E^s) = dim(E^{s,Y}).$$

(H2) The local strong unstable manifolds $W^{uu}_Y(y,\epsilon)$, $y \in H_Y$, are one-dimensional of uniform size ϵ.

We assert that $A^n \subset W$ for all n large. Indeed, suppose by contradiction that this is not true. Then, there are sequences $n_k \to \infty$ and $x^{n_k} \in A^{n_k}$ such that $x^{n_k} \notin W$ for all k. Since M is compact we can assume that $x^{n_k} \to x$ for some $x \in M$. Clearly $x \in M \setminus Int(W)$ and so $x \notin V$. Then, since $H = \bigcap_{t\in I\!R} X_t\left(M \setminus B_{\frac{\delta}{2}}(Sing(X))\right)$ and $H \subset V$, we can arrange $t \in I\!R$ such that $X_t(x) \in B_{\frac{\delta}{2}}(Sing(X))$. On the other hand, $X^n \to X$ and $x^{n_k} \to x$ so $X^{n_k}_t(x^{n_k}) \to X_t(x)$ hence $X^{n_k}_t(x^{n_k}) \in B_{\frac{\delta}{2}}(Sing(X))$ for k large. However, A^n is X^n-invariant so $X^{n_k}_t(x^{n_k}) \in A^{n_k}$ yielding $X^{n_k}_t(x^{n_k}) \in A^{n_k} \cap B_{\frac{\delta}{2}}(Sing(X))$ and so $A^{n_k} \cap B_{\frac{\delta}{2}}(Sing(X)) \neq \emptyset$ contradicting (2). Therefore, the assertion is true.

As $X^n \to X$ the assertion and (H2) with $Y = X^n$ and $H_Y = A^n$ imply that $W^{uu}_{X^n}(y,\epsilon)$ has uniform size ϵ for all $y \in A^n$ and n large.

Take $x^n \in A^n$ converging to some $x \in M$. Clearly $x \in H$. Note that the tangent vectors of the curve $W^{uu}_{X^n}(x^n,\epsilon)$ at every $c \in W^{uu}_{X^n}(x^n,\epsilon)$ belongs to E^{u,X^n}_c.

As $X^n \to X$ the angle between the directions E^{u,X^n} and E^u goes to zero as $n \to \infty$. Henceforth the manifolds $W^{uu}_X(x,\epsilon)$ and $W^{uu}_{X^n}(x^n,\epsilon)$ are almost parallel as $n \to \infty$. As $x^n \to x$ we conclude that

$$W^{uu}_{X^n}(x^n,\epsilon) \to W^{uu}_X(x,\epsilon)$$

in the sense of C^1 submanifolds.

Fix an open interval $I \subset W^{uu}_X(x,\epsilon)$ containing x. Clearly $I \subset M(X)$.

Since $\Omega(X) = M(X)$ and $I \subset M(X)$ we have $I \subset \Omega(X)$. Then, by Lemma 2, every point of I can be approximated by points for which the omega-limit set is a singularity. Since I is an interval and the stable manifolds have uniform large size we have that I intersects the stable manifold of a singularity at some point q. From this we get $T > 0$ such that

$$X_T(q) \in B_{\frac{\delta}{5}}(Sing(X)).$$

Then, there is an open set V_q containing q such that

$$X_T(V_q) \subset B_{\frac{\delta}{5}}(Sing(X)).$$

As $X^n \to X$ we have

$$X^n_T(V_q) \subset B_{\frac{\delta}{4}}(Sing(X)) \tag{3}$$

for all n large. But $W^{uu}_{X^n}(x^n, \epsilon) \to W^{uu}_X(x, \epsilon)$, $q \in I \subset W^{uu}_X(p, \epsilon)$, $q \in V_q$ and V_q is open. So,

$$W^{uu}_{X^n}(x^n, \epsilon) \cap V_q \neq \emptyset$$

for all n large. Applying (3) to X^n for n large we have

$$X^n_T(W^{uu}_{X^n}(x^n, \epsilon)) \cap B_{\frac{\delta}{4}}(Sing(X)) \neq \emptyset.$$

As $W^{uu}_{X^n}(x^n, \epsilon) \subset W^u_{X^n}(x^n)$ the invariance of $W^u_{X^n}(x^n)$ implies

$$W^u_{X^n}(x^n) \cap B_{\frac{\delta}{2}}(Sing(X)) \neq \emptyset.$$

But $W^u_X(x^n) \subset A^n$ since $x^n \in A^n$ and A^n is an attractor, so

$$A^n \cap B_\delta(Sing(X)) \neq \emptyset$$

contradicting (2). The claim follows.

Let us continue with the proof of the theorem. By the previous claim we can choose

$$\sigma \in Sing(X) \cap Cl\left(\bigcup_{n \in I\!N} A^n\right).$$

Applying [6,22] we have that σ is Lorenz-like and satisfies

$$M(X) \cap W^{ss}_X(\sigma) = \{\sigma\}.$$

Let $S^t = S^t_\sigma$ and $S^b = S^b_\sigma$ be the singular cross-sections associated to σ (c.f. Section 4 p. 278 in [6]). In particular,

$$M(X) \cap \left(\partial^h S^t \cup \partial^h S^b\right) = \emptyset.$$

As $X^n \to X$ we have that S^t, S^b are singular-cross sections of X^n too. By implicit function reasons we can assume that $\sigma(X^n) = \sigma$, where $\sigma(X^n)$ is the continuation of $\sigma = \sigma(X)$. Moreover,

$$l^t \cup l^b \subset W^s_{X^n}(\sigma), \quad \forall n. \tag{4}$$

The one-dimensional subbundle E^s of X extends to a contracting invariant subbundle in M. Take a continuous (but not necessarily invariant) extension of E^c. We still denoted by $E^s \oplus E^c$ the above-mentioned extension.

By the Invariant Manifold Theory [12] it follows that the splitting $E^s \oplus E^c$ persists by small perturbations of X. More precisely, for all n large the flow X^n has an splitting $E^{s,n} \oplus E^{c,n}$ over U such that $E^{s,n}$ is invariant contracting, $E^{s,n} \to E^s$ and $E^{c,n} \to E^c$ as $n \to \infty$. In particular, $E^{s,n} \oplus E^{c,n}$ is defined in $S^t \cup S^b$ for all n large. In what follows we denote by E^Y the subbundle in TM generated by a flow Y in M.

The dominance condition implies that for $* = t, b$ one has

$$T_x S^* \cap \left(E^s_x \oplus E^X_x \right) = T_x l^*,$$

for all $x \in l^*$.

Denote by $\angle(E, F)$ the angle between two linear subspaces. The last equality implies that there is $\rho > 0$ such that

$$\angle(T_x S^* \cap E^c_x, T_x l^*) > \rho,$$

for all $x \in l^*$ $(* = t, b)$. But $E^{c,n} \to E^c$ as $n \to \infty$. So for all n large we have

$$\angle(T_x S^* \cap E^{c,n}_x, T_x l^*) > \frac{\rho}{2}, \tag{5}$$

for all $x \in l^*$ (again $* = t, b$).

Fix $* = t, b$ and a coordinate system $(x, y) = (x^*, y^*)$ in S^* such that

$$S^* = [-1, 1] \times [-1, 1], \qquad l^* = \{0\} \times [-1, 1]$$

with respect to (x, y).

Denote by $\Pi^* : S^* \to [-1, 1]$ the projection

$$\Pi^*(x, y) = x$$

and for $\Delta > 0$ we define

$$S^{*,\Delta} = [-\Delta, \Delta] \times [-1, 1].$$

Define the line field F^n in $S^{*,\Delta}$ by

$$F^n_x = T_x S^* \cap E^{c,n}_x, \quad x \in S^{*,\Delta}.$$

The continuity of $E^{c,n}$ and (5) imply that $\exists \Delta_0 > 0$ such that $\forall n$ large the line F^n is transverse to Π^*.

Now recall that A^n is a hyperbolic attractor of X^n for all n. It follows that the periodic orbits of X^n in A^n are dense in A^n. Then, as $\sigma \in Cl\left(\cup_{n \in I\!N} A^n\right)$, there is a periodic orbit sequence $O_n \in A^n$ accumulating on σ. It follows that there is $n_0 \in I\!N$ such that either

$$O_{n_0} \cap Int(S^{t,\Delta_0}) \neq \emptyset \ \text{ or } \ O_{n_0} \cap Int(S^{b,\Delta_0}) \neq \emptyset.$$

Because $O_{n_0} \subset A_{n_0}$ we conclude that either

$$A^{n_0} \cap Int(S^{t,\Delta_0}) \neq \emptyset \ \text{ or } \ A^{n_0} \cap Int(S^{b,\Delta_0}) \neq \emptyset.$$

We denote $Z = X^{n_0}$, $A = A^{n_0}$, $F = F^{n_0}$ for simplicity.

We can assume that $A \cap Int(S^{t,\Delta_0}) \neq \emptyset$. Note that $\partial^h S^{t,\Delta_0} \subset \partial^h S^t$ by definition. Then,

$$A \cap \partial^h S^{t,\Delta_0} = \emptyset.$$

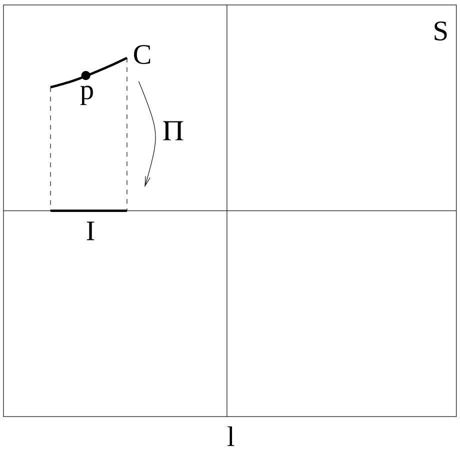

Fig. 2 Proof of Theorem 2

We denote $S = S^{t,\Delta_0}$, $(x,y) = (x^t, y^t)$, $l = l^t$ and $\Pi = \Pi^t$ for simplicity. Note that $A \cap S$ is a compact non-empty subset of S. Hence there is $p \in S \cap A$ such that

$$dist(\Pi(S^t \cap A), 0) = dist(\Pi(p), 0),$$

where $dist$ denotes the distance in $[-\Delta_0, \Delta_0]$.

Now, $p \in A$ and so $W^u_Z(p)$ is a well defined two-dimensional submanifold. The dominance condition of the sectional-hyperbolicity implies that

$$T_z(W^u_Z(p)) = E^c_z, \quad \forall z \in W^u_Z(p).$$

Hence

$$T_z(W^u_Z(p)) \cap T_z S = E^c_z \cap T_z S = F_z$$

for every $z \in W^u_Z(p) \cap S$.

As $W^u_Z(p) \cap S$ is transversal, we have that $W^u_Z(p) \cap S$ contains a curve C whose interior contains p as in Figure 2. The last equality implies that C is tangent to F.

As F is transverse to Π we have that C is transverse to Π. We conclude that $\Pi(C)$ contains an open interval $I \subset [-\Delta_0, \Delta_0]$ with $\Pi(p) \in Int(I)$. So, there is $z_0 \in C$ such that

$$dist(\Pi(z_0), 0) < dist(\Pi(p), 0).$$

Note that $C \subset S \cap A$ since A is an attractor of Z. Moreover, $p \in A$ and $C \subset W^u_Z(p)$. As $A \cap \partial^h S = \emptyset$ we conclude that

$$dist(\Pi(S \cap A), 0) = 0.$$

As A is closed, this last equality implies

$$A \cap l^t \neq \emptyset.$$

Since $l^t \subset W^s_Z(\sigma)$ and A is closed invariant for Z we conclude that $\sigma \in A$. However, this is impossible since A is a hyperbolic attractor. This implies the result. □

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