When stationary limits of spatially ergodic processes are spatially ergodic?

Spatial Birth and death processes as solutions of stochastic equations

Nancy L. Garcia
IMECC/UNICAMP
http://www.ime.unicamp.br/~nancy

Joint work with Tom Kurtz - Univ. Wisconsin - Madison – ALEA
• **General case** Open problem

• **Particle systems**
  - Spin flip systems - Andjel (1990)
  - Exclusion processes, biased voter model, multi-opinion voter model - Chen (2002)

• **Stationary measures of spatial birth and death processes**
  - Kurtz and Garcia (2006)

- Modeling spatial point processes as stationary distributions of birth and death processes
- Equating to zero the generator of the process applied to a suitable collection of statistics. (Baddeley, 2000)
- Large sample limit to prove consistency. Necessary condition is that the stationary distribution is spatially stationary and ergodic.
Birth and death rates:

- $\int_A \lambda(x, \eta_t) \beta(dx) \Delta t \approx$ probability that a point in a set $A \subset S$ is added to the configuration $\eta_t$ in $(t, t + \Delta t)$

- $\delta(x, \eta) \Delta t \approx$ the probability that a point $x \in \eta$ is deleted from the configuration $\eta_t$ in $(t, t + \Delta t)$

- $AF(\eta) = \int (F(\eta + \delta_x) - F(\eta)) \lambda(x, \eta) \beta(dx)$
  $\quad + \int (F(\eta - \delta_x) - F(\eta)) \delta(x, \eta) \eta(dx)$
ξ Poisson random measures with mean measure $\beta$

- for each $A \in \mathcal{B}(S)$, $\xi(A)$ has a Poisson distribution with expectation $\beta(A)$
- $\xi(A)$ and $\xi(B)$ are independent if $A \cap B = \emptyset$.
- Taking $\lambda = \delta \equiv 1$, unique stationary distribution for the birth and death process with generator

$$AF(\eta) = \int (F(\eta + \delta x) - F(\eta))\beta(dx) + \int (F(\eta - \delta x) - F(\eta))\eta(dx).$$

Letting $\mu_\beta^0$ denote this distribution, the stationarity can be checked by verifying that

$$\int_{\mathcal{N}(S)} AF(\eta)\mu_\beta^0(d\eta) = 0.$$
Gibbs distributions

- Radon-Nikodym derivative with respect to a Poisson point process with mean measure $\beta$,

\[ \mu_{\beta,H}(d\eta) = \frac{1}{Z_{\beta,H}} e^{-H(\eta)} \mu_{\beta}^{0}(d\eta), \]

- $\mu_{\beta,H}$ is the stationary distribution of several spatial birth and death processes. In fact, $\lambda(x, \eta) > 0$ if $H(\eta + \delta_x) < \infty$ and that $\lambda$ and $\delta$ satisfy

\[ \lambda(x, \eta)e^{-H(\eta)} = \delta(x, \eta + \delta_x)e^{-H(\eta+\delta_x)}. \]

- Again, this assertion can be verified by showing that

\[ \int AF(\eta)\mu_{\beta,H}(d\eta) = \frac{1}{Z_{\beta,H}} \int AF(\eta)e^{-H(\eta)} \mu_{\beta}^{0}(d\eta) = 0. \]
Notice that

\[ \lambda(x, \eta)e^{-H(\eta)} = \delta(x, \eta + \delta_x)e^{-H(\eta + \delta_x)} \]

is equivalent to

\[ \frac{\lambda(x, \eta)}{\delta(x, \eta + \delta_x)} = \exp\{-H(\eta + \delta_x) + H(\eta)\} \]

We can always take \( \delta(x, \eta) = 1 \)

**Example:** Pairwise interaction potential \( \rho(x_1, x_2) \geq 0 \), that is, for \( \eta = \sum_{i=1}^{m} \delta_{x_i} \),

\[ H_\rho(\eta) = \sum_{i<j} \rho(x_i, x_j) \]

Take \( \beta = \text{Lebesgue}, \delta(x, \eta) \equiv 1 \) and \( \lambda(x, \eta) = \exp\{-\int \rho(x, y)\eta(dy)\} \)
Area-interaction point process [Baddeley and Van Lieshout (1995)]

\[ H(\eta) = \eta(S) \log \rho - m_d(\eta \oplus G) \]

- Radon-Nikodym derivative

\[ L(\eta) = \frac{1}{Z} \rho^{\eta(S)} \gamma^{-m_d(\eta \oplus G)} \]

- \( \rho > 0 \) and \( \gamma > 0 \) \( G \) is a compact grain.

- \( \gamma > 1 \), the process is attractive
  - \( \gamma > 1 \), the process is repulsive
    - \( \gamma = 1 \) the Poisson random measure with mean measure \( \rho m_d \).
    - \( \gamma \to 0 \) corresponds to area-exclusion

- Take unit death rate and the birth rate given by

\[ \lambda(x, \eta) = \rho \gamma^{-m_d((x+G) \setminus (\eta \oplus G))} \]
1st approach: Graphical construction

Assume:

1. Finite range: there exists a compact set $G$ such that if $\eta_1 = \eta_2$ inside $x + G$ then $\lambda(x, \eta_1) = \lambda(x, \eta_2)$

2. Boundedness

\[ \overline{\lambda} = \sup_{x, \eta} \lambda(x, \eta) < \infty. \]

- Begin with a $\overline{\lambda}$-homogeneous Poisson point process on $\mathbb{R}^d \times \mathbb{R}$.

\[ N = \{ (\xi_1, T_1), (\xi_2, T_2), \ldots \} \]

- For each point $(\xi_i, T_i)$, associate two independent marks $S_i \sim \text{exp}(1)$ and $Z_i \sim \text{U}(0, 1)$.

- Marked cylinder $((\xi_i + G) \times [T_i, T_i + S_i], Z_i)$
Invariant measure:

Construct the process beginning at $-\infty$ in a given configuration and cut the process at time 0. If, the process is independent of the initial configuration, the process at time 0 has invariant measure.

1. Generate the free process $\alpha$ as a $\tilde{\lambda}$-homogeneous Poisson process on $\lambda$ according to Algorithm of the Poisson process.
Figure 1: Independent process
2. Construct the clan of ancestors of all points of \( \alpha \).

3. Apply the *deterministic* finite-volume “cleaning procedure” to decide which points of \( \alpha \) are going to be kept.
Figure 2: Cleaned process
2nd approach: Solutions of a system of stochastic equations

- Usually the individuals in the birth and death process are represented by points in $\mathbb{R}^d$, $\mathbb{Z}^d$ but more general spaces $S$ are OK.

- Let $\beta$ be a $\sigma$-finite, Borel measure on $S$.

**Condition 1** For each compact $\mathcal{K} \subset S$, the birth rate $\lambda$ satisfies

$$
\sup_{\zeta \in \mathcal{K}} \int_{S} c_k(x) \lambda(x, \zeta) \beta(dx) < \infty, \quad t > 0, \quad k = 1, 2, \ldots,
$$

and

$$
\delta(x, \zeta) < \infty, \quad \zeta \in S, \quad x \in \zeta.
$$
We also assume that $\lambda$ and $\delta$ satisfy the following continuity condition.

**Condition 2** If

$$\lim_{n \to \infty} \int_S c_k(x)|\zeta_n - \zeta|(dx) = 0,$$

(1)

for each $k = 1, 2, \ldots$, then

$$\lambda(x, \zeta) = \lim_{n \to \infty} \lambda(x, \zeta_n), \quad \delta(x, \zeta) = \lim_{n \to \infty} \delta(x, \zeta_n).$$

**Remark:** Since (1) implies $\zeta_n$ converges to $\zeta$ in $S$, this continuity condition is weaker than continuity in $S$.

**Equivalent:** Suppose $\zeta_0, \zeta_1, \zeta_2, \ldots \in S$ and $\zeta_n \leq \zeta_0$, $n = 1, 2, \ldots$. If $\zeta_n \to \zeta$ in $S$, then (1) holds.
• \( N \) be a PPP on \( S \times [0, \infty)^3 \) with mean measure \\
\[ \beta(dx) \times ds \times e^{-r}dr \times du. \]

• \( \eta_0 = \sum_{i=1}^{\infty} \delta_{x_i} \) PPP on \( S \) independent of \( N \),

• \( \hat{\eta}_0 = \sum_{i=1}^{\infty} \delta_{(x_i, \tau_i)} \)

• \( \{\tau_i\} \) are independent unit exponentials, independent of \( \eta_0 \) and \( N \).

(SE1)

\[
\eta_t(A) = \\
\int_{A \times [0,t] \times [0,\infty)^2} 1_{[0,\lambda(x,\eta_{s-})]}(u) 1_{(\int_s^t \delta(x,\eta_v) dv, \infty)}(r) N(dx, ds, dr, du) \\
+ \int_{A \times [0,\infty)} 1_{(\int_0^t \delta(x,\eta_s) ds, \infty)}(r) \hat{\eta}_0(dx, dr).
\]
Lemma 1 If $\eta$ is a solution of (SE1), then for each $T > 0$,

$$\int_0^T \int_S c_k(x)\lambda(x, \eta_s)\beta(dx)ds < \infty \quad a.s.,$$

$\eta^*_T$ defined by

$$\eta^*_T(B) = \int_{B \times [0,T] \times [0,\infty)^2} 1_{[0,\lambda(x,\eta_s-)]}(u)N(dx, ds, dr, du)$$

is an element of $S$,

$$\eta_t \leq \eta^*_T + \eta_0, \quad 0 \leq t \leq T,$$

and

$$\lim_{s \to t+} \int_S c_k(x)|\eta_s - \eta_t|(dx) = 0, \quad t \geq 0.$$
• If $\eta$ is a solution of (SE1)
• $x$ was born at time $s \leq t$,
• then the “residual clock time” $r - \int_s^t \delta(x, \eta_v)dv$ is an $\mathcal{F}_t$-measurable random variable.
• In particular, the counting-measure-valued process given by (SE2)

$$\hat{\eta}_t(B \times D) = \int_{B \times [0,t] \times [0,\infty)^2} 1_{[0,\lambda(x,\eta_{s-})]}(u) 1_D(r - \int_s^t \delta(x, \eta_v) dv) N(dx, ds, dr, du)$$

$$+ \int_{B \times [0,\infty)} 1_D(r - \int_0^t \delta(x, \eta_{s-}) ds) \hat{\eta}_0(dx, dr)$$

is $\{\mathcal{F}_t\}$-adapted.
\( \mathcal{S} \): counting measures \( \zeta \) on \( S \times [0, \infty) \) such that \( \zeta(\cdot \times [0, \infty)) \in \mathcal{S} \).

Alternative equation for the \( \mathcal{S} \)-valued process \( \hat{\eta} \) by requiring that

\[
\int_{S \times [0, \infty)} f(x, r) \hat{\eta}_t(dx, dr) = \int_{S \times [0, \infty)} f(x, r) \hat{\eta}_0(dx, dr) + \int_{S \times [0, t] \times [0, \infty)} f(x, r) \mathbf{1}_{[0, \lambda(x, \eta_s -)]}(u) N(dx, ds, dr, du) - \int_0^t \int_{S \times [0, \infty)} \delta(x, \eta_s) f_r(x, r) \hat{\eta}_s(dx, dr) ds, \quad (SE3)
\]

for all \( f \in \overline{C}(S \times [0, \infty)) \) such that
$f_r := \frac{\partial}{\partial r} f \in \overline{C}(S \times [0, \infty)), \quad f(x, 0) = 0,$

$\sup_r |f(\cdot, r)|, \sup_r |f_r(\cdot, r)| \in \mathcal{C}$, and there exists $r_f > 0$ such that $f_r(x, r) = 0$ for $r > r_f$. Note that if $f \in \hat{\mathcal{C}}$ and

$$f^*(x, r) = \int_0^r |f_r(x, u)| du,$$

then $f^* \in \hat{\mathcal{C}}$. In (SE3), $\hat{\eta}_0$ can be any $\hat{S}$-valued random variable that is independent of $N$. 

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Martingale problems

- \( \mathcal{D}(\hat{A}) = \{ F; F(\hat{\zeta}) = e^{-\int_{S \times [0, \infty)} f(x,r)\hat{\zeta}(dx,dr)}, \ f \geq 0 \in \hat{\mathcal{C}}. \) 
  \[ \int_0^t \int_S c_k(x) \lambda(x, \eta_s) \beta(dx)ds < \infty, \ k = 1, 2, \ldots. \]

- By Itô’s formula (RHS = local martingale)
  \[ F(\hat{\eta_t}) = F(\hat{\eta}_0) \]
  \[ + \int_{S \times [0,t] \times [0,\infty)^2} F(\hat{\eta}_{s-})(e^{-f(x,r)} - 1)1_{[0,\lambda(x,\eta_{s-})]}(u)\tilde{N}(dx, ds, dr, du) \]
  \[ + \int_0^t F(\hat{\eta}_s) \left( \int_{S \times [0,\infty)} \lambda(x, \eta_s) (e^{-f(x,r)} - 1)e^{-r} \beta(dx)dr \right. \]
  \[ \left. + \int_{S \times [0,\infty)} \delta(x, \eta_s) f_r(x, r)\hat{\eta}_s(dx, dr) \right)ds, \]
  \[ \tilde{N}(dx, ds, dr, du) = N(dx, ds, dr, du) - \beta(dx) \times ds \times e^{-r} dr \times du. \]
\[
\int_0^t \int_{S \times [0, \infty)} \delta(x, \eta_s) |f_r(x, r)| \tilde{\eta}_s (dx, dr) ds < \infty, \quad t > 0. \quad (2)
\]

Consequently, defining

\[
\hat{A}F(\hat{\zeta}) = F(\hat{\zeta}) \left( \int_{S \times [0, \infty)} \lambda(x, \zeta) (e^{-f(x, r)} - 1) e^{-r} \beta(dx) dr 
\right.
\]

\[
+ \left. \int_{S \times [0, \infty)} \delta(x, \zeta) f_r(x, r) \hat{\zeta}(dx, dr) \right), \quad (3)
\]

any solution of \((\text{SE3})\) must be a solution of the local martingale problem for \(\hat{A}\).
We say that \( \hat{\eta} \) is a solution of the local martingale problem for \( \hat{A} \) if there exists a filtration \( \{ \mathcal{F}_t \} \) such that \( \hat{\eta} \) is \( \{ \mathcal{F}_t \} \)-adapted and

\[
M_F(t) = F(\hat{\eta}_t) - F(\hat{\eta}_0) - \int_0^t \hat{A}F(\hat{\eta}_s)ds
\]

is a \( \{ \mathcal{F}_t \} \)-local martingale for each \( F \in \mathcal{D}(\hat{A}) \), that is, for each \( F \) of the form \( F(\hat{\zeta}) = e^{-\int f d\hat{\zeta}} \), \( f \in \hat{C} \), \( f \geq 0 \). In particular, let \( \overline{f}(x) = \sup_r f(x, r) \) and

\[
\tau_{f,c} = \inf\{t : \int_0^t \int_S \overline{f}(x) \lambda(x, \eta_s) \beta(dx)ds > c\}.
\]

Then \( M_F(\cdot \wedge \tau_{f,c}) \) is a martingale. Note that \( \tau_{f,c} \) is a \( \mathcal{F}_t^n \)-stopping time.
Main theorem: Suppose that $\lambda$ and $\delta$ satisfy Conditions 1 and 2. Then each solution of the stochastic equation (SE2) (or equivalently, (SE3)) is a solution of the local martingale problem for $\hat{A}$ defined by (3), and each solution of the local martingale problem for $\hat{A}$ is a weak solution of the stochastic equation.
Existence of the solution:

- \((\lambda, \delta)\) is **attractive** if \(\zeta_1 \subset \zeta_2\) implies \(\lambda(x, \zeta_1) \leq \lambda(x, \zeta_2)\) and \(\delta(x, \zeta_1) \geq \delta(x, \zeta_2)\).

- If \((\lambda, \delta)\) is attractive and we set \(\eta^0 \equiv 0\), then \(\eta^n\) defined by

\[
\eta_{t+1}(B) = \int B \times [0, t] \times [0, \infty)^2 1_{[0, \lambda(x, \eta^n_s)]}(u) 1_{[\int_s^t \delta(x, \eta^n_v) dv, \infty]}(r) N(dx, ds, dr, du)
\]

\[
+ \int B \times [0, \infty) 1_{[\int_0^t \delta(x, \eta^n_s) ds, \infty]}(r) \hat{\eta}_0(dx, dr)
\]

is monotone increasing and either \(\eta^n\) converges to a process with values in \(S\), or

\[
\int_0^T \int_S c_k(x) \lambda(x, \eta^n_s) \beta(dx) ds \to \infty,
\]

for some \(T\) and \(k\).
• For an arbitrary \((\lambda, \delta)\) define an attractive pair by setting

\[
\bar{\lambda}(x, \zeta) = \sup_{\zeta' \subset \zeta} \lambda(x, \zeta') \quad \bar{\delta}(x, \zeta) = \inf_{\zeta' \subset \zeta} \delta(x, \zeta').
\]

Let \(\eta_0\) be an \(S\)-valued random variable independent of \(N\), and let \(\hat{\eta}_0\) be defined as before. We assume that \(\bar{\lambda}\)

\[
\int c_k(x) \bar{\lambda}(x, \zeta) \beta(dx) < \infty, \quad \zeta \in S, k = 1, 2, \ldots, \tag{7}
\]

and that there exists a solution \(\bar{\eta}\) for the pair \((\bar{\lambda}, \bar{\delta})\).
\[ \eta_t^n(B) = \int_{B \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_s^{-} \cap K_n \cap \eta_s^n)]}(u) \mathbf{1}(\int_s^t \delta(x,\bar{\eta}_v \cap K_n \cap \eta_v^n) \, dv, \infty)(r) N(dx, ds, dr, du) \]

\[ + \int_{B \times [0,\infty)} \mathbf{1}(\int_0^t \delta(x,\bar{\eta}_v \cap K_n \cap \eta_v^n) \, dv, \infty)(r) \hat{\eta}_0(dx, dr). \]

- Existence and uniqueness for this equation follow from the fact that only finitely many births can occur in a bounded time interval in \( K_n \).

- Consequently, the equation can be solved from one such birth to the next.
Since

- \( \lambda(x, \bar{\eta}_{s-} \cap K_n \cap \eta^n_{s-}) \leq \bar{\lambda}(x, \bar{\eta}_{s-}) \)
- \( \delta(x, \bar{\eta}_s \cap K_n \cap \eta^n_v) \geq \bar{\delta}(x, \bar{\eta}_s) \),
- it follows that \( \eta^n_t \subset \bar{\eta}_t \) and hence that

\[
\eta^n_t(B) = (SE2b)
\]

\[
\int_B \mathbf{1}_{[0,\lambda(x,K_n \cap \eta^n_{s-})]}(u) \mathbf{1}_{(\int_{s}^{t} \delta(x,K_n \cap \eta^n_v) \, dv, \infty)}(r) N(dx, ds, dr, du)
\]

\[
+ \int_B \mathbf{1}_{(\int_{0}^{t} \delta(x,K_n \cap \eta^n_v) \, dv, \infty)}(r) \hat{\eta}_0(dx, dr).
\]
Uniqueness for (SE2b) implies that the residual clock times at time $t$ are conditionally independent, unit exponentials given $\mathcal{F}_t^\eta$.

For $G(\zeta) = e^{-\int_S g(x)\zeta(dx)}$, $g \in \mathcal{C}$ nonnegative, and

$$A_n G(\zeta) = \int (G(\zeta + \delta_x) - G(\zeta)) \lambda(x, K_n \cap \zeta) \beta(dx)$$

$$+ \int (G(\zeta - \delta_x) - G(\zeta)) \delta(x, K_n \cap \zeta) \zeta(dx),$$

$$G(\eta_t^n) - G(\eta_0^t) - \int_0^t A_n G(\eta_s^n) ds$$

is a local martingale.
Exploiting the fact that $\eta^n_t \subset \bar{\eta}_t$, the relative compactness of $\{\eta^n\}$, in the sense of convergence in distribution in $D_C[0, \infty)$ follows.

**Proposition 0.1** Suppose that Conditions 1 and 2 hold. If 
$(x, \zeta) \to \lambda(x, \zeta)$ and $(x, \zeta) \to \delta(x, \zeta)$ are continuous on $S \times C$, then $\zeta \to AG(\zeta)$ is continuous, and any limit point of $\{\eta^n\}$ is a solution of the local martingale problem for $A$, and hence a weak solution of (SE2).
• If \( \sup_{\zeta \in S} \int_S \lambda(x, \zeta) \beta(dx) < \infty \), then a solution of (SE1) has only finitely many births per unit time and it is easy to see that (SE1) has a unique solution.

• Condition 1, however, only ensures that there are finitely many births per unit time in each \( K_k \), and uniqueness requires additional conditions.

• The conditions we use are essentially the same as those used for existence and uniqueness of the solution of the time change system in Garcia (1995).

• From now on, we are going to assume that \( \delta(x, \eta) = 1 \), for all \( x \in S \) and \( \eta \in S \).
• $N$ PPP $S \times [0, \infty)^3$ with mean measure 
  \[ \beta(dx) \times ds \times e^{-r}dr \times du. \]
• $\eta_0$ be an $S$-valued random variable independent of $N$,
• $\hat{\eta}_0$ be defined as before.
• $\{\mathcal{F}_t\}$ is a filtration such that $\hat{\eta}_0$ is $\mathcal{F}_0$-measurable and $N$ is 
  $\{\mathcal{F}_t\}$-compatible.

\[ \eta_t(B) = \int_{B \times [0,t] \times [0,\infty)^2} 1_{[0,\lambda(x,\eta_{s-})]}(u)1_{(t-s,\infty)}(r)N(dx,ds,dr,du) \]
\[ + \int_{B \times [0,\infty)} 1_{(t,\infty)}(r)\hat{\eta}_0(dx,dr). \quad \text{(SE3)} \]
**Theorem:** Suppose that

\[ a(x, y) \geq \sup_{\eta} |\lambda(x, \eta + \delta y) - \lambda(x, \eta)| \]

and that there exists a positive function \( c \) such that

\[ M = \sup_{x} \int_{S} \frac{c(x)a(x, y)}{c(y)} \beta(dy) < \infty. \]

Then, there exists a unique solution of (SE3).

**Example:** Let \( d(x, \eta) = \inf\{d_S(x, y) : y \in \eta\} \), where \( d_S \) is a distance in \( S \). Suppose \( \lambda(x, \eta) = h(d(x, \eta)) \). Then

\[ a(x, y) = \sup_{r > d_S(x, y)} |h(r) - h(d_S(x, y))|. \]

If \( h \) is increasing, then

\[ a(x, y) = h(\infty) - h(d_S(x, y)) \]

if \( h \) is decreasing, then

\[ a(x, y) = h(d_S(x, y)) - h(\infty). \]
Temporal ergodicity

1. There exists an unique stationary distribution for the process. Under this condition, the corresponding stationary process is ergodic in the sense of triviality of its tail $\sigma$-algebra.

2. (Stronger) For all initial distributions, the distribution of the process at time $t$ converges to the (unique) stationary distribution as $t \to \infty$.

1.- (CFTP) Kendall and Møller (2000), for $\eta^1 \subset \eta^2$, define

$$\bar{\lambda}(x, \eta^1, \eta^2) = \sup_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta) \quad \underline{\lambda}(x, \eta^1, \eta^2) = \inf_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta).$$

Note that for $\eta^1 \subset \eta^2$

$$|\bar{\lambda}(x, \eta^1, \eta^2) - \underline{\lambda}(x, \eta^1, \eta^2)| \leq \int_S a(x, y)|\eta^1 - \eta^2|(dy).$$
We assume that $N$ is defined on $S \times (-\infty, \infty) \times [0, \infty)^2$, that is, for all positive and negative time, and consider a system starting from time $-T$, that is, for $t \geq -T$

$$\eta_{t}^{1,T}(B) = \int_{B \times [-T,t] \times [0,\infty)^2} \mathbf{1}_{[0,h(x,1^{T},\eta_{s}^{1,T},\eta_{s}^{2,T})]}(u) \mathbf{1}_{(t-s,\infty)}(r) N(dx, ds, dr, du)$$

$$+ \int_{B \times [0,\infty)} \mathbf{1}_{(t+T,\infty)}(r) \hat{\eta}_{-T}^{1,T}(dx, dr)$$

$$\eta_{t}^{2,T}(B) = \int_{B \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0,h(x,1^{T},\eta_{s}^{1,T},\eta_{s}^{2,T})]}(u) \mathbf{1}_{(t-s,\infty)}(r) N(dx, ds, dr, du)$$

$$+ \int_{B \times [0,\infty)} \mathbf{1}_{(t+T,\infty)}(r) \hat{\eta}_{-T}^{2,T}(dx, dr), \quad (9)$$

where we require $\eta_{-T}^{1} \subset \eta_{-T}^{2,T}$. 
Suppose $\lambda(x, \eta) \leq \Lambda(x)$ for all $\eta$.

Then we can obtain a solution of (9) by iterating

$$
\eta_{t}^{1,T,n+1}(B) = \int_{B \times [-T,t] \times [0,\infty)} 1_{[0,\lambda(x,\eta_{s}^{1,T,n},\eta_{s}^{2,T,n})]}(u) 1_{(t-s,\infty)}(r) dN
$$

$$
+ \int_{B \times [0,\infty)} 1_{(t,T,\infty)}(r) \hat{\eta}_{-T}^{1}(d\eta, dr)
$$

$$
\eta_{t}^{2,T,n+1}(B) = \int_{B \times [-T,t] \times [0,\infty)} 1_{[0,\lambda(x,\eta_{s}^{1,T,n},\eta_{s}^{2,T,n})]}(u) 1_{(t-s,\infty)}(r) dN
$$

$$
+ \int_{B \times [0,\infty)} 1_{(t,T,\infty)}(r) \hat{\eta}_{-T}^{2}(d\eta, dr),
$$

where we take $\eta_{t}^{1,T,1} \equiv \emptyset$ and

$$
\eta_{t}^{2,T,1}(B) = \int_{B \times [-T,t] \times [0,\infty)} 1_{[0,\Lambda(x)]}(u) 1_{(t-s,\infty)}(r) N(dx, ds, dr, du)
$$

$$
+ \int_{B \times [0,\infty)} 1_{(t,T,\infty)}(r) \hat{\eta}_{-T}^{2}(dx, dr).
$$
Note that $\eta^{1,T,n} \subset \eta^{2,T,n}$, $\{\eta^{1,T,n}\}$ is monotone increasing, and $\{\eta^{2,T,n}\}$ is monotone decreasing, and the limit, which must exist, will be a solution of (9).
• For $C \subset \mathbb{R}$, define $(C + t) = \{(s + t) : s \in C\},$

• time-shift of $N$ by
  $R_t N(B \times C \times D \times E) = N(B \times (C + t) \times D \times E)$.

• Taking $T = \infty$ in

\[
\eta^{1,\infty,n+1}_t(B) = \int_{B \times (-\infty,t] \times [0,\infty)^2} \mathbf{1}_{[0,\Lambda(x,\eta^{1,\infty,n}_s,\eta^{2,\infty,n}_s)]}(u) \mathbf{1}_{(t-s,\infty)}(r) dN,
\]

(10)

\[
\eta^{2,\infty,n+1}_t(B) = \int_{B \times (-\infty,t] \times [0,\infty)^2} \mathbf{1}_{[0,\bar{\Lambda}(x,\eta^{1,\infty,n}_s,\eta^{2,\infty,n}_s)]}(u) \mathbf{1}_{(t-s,\infty)}(r) dN,
\]

satisfy $\eta^{m,\infty,n}_t = H^{m,n}(R_t N), m = 1, 2$, for deterministic mappings $H^{m,n}$

• $\eta^{m,\infty}_t = H^{m}(R_t N)$ where $H^{m} = \lim_{n \to \infty} H^{m,n}$.

• It follows that $(\eta^{1,\infty}_t, \eta^{2,\infty}_t)$ is stationary and ergodic.
• Any stationary solution of the martingale problem can be represented as a weak solution $\eta$ of the stochastic equation on the doubly infinite time interval and hence coupled to versions of $\eta^{1,\infty,n}$ and $\eta^{2,\infty,n}$ so that $\eta_t^{1,\infty,n} \subset \eta_t \subset \eta_t^{2,\infty,n}$, $-\infty < t < \infty$.

• Let $\lambda : S \times \mathcal{N}(S) \to [0, \infty)$ satisfy

$$a(x, y) \geq \sup_{\eta} |\lambda(x, \eta + \delta_y) - \lambda(x, \eta)|$$

and that there exists a positive function $c$ such that

$$M = \sup_{x} \int_{S} \frac{c(x)a(x, y)}{c(y)} \beta(dy) < 1.$$

Then $\eta \equiv \eta^{2,\infty} = \eta^{1,\infty}$ a.s. is a stationary solution of (SE3) and the distribution of $\eta_{t}^{2,\infty}$ is the unique stationary distribution for $A$. 

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2.- We can also use the stochastic equation and estimates similar to those used in the proof of uniqueness to give conditions for ergodicity in the sense of convergence as $t \to \infty$ for all initial distributions.

**Theorem:** Let $\lambda : S \times \mathcal{N}(S) \to [0, \infty)$ satisfy $M < 1$. Then the process obtained as a solution of the system of stochastic equations (SE3) is temporally ergodic and the rate of convergence is exponential.
Spatial ergodicity

- \( S = \mathbb{R}^d \)

- \( \lambda \) is translation invariant: \( \lambda(x + y, \eta) = \lambda(x, S_y \eta) \) for \( x, y \in \mathbb{R}^d, \eta \in \mathcal{N}(\mathbb{R}^d) \).

\[
(S_x \eta)(B) = \eta(T_x B), \quad \eta \in \mathcal{N}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}^d).
\] (11)

Note that if \( \eta = \sum \delta_{x_i} \), then \( S_x \eta = \sum \delta_{x_i - x} \).
Let $\eta$ be a translation invariant, $\mathcal{N}(\mathbb{R}^d)$-valued random variable.

- A measurable subset $G \subset \mathcal{N}(\mathbb{R}^d)$ is \textit{almost surely translation invariant} for $\eta$, if
  
  $$1_G(\eta) = 1_G(S_x \eta) \text{ a.s.}$$

  for every $x \in \mathbb{R}^d$.

- $\eta$ is \textit{spatially ergodic} if $P\{\eta \in G\}$ is 0 or 1 for each almost surely translation invariant $G \subset \mathcal{N}(\mathbb{R}^d)$.

- For $x \in \mathbb{R}^d$, we define $S_x N$ so that the spatial coordinate of each point is shifted by $-x$. Almost sure translation invariance of a set $G \subset \mathcal{N}(\mathbb{R}^d \times [0, \infty)^3)$ and spatial ergodicity are defined analogously.

- Spatial ergodicity for $N$ follows from its independence properties.
1. Suppose $\lambda$ is translation invariant. If $\eta_0$ is translation invariant and spatially ergodic and the solution of $\text{(SE3)}$ is unique, then for each $t > 0$, $\eta_t$ is translation invariant and spatially ergodic.

**Remark:** Unfortunately, it is not clear, in general, how to carry this conclusion over to $t = \infty$, that is, to the limiting distribution of the solution.

2. If $\eta$ is temporally ergodic and $\pi$ is the unique stationary distribution, then it must be translation invariant since $\{\eta_t\}$ stationary (in time) implies $\{S_x \eta_t\}$ is stationary.
3. Suppose that $\lambda$ is translation invariant, then for each $t$, 
$\eta_{t,1,\infty} \equiv \lim_{n \to \infty} \eta_{t,1,\infty,n}$ and $\eta_{t,2,\infty} \equiv \lim_{n \to \infty} \eta_{t,2,\infty,n}$ are spatially ergodic.

4. If $\lambda$ satisfies the conditions $M < 1$, then the unique stationary distribution is spatially ergodic.
Future work:

1. Is the condition $M < 1$ necessary?

2. Can we work with variable death rate?