When stationary limits of spatially ergodic processes are spatially ergodic?

Spatial Birth and death processes as solutions of stochastic equations

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Joint work with Tom Kurtz - Univ. Wisconsin - Madison – ALEA

- **General case** Open problem
- **Particle systems**
  Spin flip systems - Andjel (1990)
  Exclusion processes, biased voter model, multi-opinion voter model - Chen (2002)
- **Stationary measures of spatial birth and death processes**
  Fernández, Ferrari and Garcia (2000, 2002)
  Kurtz and Garcia (2006)

Motivation: Time-invariance estimation. Kurtz and Li (2004)

- Modeling spatial point processes as stationary distributions of birth and death processes
- Equating to zero the generator of the process applied to a suitable collection of statistics. (Baddeley, 2000)
- Large sample limit to prove consistency. Necessary condition is that the stationary distribution is spatially stationary and ergodic.

Birth and death rates:

- $\int_A \lambda(x, \eta_t)\beta(dx)\Delta t \approx$ probability that a point in a set $A \subset S$ is added to the configuration η_t in $(t, t + \Delta t)$
- $\delta(x, \eta)\Delta t \approx$ the probability that a point $x \in \eta$ is deleted from the configuration η_t in $(t, t + \Delta t)$
- $$\begin{aligned} AF(\eta) &= \int (F(\eta + \delta_x) - F(\eta))\lambda(x, \eta)\beta(dx) \\ &\quad + \int (F(\eta - \delta_x) - F(\eta))\delta(x, \eta)\eta(dx) \end{aligned}$$

ξ **Poisson random measures with mean measure** β

- for each $A \in \mathcal{B}(S)$, $\xi(A)$ has a Poisson distribution with expectation $\beta(A)$
- $\xi(A)$ and $\xi(B)$ are independent if $A \cap B = \emptyset$.
- Taking $\lambda = \delta \equiv 1$, unique stationary distribution for the birth and death process with generator

  $$AF(\eta) = \int (F(\eta+\delta_x) - F(\eta))\beta(dx) + \int (F(\eta-\delta_x) - F(\eta))\eta(dx).$$

  Letting μ_β^0 denote this distribution, the stationarity can be checked by verifying that

  $$\int_{\mathcal{N}(S)} AF(\eta)\mu_\beta^0(d\eta) = 0.$$

Gibbs distributions

- Radon-Nikodym derivative with respect to a Poisson point process with mean measure β,

$$\mu_{\beta,H}(d\eta) = \frac{1}{Z_{\beta,H}} e^{-H(\eta)} \mu_{\beta}^{0}(d\eta),$$

- $\mu_{\beta,H}$ is the stationary distribution of several spatial birth and death processes. In fact, $\lambda(x,\eta) > 0$ if $H(\eta + \delta_x) < \infty$ and that λ and δ satisfy

$$\lambda(x,\eta) e^{-H(\eta)} = \delta(x, \eta + \delta_x) e^{-H(\eta+\delta_x)}.$$

- Again, this assertion can be verified by showing that

$$\int AF(\eta)\mu_{\beta,H}(d\eta) = \frac{1}{Z_{\beta,H}} \int AF(\eta) e^{-H(\eta)} \mu_{\beta}^{0}(d\eta) = 0.$$

Notice that

$$\lambda(x,\eta)e^{-H(\eta)} = \delta(x,\eta+\delta_x)e^{-H(\eta+\delta_x)}$$

is equivalent to

$$\frac{\lambda(x,\eta)}{\delta(x,\eta+\delta_x)} = \exp\{-H(\eta+\delta_x)+H(\eta)\}$$

We can always take $\delta(x,\eta) = 1$

Example: Pairwise interaction potential $\rho(x_1,x_2) \geq 0$, that is, for $\eta = \sum_{i=1}^m \delta_{x_i}$,

$$H_\rho(\eta) = \sum_{i<j} \rho(x_i,x_j)$$

Take $\beta =$ Lebesgue, $\delta(x,\eta) \equiv 1$ and $\lambda(x,\eta) = \exp\{-\int \rho(x,y)\eta(dy)\}$

Area-interaction point process [Baddeley and Van Lieshout (1995)]

$$H(\eta) = \eta(S)\log\rho - m_d(\eta \oplus G)$$

- Radon-Nikodym derivative

$$L(\eta) = \frac{1}{Z}\rho^{\eta(S)}\gamma^{-m_d(\eta\oplus G)}$$

.

- $\rho > 0$ and $\gamma > 0$ G is a compact *grain*.
- $\gamma > 1$, the process is *attractive*
  $\gamma > 1$, the process is*repulsive*
  $\gamma = 1$ the Poisson random measure with mean measure ρm_d.
  $\gamma \to 0$ corresponds to *area-exclusion*
- Take unit death rate and the birth rate given by

$$\lambda(x,\eta) = \rho\,\gamma^{-m_d((x+G)\setminus(\eta\oplus G))}$$

1st approach: Graphical construction

Assume:

1. Finite range: there exists a compact set G such that if $\eta_1 = \eta_2$ inside $x + G$ then $\lambda(x, \eta_1) = \lambda(x, \eta_2)$
2. Boundedness

$$\overline{\lambda} = \sup_{x,\eta} \lambda(x, \eta) < \infty.$$

- Begin with a $\overline{\lambda}$-homogeneous Poisson point process on $\mathbb{R}^d \times \mathbb{R}$.

$$N = \{(\xi_1, T_1), (\xi_2, T_2), \ldots\}$$

- For each point (ξ_i, T_i), associate two independent marks $S_i \sim \exp(1)$ and $Z_i \sim U(0, 1)$.

- Marked *cylinder* $((\xi_i + G) \times [T_i, T_i + S_i), Z_i)$

Invariant measure:

Construct the process beginning at $-\infty$ in a given configuration and cut the process at time 0. If, the process is independent of the initial configuration, the process at time 0 has invariant measure.

1. Generate the *free* process α as a $\overline{\lambda}$-homogeneous Poisson process on λ according to Algorithm of the Poisson process.

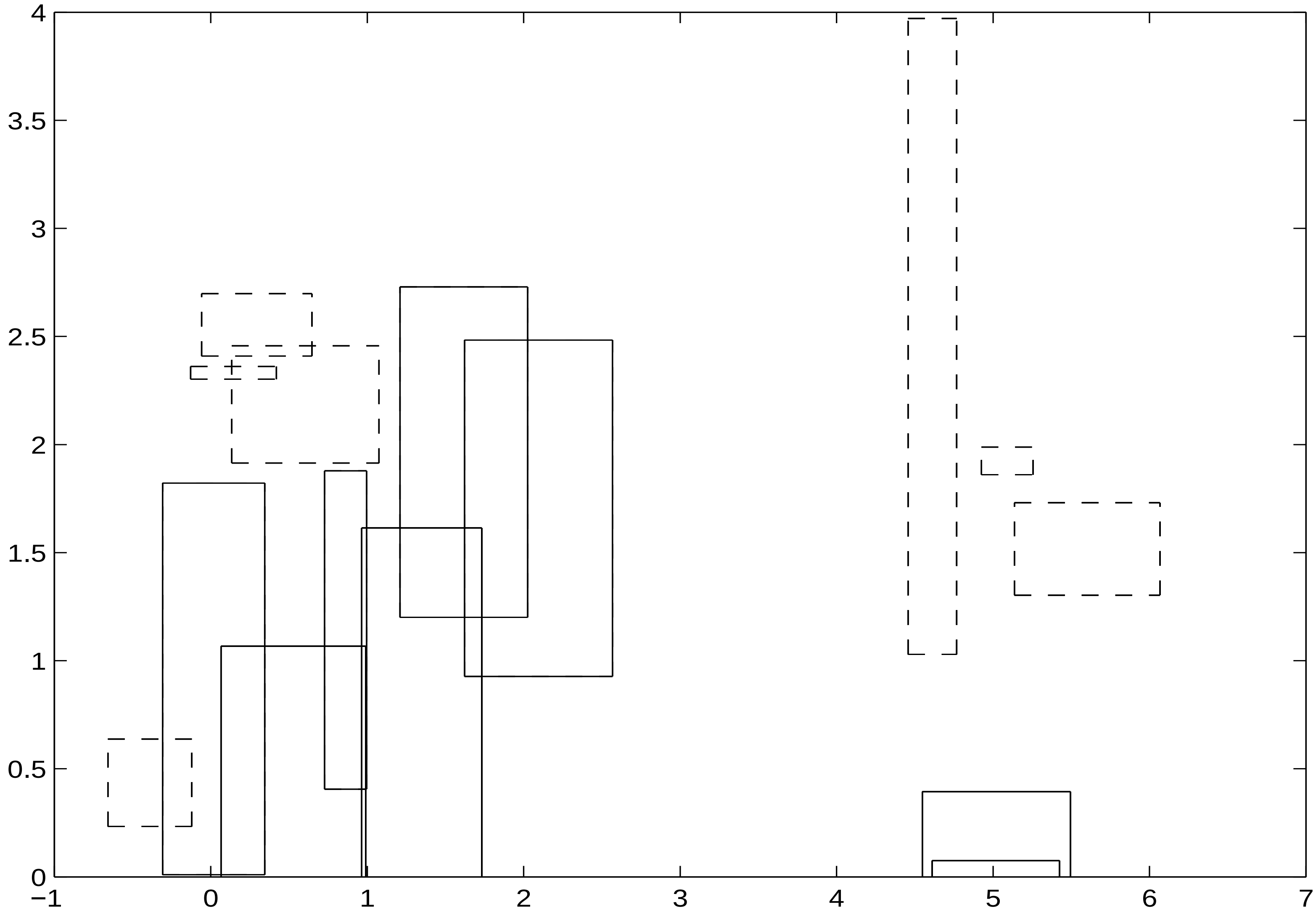

Figure 1: Independent process

2. Construct the clan of ancestors of all points of α.
3. Apply the *deterministic* finite-volume "cleaning procedure" to decide which points of α are going to be kept.

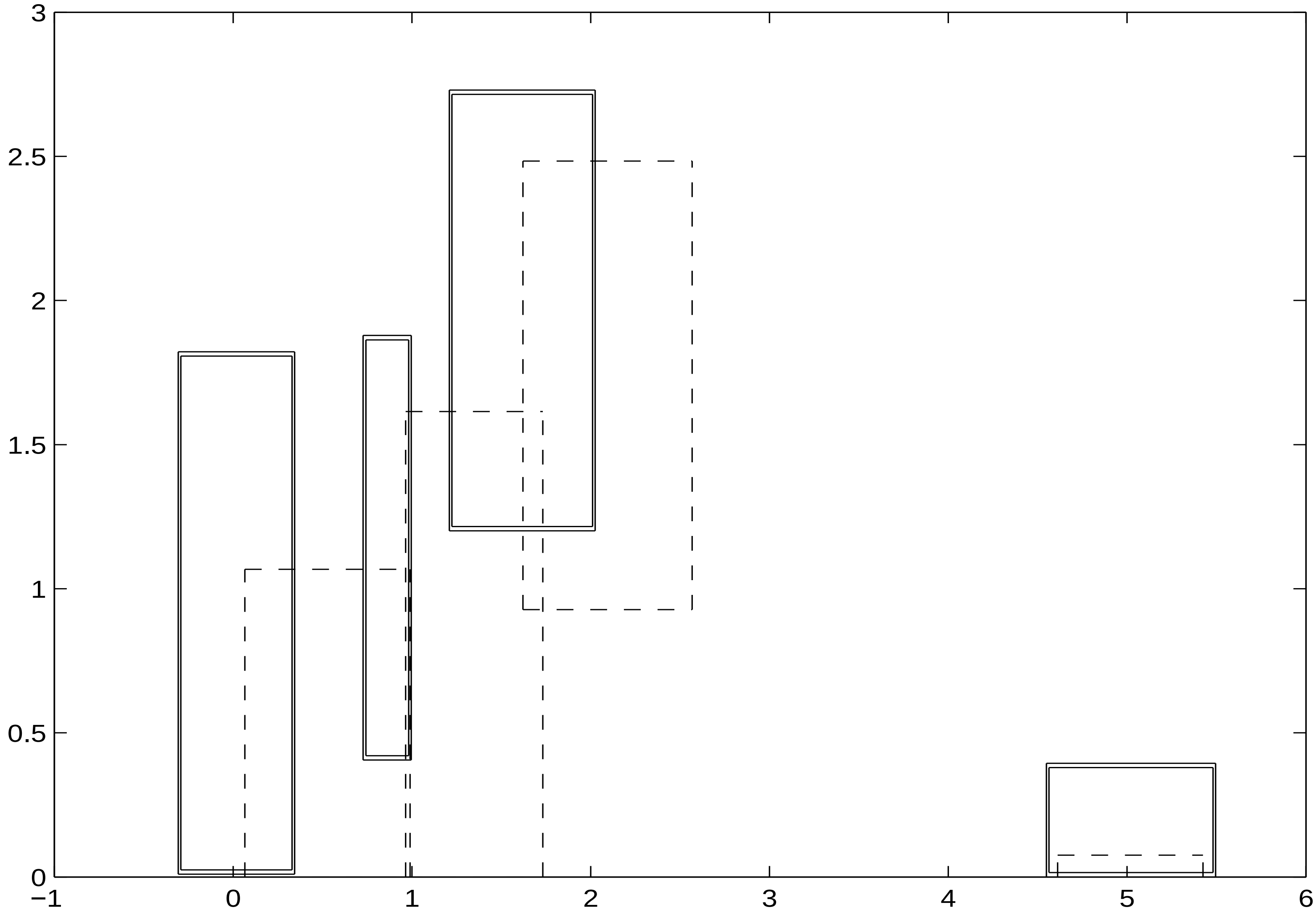

Figure 2: Cleaned process

2nd approach: Solutions of a system of stochastic equations

- Usually the individuals in the birth and death process are represented by points in $\mathbb{R}^d$, $\mathbb{Z}^d$ but more general spaces S are OK.
- Let β be a σ-finite, Borel measure on S.

Condition 1 *For each compact $\mathcal{K} \subset \mathcal{S}$, the birth rate λ satisfies*

$$\sup_{\zeta \in \mathcal{K}} \int_S c_k(x) \lambda(x, \zeta) \beta(dx) < \infty, \quad t > 0, \quad k = 1, 2, \ldots,$$

and

$$\delta(x, \zeta) < \infty, \quad \zeta \in \mathcal{S}, \quad x \in \zeta.$$

We also assume that λ and δ satisfy the following continuity condition.

Condition 2 *If*

$$\lim_{n\to\infty}\int_S c_k(x)|\zeta_n-\zeta|(dx)=0, \tag{1}$$

for each $k=1,2,\ldots,$ *then*

$$\lambda(x,\zeta)=\lim_{n\to\infty}\lambda(x,\zeta_n), \quad \delta(x,\zeta)=\lim_{n\to\infty}\delta(x,\zeta_n).$$

Remark: Since (1) implies ζ_n converges to ζ in $\mathcal{S}$, this continuity condition is weaker than continuity in $\mathcal{S}$.

Equivalent: Suppose $\zeta_0,\zeta_1,\zeta_2,\ldots\in\mathcal{S}$ and $\zeta_n\le\zeta_0$, $n=1,2,\ldots$. If $\zeta_n\to\zeta$ in $\mathcal{S}$, then (1) holds.

- N be a PPP on $S \times [0, \infty)^3$ with mean measure $\beta(dx) \times ds \times e^{-r} dr \times du$.
- $\eta_0 = \sum_{i=1}^{\infty} \delta_{x_i}$ PPP on S independent of N,
- $\widehat{\eta}_0 = \sum_{i=1}^{\infty} \delta_{(x_i, \tau_i)}$
- $\{\tau_i\}$ are independent unit exponentials, independent of η_0 and N.

(SE1)

$$\eta_t(A) = \int_{A \times [0,t] \times [0,\infty)^2} \mathbf{1}_{[0, \lambda(x, \eta_{s-})]}(u) \mathbf{1}_{(\int_s^t \delta(x, \eta_v)\, dv, \infty)}(r) N(dx, ds, dr, du) + \int_{A \times [0,\infty)} \mathbf{1}_{(\int_0^t \delta(x, \eta_s)\, ds, \infty)}(r) \widehat{\eta}_0(dx, dr).$$

Lemma 1 *If η is a solution of* **(SE1)**, *then for each $T > 0$,*

$$\int_0^T \int_S c_k(x)\lambda(x, \eta_s)\beta(dx)ds < \infty \quad a.s.,$$

η_T^* defined by

$$\eta_T^*(B) = \int_{B\times[0,T]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)N(dx, ds, dr, du)$$

is an element of $\mathcal{S}$,

$$\eta_t \leq \eta_T^* + \eta_0, \quad 0 \leq t \leq T,$$

and

$$\lim_{s\to t+} \int_S c_k(x)|\eta_s - \eta_t|(dx) = 0, \quad t \geq 0.$$

- If η is a solution of **(SE1)**
- x was born at time $s \leq t$,
- then the "residual clock time" $r - \int_s^t \delta(x, \eta_v) dv$ is an $\mathcal{F}_t$-measurable random variable.
- In particular, the counting-measure-valued process given by **(SE2)**

$$\widehat{\eta}_t(B \times D)$$

$$= \int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u) \mathbf{1}_D(r - \int_s^t \delta(x, \eta_v)\, dv) N(dx, ds, dr, du)$$

$$+ \int_{B\times[0,\infty)} \mathbf{1}_D(r - \int_0^t \delta(x, \eta_{s-})\, ds) \widehat{\eta}_0(dx, dr)$$

is $\{\mathcal{F}_t\}$-adapted.

$\widehat{\mathcal{S}}$: counting measures ζ on $S \times [0, \infty)$ such that $\zeta(\cdot \times [0, \infty)) \in \mathcal{S}$.

Alternative equation for the $\widehat{\mathcal{S}}$-valued process $\widehat{\eta}$ by requiring that

$$\begin{aligned}\int_{S\times[0,\infty)} f(x,r)\widehat{\eta}_t(dx,dr) &= \int_{S\times[0,\infty)} f(x,r)\widehat{\eta}_0(dx,dr) \\ &\quad + \int_{S\times[0,t]\times[0,\infty)^2} f(x,r)\mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)N(dx,ds,dr,du) \\ &\quad - \int_0^t \int_{S\times[0,\infty)} \delta(x,\eta_s)f_r(x,r)\widehat{\eta}_s(dx,dr)ds, \qquad \textbf{(SE3)}\end{aligned}$$

for all $f \in \overline{C}(S \times [0, \infty))$ such that

$f_r \equiv \frac{\partial}{\partial r} f \in \overline{C}(S \times [0,\infty))$, $f(x,0) = 0$,
$\sup_r |f(\cdot,r)|, \sup_r |f_r(\cdot,r)| \in \mathcal{C}$, and there exists $r_f > 0$ such that $f_r(x,r) = 0$ for $r > r_f$. Note that if $f \in \widehat{\mathcal{C}}$ and

$$f^*(x,r) = \int_0^r |f_r(x,u)| du,$$

then $f^* \in \widehat{\mathcal{C}}$. In **(SE3)**, $\widehat{\eta}_0$ can be any $\widehat{\mathcal{S}}$-valued random variable that is independent of N.

Martingale problems

- $\mathcal{D}(\widehat{A}) = \{F; F(\widehat{\zeta}) = e^{-\int_{S\times[0,\infty)} f(x,r)\widehat{\zeta}(dx,dr)}, \ f \geq 0 \in \widehat{\mathcal{C}}.$

$$\int_0^t \int_S c_k(x)\lambda(x,\eta_s)\beta(dx)ds < \infty, \quad k = 1, 2, \ldots.$$

- By Itô's formula (RHS = local martingale)

$$F(\widehat{\eta}_t) = F(\widehat{\eta}_0)$$
$$+ \int_{S\times[0,t]\times[0,\infty)^2} F(\widehat{\eta}_{s-})(e^{-f(x,r)} - 1)\mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\tilde{N}(dx,ds,dr,du)$$
$$+ \int_0^t F(\widehat{\eta}_s)\Big(\int_{S\times[0,\infty)} \lambda(x,\eta_s)(e^{-f(x,r)} - 1)e^{-r}\beta(dx)dr$$
$$+ \int_{S\times[0,\infty)} \delta(x,\eta_s)f_r(x,r)\widehat{\eta}_s(dx,dr)\Big)ds,$$

$$\tilde{N}(dx,ds,dr,du) = N(dx,ds,dr,du) - \beta(dx) \times ds \times e^{-r}dr \times du.$$

$$\int_0^t \int_{S\times[0,\infty)} \delta(x,\eta_s)|f_r(x,r)|\widehat{\eta}_s(dx,dr)ds < \infty, \quad t > 0. \tag{2}$$

Consequently, defining

$$\begin{aligned} \widehat{A}F(\widehat{\zeta}) \quad &= \quad F(\widehat{\zeta})\Big(\int_{S\times[0,\infty)} \lambda(x,\zeta)(e^{-f(x,r)}-1)e^{-r}\beta(dx)dr \\ &\quad + \int_{S\times[0,\infty)} \delta(x,\zeta)f_r(x,r)\widehat{\zeta}(dx,dr)\Big), \end{aligned} \tag{3}$$

any solution of **(SE3)** must be a solution of the local martingale problem for $\widehat{A}$.

We say that $\widehat{\eta}$ is a solution of the *local martingale problem* for $\widehat{A}$ if there exists a filtration $\{\mathcal{F}_t\}$ such that $\widehat{\eta}$ is $\{\mathcal{F}_t\}$-adapted and

$$M_F(t) = F(\widehat{\eta}_t) - F(\widehat{\eta}_0) - \int_0^t \widehat{A}F(\widehat{\eta}_s)ds \qquad (4)$$

is a $\{\mathcal{F}_t\}$-local martingale for each $F \in \mathcal{D}(\widehat{A})$, that is, for each F of the form $F(\widehat{\zeta}) = e^{-\int f d\widehat{\zeta}}$, $f \in \widehat{\mathcal{C}}$, $f \geq 0$. In particular, let $\overline{f}(x) = \sup_r f(x,r)$ and

$$\tau_{f,c} = \inf\{t : \int_0^t \int_S \overline{f}(x)\lambda(x,\eta_s)\beta(dx)ds > c\}.$$

Then $M_F(\cdot \wedge \tau_{f,c})$ is a martingale. Note that $\tau_{f,c}$ is a $\{\mathcal{F}_t^\eta\}$-stopping time.

Main theorem: Suppose that λ and δ satisfy Conditions 1 and 2. Then each solution of the stochastic equation **(SE2)** (or equivalently, **(SE3)**) is a solution of the local martingale problem for $\widehat{A}$ defined by (3), and each solution of the local martingale problem for $\widehat{A}$ is a weak solution of the stochastic equation.

Existence of the solution:

- (λ, δ) is *attractive* if $\zeta_1 \subset \zeta_2$ implies $\lambda(x, \zeta_1) \leq \lambda(x, \zeta_2)$ and $\delta(x, \zeta_1) \geq \delta(x, \zeta_2)$.

- If (λ, δ) is attractive and we set $\eta^0 \equiv 0$, then η^n defined by

$$\eta_t^{n+1}(B) = \int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-}^n)]}(u)\mathbf{1}_{(\int_s^t \delta(x,\eta_v^n)\,dv,\infty)}(r)N(dx, ds, dr, du) + \int_{B\times[0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\eta_s^n)\,ds,\infty)}(r)\widehat{\eta}_0(dx, dr) \qquad (5)$$

is monotone increasing and either η^n converges to a process with values in $\mathcal{S}$, or

$$\int_0^T \int_S c_k(x)\lambda(x, \eta_s^n)\beta(dx)ds \to \infty, \qquad (6)$$

for some T and k.

- For an arbitrary (λ, δ) define an attractive pair by setting

$$\overline{\lambda}(x,\zeta) = \sup_{\zeta' \subset \zeta} \lambda(x,\zeta') \qquad \underline{\delta}(x,\zeta) = \inf_{\zeta' \subset \zeta} \delta(x,\zeta').$$

Let η_0 be an $\mathcal{S}$-valued random variable independent of N, and let $\widehat{\eta}_0$ be defined as before. We assume that $\overline{\lambda}$

$$\int c_k(x)\overline{\lambda}(x,\zeta)\beta(dx) < \infty, \quad \zeta \in S, k = 1, 2, \ldots, \qquad (7)$$

and that there exists a solution $\overline{\eta}$ for the pair $(\overline{\lambda}, \underline{\delta})$.

$$\eta_t^n(B) = \int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\overline{\eta}_{s-}\cap K_n\cap\eta^n_{s-})]}(u)\mathbf{1}_{(\int_s^t \delta(x,\overline{\eta}_v\cap K_n\cap\eta^n_v)\,dv,\infty)}(r)N(dx,ds,dr,du)$$
$$+\int_{B\times[0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\overline{\eta}_v\cap K_n\cap\eta^n_v)\,dv,\infty)}(r)\widehat{\eta}_0(dx,dr).$$

- Existence and uniqueness for this equation follow from the fact that only finitely many births can occur in a bounded time interval in K_n.
- Consequently, the equation can be solved from one such birth to the next.

Since

- $\lambda(x, \overline{\eta}_{s-} \cap K_n \cap \eta^n_{s-}) \leq \overline{\lambda}(x, \overline{\eta}_{s-})$
- $\delta(x, \overline{\eta}_s \cap K_n \cap \eta^n_v) \geq \underline{\delta}(x, \overline{\eta}_s)$,
- it follows that $\eta^n_t \subset \overline{\eta}_t$ and hence that

$$\eta^n_t(B) = \textbf{(SE2b)}$$

$$\int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,K_n\cap\eta^n_{s-})]}(u)\mathbf{1}_{(\int_s^t \delta(x,K_n\cap\eta^n_v)\,dv,\infty)}(r)N(dx,ds,dr,du)$$

$$+\int_{B\times[0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,K_n\cap\eta^n_v)\,dv,\infty)}(r)\widehat{\eta}_0(dx,dr).$$

Uniqueness for **(SE2b)** implies that the residual clock times at time t are conditionally independent, unit exponentials given $\mathcal{F}_t^\eta$.

For $G(\zeta) = e^{-\int_S g(x)\zeta(dx)}$, $g \in \mathcal{C}$ nonnegative, and

$$\begin{aligned} A_n G(\zeta) &= \int (G(\zeta + \delta_x) - G(\zeta))\lambda(x, K_n \cap \zeta)\beta(dx) \\ &\quad + \int (G(\zeta - \delta_x) - G(\zeta))\delta(x, K_n \cap \zeta)\zeta(dx), \end{aligned}$$

$$G(\eta_t^n) - G(\eta_0^t) - \int_0^t A_n G(\eta_s^n)ds \tag{8}$$

is a local martingale.

Exploiting the fact that $\eta_t^n \subset \overline{\eta}_t$, the relative compactness of $\{\eta^n\}$, in the sense of convergence in distribution in $D_{\mathcal{C}}[0, \infty)$ follows.

Proposition 0.1 *Suppose that Conditions 1 and 2 hold. If* $(x, \zeta) \to \lambda(x, \zeta)$ *and* $(x, \zeta) \to \delta(x, \zeta)$ *are continuous on* $S \times \mathcal{C}$*, then* $\zeta \to AG(\zeta)$ *is continuous, and any limit point of* $\{\eta^n\}$ *is a solution of the local martingale problem for* A*, and hence a weak solution of* **(SE2)**.

- If $\sup_{\zeta \in \mathcal{S}} \int_S \lambda(x,\zeta)\beta(dx) < \infty$, then a solution of **(SE1)** has only finitely many births per unit time and it is easy to see that **(SE1)** has a unique solution.
- Condition 1, however, only ensures that there are finitely many births per unit time in each K_k, and uniqueness requires additional conditions.
- The conditions we use are essentially the same as those used for existence and uniqueness of the solution of the time change system in Garcia (1995).
- From now on, we are going to assume that $\delta(x,\eta) = 1$, for all $x \in S$ and $\eta \in \mathcal{S}$.

- N PPP $S \times [0,\infty)^3$ with mean measure $\beta(dx) \times ds \times e^{-r} dr \times du$.
- η_0 be an $\mathcal{S}$-valued random variable independent of N,
- $\widehat{\eta}_0$ be defined as before.
- $\{\mathcal{F}_t\}$ is a filtration such that $\widehat{\eta}_0$ is $\mathcal{F}_0$-measurable and N is $\{\mathcal{F}_t\}$-compatible.

$$\begin{aligned}\eta_t(B) &= \int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx,ds,dr,du) \\ &\quad + \int_{B\times[0,\infty)} \mathbf{1}_{(t,\infty)}(r)\widehat{\eta}_0(dx,dr). \qquad \textbf{(SE3)}\end{aligned}$$

Theorem: Suppose that

$$a(x,y) \geq \sup_{\eta} |\lambda(x, \eta + \delta_y) - \lambda(x, \eta)|$$

and that there exists a positive function c such that

$$M = \sup_x \int_S \frac{c(x)a(x,y)}{c(y)} \beta(dy) < \infty.$$

Then, there exists a unique solution of **(SE3)**.

Example: Let $d(x,\eta) = \inf\{d_S(x,y) : y \in \eta\}$, where d_S is a distance in S. Suppose $\lambda(x,\eta) = h(d(x,\eta))$. Then $a(x,y) = \sup_{r > d_S(x,y)} |h(r) - h(d_S(x,y))|$. If h is increasing, then $a(x,y) = h(\infty) - h(d_S(x,y))$ if h is decreasing, then $a(x,y) = h(d_S(x,y)) - h(\infty)$.

Temporal ergodicity

1. There exists an unique stationary distribution for the process. Under this condition, the corresponding stationary process is ergodic in the sense of triviality of its tail σ-algebra.

2. (Stronger) For all initial distributions, the distribution of the process at time t converges to the (unique) stationary distribution as $t \to \infty$.

1.- (CFTP) Kendall and Møller (2000), for $\eta^1 \subset \eta^2$, define

$$\overline{\lambda}(x, \eta^1, \eta^2) = \sup_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta) \qquad \underline{\lambda}(x, \eta^1, \eta^2) = \inf_{\eta^1 \subset \eta \subset \eta^2} \lambda(x, \eta).$$

Note that for $\eta^1 \subset \eta^2$

$$|\overline{\lambda}(x, \eta^1, \eta^2) - \underline{\lambda}(x, \eta^1, \eta^2)| \leq \int_S a(x, y)|\eta^1 - \eta^2|(dy).$$

We assume that N is defined on $S \times (-\infty, \infty) \times [0, \infty)^2$, that is, for all positive and negative time, and consider a system starting from time $-T$, that is, for $t \geq -T$

$$\eta_t^{1,T}(B) = \int_{B\times[-T,t]\times[0,\infty)^2} \mathbf{1}_{[0,\underline{\lambda}(x,\eta_{s-}^{1,T},\eta_{s-}^{2,T}))}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx,ds,dr,du$$
$$+ \int_{B\times[0,\infty)} \mathbf{1}_{(t+T,\infty)}(r)\widehat{\eta}_{-T}^{1,T}(dx,dr)$$
$$\eta_t^{2,T}(B) = \int_{B\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\overline{\lambda}(x,\eta_{s-}^{1,T},\eta_{s-}^{2,T})]}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx,ds,dr,du)$$
$$+ \int_{B\times[0,\infty)} \mathbf{1}_{(t+T,\infty)}(r)\widehat{\eta}_{-T}^{2,T}(dx,dr), \qquad (9$$

where we require $\eta_{-T}^{1,T} \subset \eta_{-T}^{2,T}$.

Suppose $\lambda(x,\eta) \le \Lambda(x)$ for all η.

Then we can obtain a solution of (9) by iterating

$$\begin{aligned}
\eta_t^{1,T,n+1}(B) &= \int_{B\times[-T,t]\times[0,\infty)^2} \mathbf{1}_{[0,\underline{\lambda}(x,\eta_{s-}^{1,T,n},\eta_{s-}^{2,T,n}))}(u)\mathbf{1}_{(t-s,\infty)}(r)dN \\
&\qquad + \int_{B\times[0,\infty)} \mathbf{1}_{(t+T,\infty)}(r)\widehat{\eta}_{-T}^{1,T}(dx,dr) \\
\eta_t^{2,T,n+1}(B) &= \int_{B\times[-T,t]\times[0,\infty)^2} \mathbf{1}_{[0,\overline{\lambda}(x,\eta_{s-}^{1,T,n},\eta_{s-}^{2,T,n})]}(u)\mathbf{1}_{(t-s,\infty)}(r)dN \\
&\qquad + \int_{B\times[0,\infty)} \mathbf{1}_{(t+T,\infty)}(r)\widehat{\eta}_{-T}^{2,T}(dx,dr),
\end{aligned}$$

where we take $\eta_t^{1,T,1} \equiv \emptyset$ and

$$\begin{aligned}
\eta_t^{2,T,1}(B) &= \int_{B\times[-T,t]\times[0,\infty)^2} \mathbf{1}_{[0,\Lambda(x)]}(u)\mathbf{1}_{(t-s,\infty)}(r)N(dx,ds,dr,du) \\
&\qquad + \int_{B\times[0,\infty)} \mathbf{1}_{(t+T,\infty)}(r)\widehat{\eta}_{-T}^{2,T}(dx,dr).
\end{aligned}$$

Note that $\eta^{1,T,n} \subset \eta^{2,T,n}$, $\{\eta^{1,T,n}\}$ is monotone increasing, and $\{\eta^{2,T,n}\}$ is monotone decreasing, and the limit, which must exist, will be a solution of (9).

- For $C \subset \mathbb{R}$, define $(C+t) = \{(s+t) : s \in C\}$,
- time-shift of N by
  $R_t N(B \times C \times D \times E) = N(B \times (C+t) \times D \times E)$.
- Taking $T = \infty$ in

$$\eta_t^{1,\infty,n+1}(B) = \int_{B\times(-\infty,t]\times[0,\infty)^2} \mathbf{1}_{[0,\underline{\lambda}(x,\eta_{s-}^{1,\infty,n},\eta_{s-}^{2,\infty,n})}(u)\mathbf{1}_{(t-s,\infty)}(r)dN \tag{10}$$

$$\eta_t^{2,\infty,n+1}(B) = \int_{B\times(-\infty,t]\times[0,\infty)^2} \mathbf{1}_{[0,\overline{\lambda}(x,\eta_{s-}^{1,\infty,n},\eta_{s-}^{2,\infty,n})]}(u)\mathbf{1}_{(t-s,\infty)}(r)dN,$$

  satisfy $\eta_t^{m,\infty,n} = H^{m,n}(R_t N)$, $m = 1, 2$, for deterministic mappings $H^{m,n}$
- $\eta_t^{m,\infty} = H^m(R_t N)$ where $H^m = \lim_{n\to\infty} H^{m,n}$.
- It follows that $(\eta_t^{1,\infty}, \eta_t^{2,\infty})$ is stationary and ergodic.

- Any stationary solution of the martingale problem can be represented as a weak solution η of the stochastic equation on the doubly infinite time interval and hence coupled to versions of $\eta^{1,\infty,n}$ and $\eta^{2,\infty,n}$ so that $\eta_t^{1,\infty,n} \subset \eta_t \subset \eta_t^{2,\infty,n}$, $-\infty < t < \infty$.

- Let $\lambda : S \times \mathcal{N}(S) \to [0, \infty)$ satisfy

$$a(x,y) \geq \sup_{\eta} |\lambda(x, \eta + \delta_y) - \lambda(x, \eta)|$$

and that there exists a positive function c such that

$$M = \sup_x \int_S \frac{c(x)a(x,y)}{c(y)} \beta(dy) < 1.$$

Then $\eta \equiv \eta^{2,\infty} = \eta^{1,\infty}$ a.s. is a stationary solution of **(SE3)** and the distribution of $\eta_t^{2,\infty}$ is the unique stationary distribution for A.

2.- We can also use the stochastic equation and estimates similar to those used in the proof of uniqueness to give conditions for ergodicity in the sense of convergence as $t \to \infty$ for all initial distributions.

Theorem: Let $\lambda : S \times \mathcal{N}(S) \to [0, \infty)$ satisfy $M < 1$. Then the process obtained as a solution of the system of stochastic equations **(SE3)** is temporally ergodic and the rate of convergence is exponential.

Spatial ergodicity

- $S = \mathbb{R}^d$
- λ is translation invariant: $\lambda(x+y,\eta) = \lambda(x, S_y\eta)$ for $x, y \in \mathbb{R}^d, \eta \in \mathcal{N}(\mathbb{R}^d)$.

$$(S_x\eta)(B) = \eta(T_xB), \quad \eta \in \mathcal{N}(\mathbb{R}^d), B \in \mathcal{B}(\mathbb{R}^d). \tag{11}$$

Note that if $\eta = \sum \delta_{x_i}$, then $S_x\eta = \sum \delta_{x_i - x}$.

Let η be a translation invariant, $\mathcal{N}(\mathbb{R}^d)$-valued random variable.

- A measurable subset $G \subset \mathcal{N}(\mathbb{R}^d)$ is *almost surely translation invariant* for η, if

$$\mathbf{1}_G(\eta) = \mathbf{1}_G(S_x\eta) \quad a.s.$$

for every $x \in \mathbb{R}^d$.

- η is *spatially ergodic* if $P\{\eta \in G\}$ is 0 or 1 for each almost surely translation invariant $G \subset \mathcal{N}(\mathbb{R}^d)$.

- For $x \in \mathbb{R}^d$, we define S_xN so that the spatial coordinate of each point is shifted by $-x$. Almost sure translation invariance of a set $G \subset \mathcal{N}(\mathbb{R}^d \times [0, \infty)^3)$ and spatial ergodicity are defined analogously

- Spatial ergodicity for N follows from its independence properties.

1. Suppose λ is translation invariant. If η_0 is translation invariant and spatially ergodic and the solution of **(SE3)** is unique, then for each $t > 0$, η_t is translation invariant and spatially ergodic.

   Remark: Unfortunately, it is not clear, in general, how to carry this conclusion over to $t = \infty$, that is, to the limiting distribution of the solution

2. If η is temporally ergodic and π is the unique stationary distribution, then it must be translation invariant since $\{\eta_t\}$ stationary (in time) implies $\{S_x\eta_t\}$ is stationary.

3. Suppose that λ is translation invariant, then for each t, $\eta_t^{1,\infty} \equiv \lim_{n\to\infty} \eta_t^{1,\infty,n}$ and $\eta_t^{2,\infty} \equiv \lim_{n\to\infty} \eta_t^{2,\infty,n}$ are spatially ergodic.

4. If λ satisfies the conditions $M < 1$, then the unique stationary distribution is spatially ergodic.

Future work:

1. Is the condition $M < 1$ necessary?

2. Can we work with variable death rate?