

Probabilistic Methods in Asymptotic Geometric Analysis.

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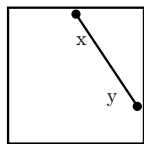
September 21st, 2016. Colmea. RJ

- 1 Origin
- 2 Normed Spaces
- 3 Distribution of volume of high dimensional convex bodies
- 4 Concentration of measure
- 5 Geometry of Log-Concave Functions

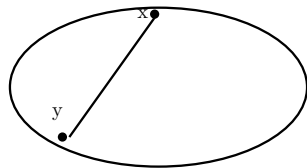
Asymptotic Geometric Analysis. Origin

Asymptotic Geometric Analysis has its origin in the interaction of Convex Geometry and Functional Analysis.

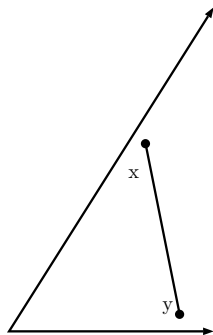
Convexity



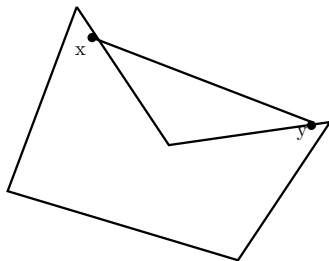
YES



YES



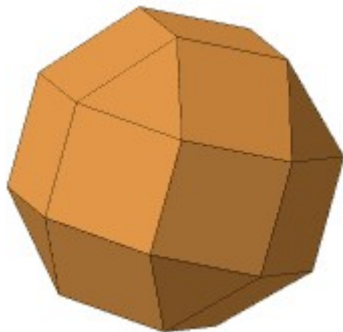
YES



NO!

Convex bodies

A convex body is a subset $K \subseteq \mathbb{R}^n$ which is convex, compact and has non-empty interior.



Classical Convex Geometry

The main interest lies in the study of the geometry of convex bodies and related geometric inequalities in Euclidean Space of fixed dimension.

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The Minkowski sum

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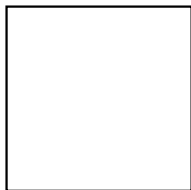
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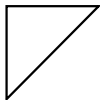
$$\begin{aligned} A + B &= \{x + y : x \in A, y \in B\} \\ &= \bigcup_{x \in A} (x + B) \\ &= \{x \in \mathbb{R}^n : A \cap (x - B) \neq \emptyset\}. \end{aligned}$$

The Minkowski sum

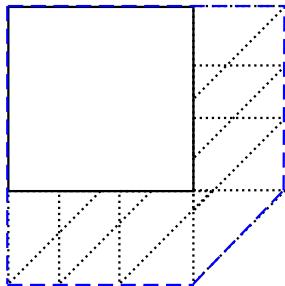
A



B



$A + B$



Brunn-Minkowski inequality

Brunn-Minkowski inequality (1887)

For any convex bodies K, L

$$|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}.$$

with equality if and only if K and L are homothetic.

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Equivalent statement:

Brunn-Minkowski inequality (1887)

For any $0 \leq \lambda \leq 1$ and any convex bodies K, L ,

$$|\lambda K + (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda}.$$

Functional Analysis

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Connection of Convex Geometry with normed spaces

Given a convex body $0 \in \text{int}(K)$ we define its Minkowski Functional denoted by $\|\cdot\|_K$ as

$$\|x\|_K = \min\{\lambda > 0 : x \in \lambda K\} \quad \forall x \in \mathbb{R}^n$$

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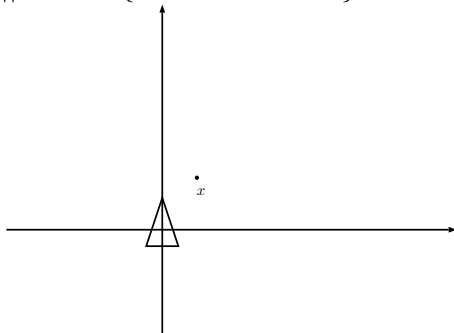
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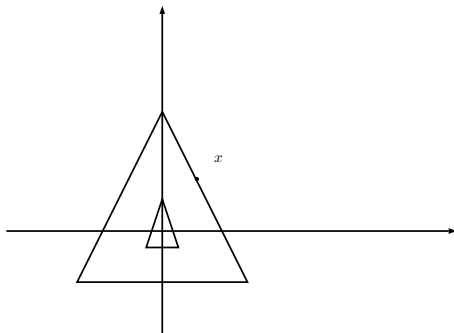
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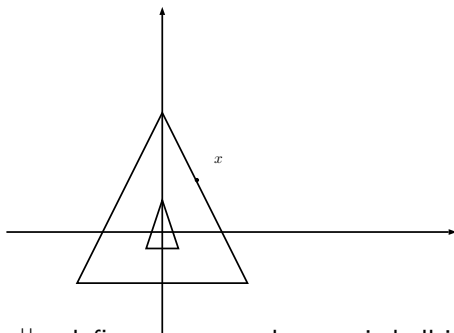
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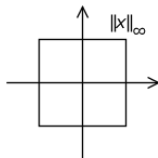
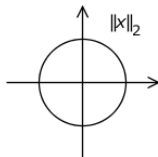
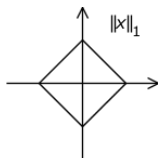
$$\|x\|_K = \min\{\lambda > 0 : x \in \lambda K\} \quad \forall x \in \mathbb{R}^n$$



If $K = -K$ then $\|\cdot\|_K$ defines a norm whose unit ball is precisely K . On the other hand, the unit ball of any normed space in \mathbb{R}^n is a centrally symmetric convex body

Some examples

- $\left(\mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i| \right)$
- $\left(\mathbb{R}^n, \|x\|_2 = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \right)$
- $\left(\mathbb{R}^n, \|x\|_\infty = \max_i |x_i| \right)$



Banach-Mazur distance

Definition

X, Y Banach Spaces. The Banach-Mazur distance between them is defined as

$$d_{\text{BM}}(X, Y) = \inf\{\|T\| \|T^{-1}\| : T : X \rightarrow Y \text{ is an isomorphism}\}$$

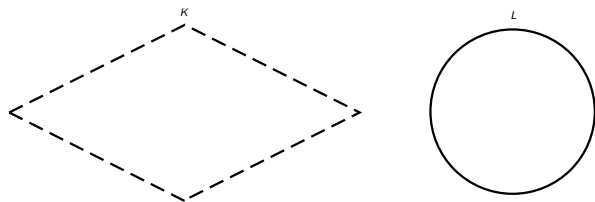
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Definition

$K = -K, L = -L \subset \mathbb{R}^n$ convex *o-symmetric* bodies.

$$d_{\text{BM}}(K, L) = \inf\{\lambda > 0 : K \subset T(L) \subset \lambda K\},$$

infimum: over all $T \in GL_n$.



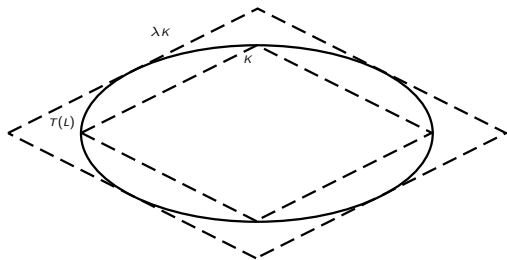
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(John)(1948) showed $d_{\text{BM}}(K, B_2^n) \leq \sqrt{n}$. As a consequence

$$d_{\text{BM}}(K, L) \leq n$$

Banach-Mazur distance

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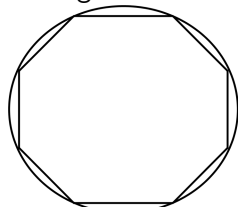
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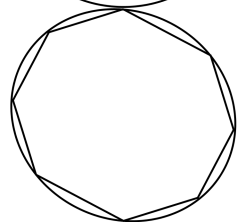
(Gluskin) There exists a universal constant $c > 0$ such that $\forall n$ we can always find $K, L \subset \mathbb{R}^n$ symmetric convex bodies with $d_{\text{BM}}(K, L) \geq cn$.

Idea

He constructed n -dimensional convex bodies for which $d_{\text{BM}}(K, L) \geq cn$ in the following way. Let $K = \text{conv}\{\mp k_i\}_{i=1}^n$ and $L = \text{conv}\{\mp l_i\}_{i=1}^n$ where every $k_i \in S^{n-1}$ is taken independently and uniformly with respect to the Lebesgue measure.



$$k_i \in S^{n-1}$$



$$l_i \in S^{n-1}$$

Hyperplane Conjecture

A convex body $K \subset \mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, it is centered (i.e. its barycenter is at the origin) and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, y \rangle^2 dx = L_K \|y\|_2^2,$$

for all $y \in \mathbb{R}^n$.

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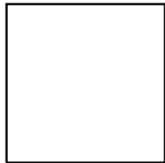
for all $y \in \mathbb{R}^n$. Equivalently,

$$\int_K \langle x, \theta \rangle^2 dx = L_K,$$

for all $\theta \in S^{n-1}$. The constant L_K is called isotropic constant of K .

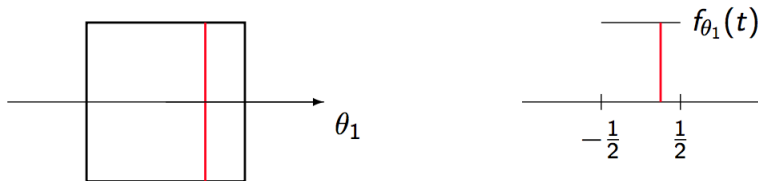
Hyperplane Conjecture

Given K we consider a random vector X uniformly distributed in K and, for every $\theta \in S^{n-1}$ the real random variable $\langle x, \theta \rangle$ with density $f_\theta(t) = |K \cap \theta^\perp + t\theta|$



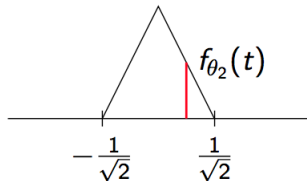
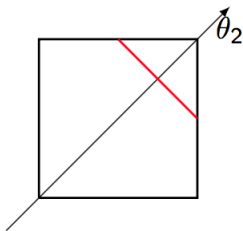
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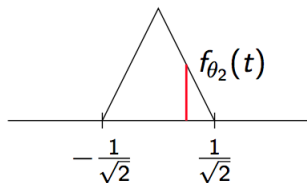
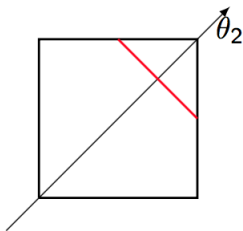
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The previous conjecture is equivalent with the following.

There exists an absolute constant $c > 0$ with the following property: for every $n \geq 1$ and every centered convex body K of volume 1 in \mathbb{R}^n there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^\perp| \geq c.$$

Hyperplane conjecture

This question is rather hard and the only successful approach uses random polytopes.

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Klartag and Kozma proved that if $N > n$ and if G_1, \dots, G_N are independent standard Gaussian random vectors in R^n , then the isotropic constant of the random polytopes $K_N := \text{conv}\{\pm G_1, \dots, \pm G_N\}$ and $C_N := \text{conv}\{G_1, \dots, G_N\}$ is bounded by an absolute constant $C > 0$ with probability greater than $1 - Ce^{-cn}$

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Other examples are when the vertices are uniformly distributed on the cube $[-1/2, 1/2]^n$ or on the Euclidean sphere S^{n-1} .

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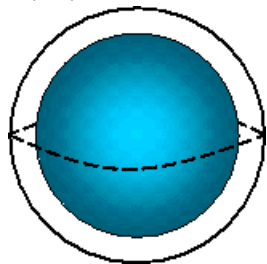
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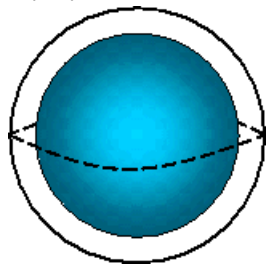
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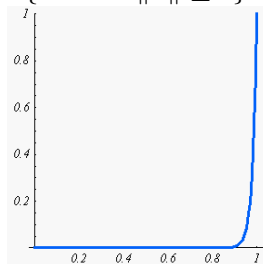
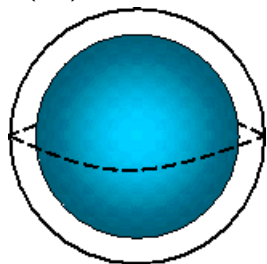
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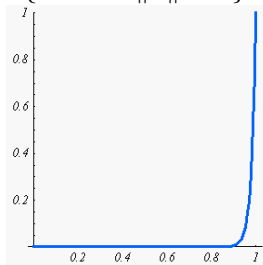
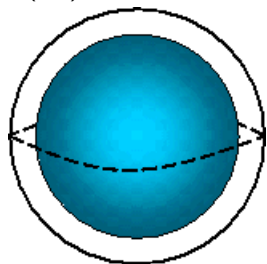
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Almost all near the surface!

The Euclidean Ball

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$$\int_{\mathbb{R}^n} f = n\omega_n \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta)r^{n-1}d\sigma(\theta)dr,$$

where in the previous integral we are normalizing by pulling the surface area on the sphere $n\omega_n$ so that $\sigma = \sigma_{n-1}$ is the rotation-invariant measure on S^{n-1} of total mass 1.

The Euclidean Ball

To find ω_n we integrate the function

$$x \mapsto \exp\left(-\frac{1}{2} \sum_1^n x_i^2\right)$$

with both sides of the equality.

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \prod_1^n e^{-x_i^2/2} dx = \prod_1^n \left(\int_{-\infty}^{\infty} e^{-x_i^2/2} dx_i \right) = (\sqrt{2\pi})^n$$

and

$$n\omega_n \int_0^{\infty} \int_{S^{n-1}} e^{-r^2/2} r^{n-1} d\sigma dr = \omega_n 2^{n/2} \Gamma\left(\frac{n}{2} + 1\right).$$

Hence

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}.$$

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So that ω_n is roughly $\left(\sqrt{\frac{2\pi e}{n}}\right)^n$, or equivalently, the Euclidean ball of volume 1 has radius about

$$\sqrt{\frac{n}{2\pi e}},$$

which is pretty big.

The Euclidean Ball

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The slice at a distance x from the center has volume

$$\sqrt{e} \left(1 - \frac{x^2}{r^2} \right)^{(n-1)/2}$$

Since r is roughly $\sqrt{n/(2\pi e)}$, we get that

$$\sqrt{e} \left(1 - \frac{2\pi e x^2}{n} \right)^{(n-1)/2} \approx \sqrt{e} \exp(-\pi e x^2).$$

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Let's note that the variance doesn't depend on n , whereas the radius of the ball of volume 1 grows like $\sqrt{n/(2\pi e)}$. So almost all the volume stays within a slab of fixed width.

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Therefore, if we project the mass distribution of the ball of volume 1 onto a single direction, we get a distribution that is approximately Gaussian with variance $1/(2\pi e)$.

Let's note that the variance doesn't depend on n , whereas the radius of the ball of volume 1 grows like $\sqrt{n/(2\pi e)}$. So almost all the volume stays within a slab of fixed width.

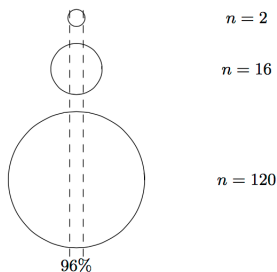


Figure 6. Balls in various dimensions, and the slab that contains about 96% of each of them.

For the n -dimensional cube

Let $K \subset \mathbb{R}^n$ a convex body with $0 \in \text{int}(K)$. If we denote by $r(\theta)$ the radius of K in the direction θ then the volume of K is

$$n\omega_n \int_{S^{n-1}} \int_0^{r(\theta)} s^{n-1} ds d\sigma = \omega_n \int_{S^{n-1}} r(\theta)^n d\sigma(\theta).$$

For the cube $K = [-1, 1]^n$ that has volume 2^n the equality above says that

$$\int_{S^{n-1}} r(\theta)^n = \frac{2^n}{\omega_n} \approx \left(\sqrt{\frac{2n}{\pi e}} \right)^n.$$

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For the ball of ℓ_1 (B_1^n)

$$\int_{S^{n-1}} r(\theta)^n = \frac{2^n}{n!\omega_n} \leq \left(\frac{2}{\sqrt{n}} \right)^n.$$

Volume distribution

The volume on high dimensional convex bodies concentrates in places our low dimensional intuition considers small!

Central Limit Theorem

[Klartag] Uniform distribution on high dimensional convex bodies has marginals that are approximately gaussian. Indeed, for an isotropic convex body $K \subset \mathbb{R}^n$ and a X uniformly distributed in K with some mild additional assumptions ($k \leq cn^\kappa$) we have that there exists $\mathcal{E} \subset G_{n,k}$ with $\sigma_{n,k}(\mathcal{E}) \geq 1 - e^{-c\sqrt{n}}$ such that for any $E \in \mathcal{E}$

$$\sup_{A \subseteq E} \left| \text{Prob}\{\text{Proj}_E(X) \in A\} - \frac{1}{(2\pi)^{k/2}} \int_A e^{-\frac{|x|^2}{2}} dx \right| \leq \frac{1}{n^\kappa},$$

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(Paouris) For vector a X as before it is also known that

$$\text{Prob}\{|X| \geq C\sqrt{n}\} \leq e^{-\sqrt{n}}$$

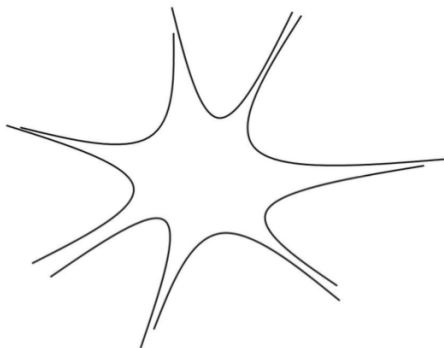
Volumetric shape of Convex bodies

A valid question seems to be: How do convex bodies in high dimension "look" like?

V. Milman's picture of high dimensional convex body

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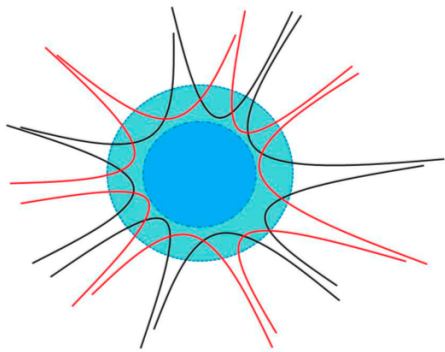
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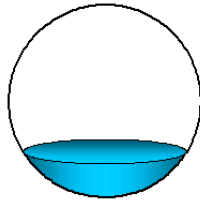
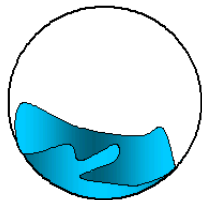
Isoperimetric Inequality [Lévy]

$$A \subset S^{n+1} \subset \mathbb{R}^{n+2}$$

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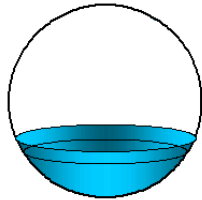
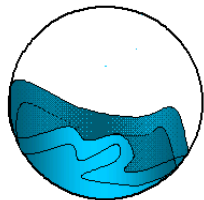
B spherical cap with $\mu_{n+1}(B) = \mu_{n+1}(A)$



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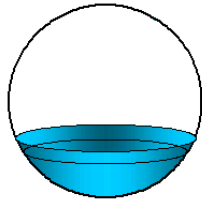
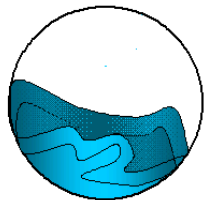
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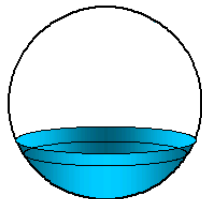
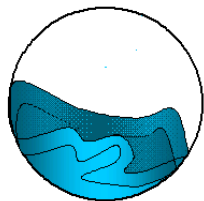


$$\mu_{n+1}(B_\varepsilon) \leq \mu_{n+1}(A_\varepsilon)$$

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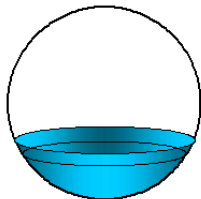
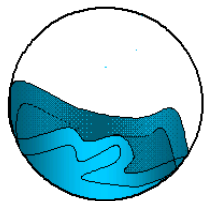
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General Problem

(Ω, d, μ) metric probability space. For $A \subset \Omega$ and $\epsilon > 0$, the ϵ -expansion of A is defined by

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f L -Lipschitz then

$$\mu\{|f - M_f| \geq \epsilon\} \leq 2\alpha(\epsilon/L),$$

where M_f denotes a median of f .

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$(X, \|\cdot\|)$ is **uniformly convex** if $\forall \varepsilon > 0, \delta(\varepsilon) > 0$

Result by Arias de Reyna, Ball, and Villa

- X n -dimensional uniformly convex normed space

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Geometric Brunn-Minkowski inequality

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Dvoretzky's Theorem

Theorem

Let $X = \mathbb{R}^n$, $\|\cdot\|$ be an n -dimensional normed space with unit ball K . Consider $M = \int_{S^{n-1}} \|x\| d\mu$, $b = \sup_{x \in S^{n-1}} \|x\|$. Then there exists a subspace E of dimension $k \approx c(\varepsilon)n \left(\frac{M}{b}\right)^2$ such that

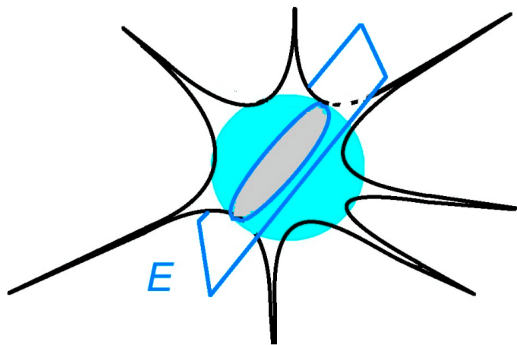
$$(1 - \varepsilon)M|x| \leq \|x\| \leq (1 + \varepsilon)M|x|$$

for all $x \in E$ or equivalently

$$\frac{1}{(1 + \varepsilon)M}(B_2^n \cap E) \subseteq K \cap E \subseteq \frac{1}{(1 - \varepsilon)M}(B_2^n \cap E)$$

which implies $d_{BM}(E, \ell_2^k) \leq \frac{1+\varepsilon}{1-\varepsilon}$.

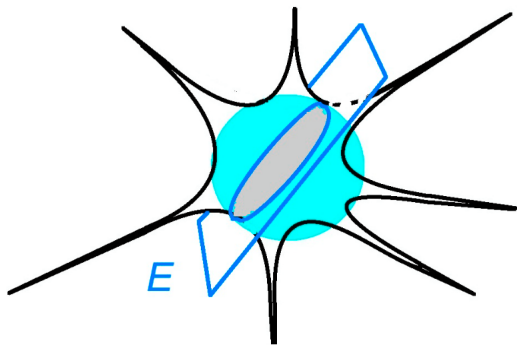
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Milman's picture of high dimensional convex body adapted by R. Vershynin

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Log-concave functions

$f : \mathbb{R}^n \rightarrow [0, \infty)$ is log-concave if

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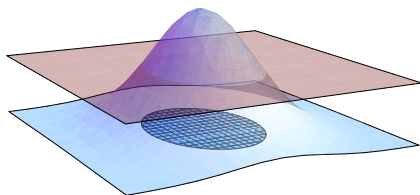
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- Level sets $\{x \in \text{supp } f : f(x) \geq t\}$ are convex.



Prékopa-Leindler inequality

Brunn-Minkowski inequality (1887)

For any $0 \leq \lambda \leq 1$

$$|\lambda K + (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda}.$$

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Prékopa-Leindler inequality (1971)

For any three integrable functions $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and $\lambda \in [0, 1]$ such that for any $x, y \in \mathbb{R}^n$

$$h(\lambda x + (1 - \lambda)y) \geq f^\lambda(x)g^{1-\lambda}(y)$$

we have

$$\int h(z)dz \geq \left(\int f(x)dx \right)^\lambda \left(\int g(y)dy \right)^{1-\lambda}.$$

Prékopa-Leindler inequality

If K, L convex bodies, taking $f = \chi_K$, $g = \chi_L$, $h = \chi_{\lambda K + (1-\lambda)L}$ we obtain

$$|\lambda K + (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda}.$$

Multiplicative Brunn-Minkowski inequality

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$$\chi_K * \chi_L(x) = |K \cap (x - L)|$$

Thank you!