Probabilistic Methods in Asymptotic Geometric Analysis.

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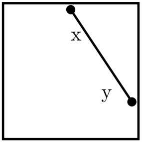
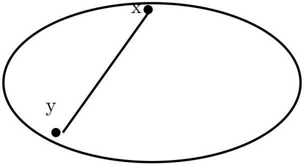
Asymptotic Geometric Analysis. Origin

Asymptotic Geometric Analysis has its origin in the interaction of Convex Geometry and Functional Analysis.

Convexity



YES



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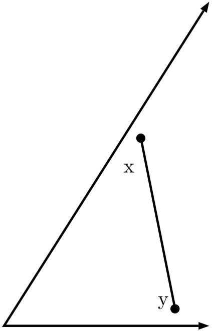

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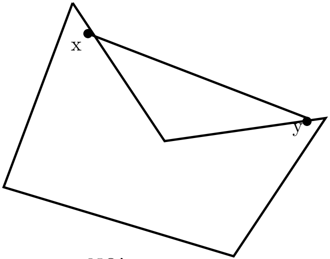
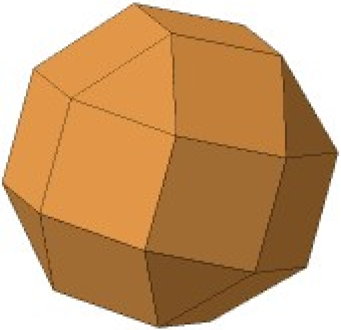


NO!

Convex bodies

A convex body is a subset $K \subseteq \mathbb{R}^n$ which is convex, compact and has non-empty interior.



Classical Convex Geometry

The main interest lies in the study of the geometry of convex bodies and related geometric inequalities in Euclidean Space of fixed dimension.

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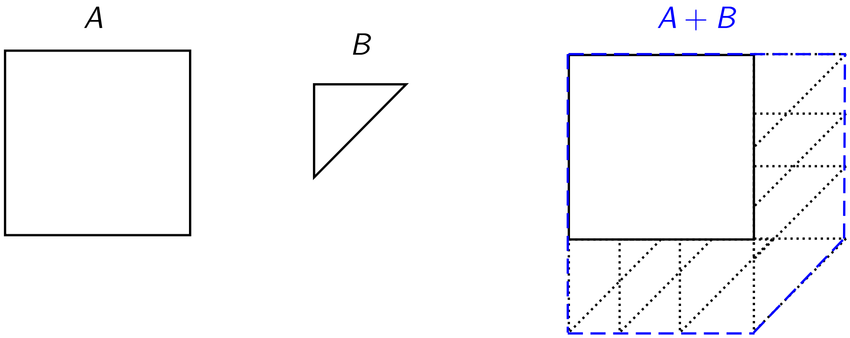

Brunn-Minkowski inequality

Brunn-Minkowski inequality (1887)

For any convex bodies K, L

$$|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}.$$

with equality if and only if K and L are homothetic.

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Equivalent statement:

Brunn-Minkowski inequality (1887)

For any $0 \leq \lambda \leq 1$ and any convex bodies K, L,

$$|\lambda K + (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}.$$

Functional Analysis

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Connection of Convex Geometry with normed spaces

Given a convex body $0 \in int(K)$ we define its Minkowski Functional denoted by $||\cdot||_K$ as

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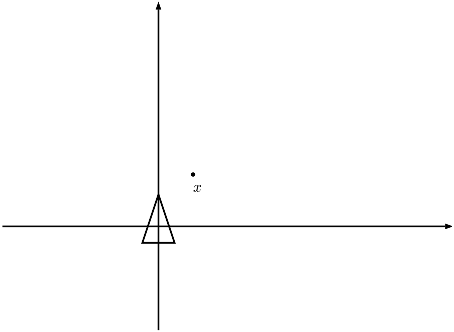

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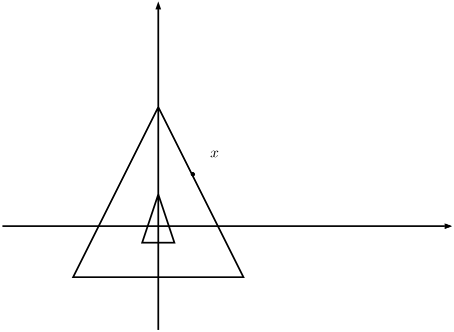

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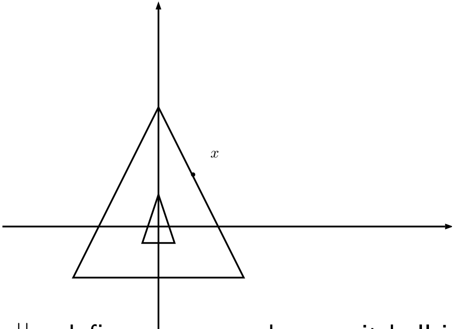
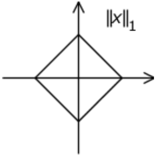


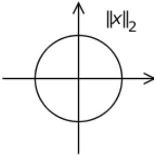
If $K = -K$ then $||\cdot||_K$ defines a norm whose unit ball is precisely K.On the other hand, the unit ball of any normed space in $\mathbb{R}^n$ is a centrally symmetric convex body

Some examples

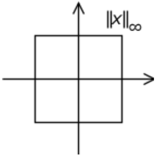
- $\left(\mathbb{R}^n, ||x||_1 = \sum_{i=1}^{n} |x_i|\right)$



- $\left(\mathbb{R}^n, ||x||_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\right)$



- $\left(\mathbb{R}^n, ||x||_\infty = \max_i |x_i|\right)$



Banach-Mazur distance

Definition

X, Y Banach Spaces. The Banach-Mazur distance between them is defined as

$$d_{\mathrm{BM}}(X, Y) = \inf\{\|T\|\|T^{-1}\| : T : X \to Y \text{ is an isomorphism}\}$$

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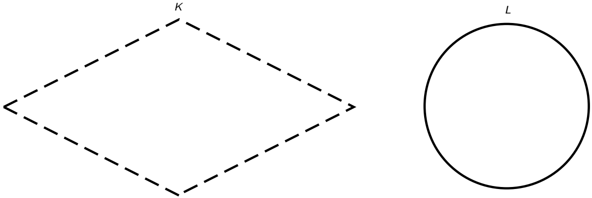
$K = -K, L = -L \subset \mathbb{R}^n$ convex *o-symmetric* bodies.

$$d_{\mathrm{BM}}(K, L) = \inf\{\lambda > 0 : \; K \subset T(L) \subset \lambda K\},$$

infimum: over all $T \in GL_n$.



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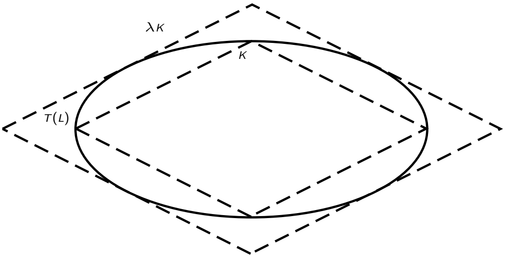

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(John)(1948) showed $d_{\mathrm{BM}}(K, B_2^n) \leq \sqrt{n}$. As a consequence

$$d_{\mathrm{BM}}(K, L) \leq n$$

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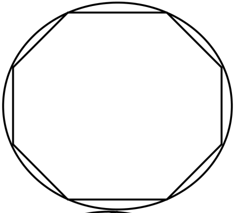
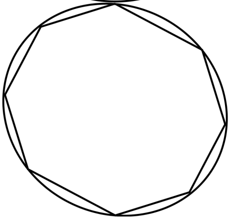
$d_{\mathrm{BM}}(B_2^n, B_1^n) = d_{\mathrm{BM}}(B_2^n, B_\infty^n) = \sqrt{n}$
(Gluskin) There exists a universal constant $c > 0$ such that $\forall n$ we can always find $K, L \subset \mathbb{R}^n$ symmetric convex bodies with $d_{\mathrm{BM}}(K, L) \geq cn$.

Idea

He constructed n-dimensional convex bodies for which $d_{\mathrm{BM}}(K, L) \geq cn$ in the following way. Let $K = conv\{\mp k_i\}_{i=1}^{n}$ and $L = conv\{\mp l_i\}_{i=1}^{n}$ where every $k_i \in S^{n-1}$ is taken independently and uniformly with respect to the Lebesgue measure.



$k_i \in S^{n-1}$



$l_i \in S^{n-1}$

Hyperplane Conjecture

A convex body $K \subset \mathbb{R}^n$ is called isotropic if it has volume $|K| = 1$, it is centered (i.e. its barycenter is at the origin) and there exists a constant $L_K > 0$ such that

$$\int_K \langle x, y \rangle^2 dx = L_K \|y\|_2^2,$$

for all $y \in \mathbb{R}^n$.

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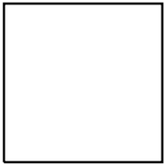
for all $y \in \mathbb{R}^n$. Equivalently,

$$\int_K \langle x, \theta \rangle^2 dx = L_K,$$

for all $\theta \in S^{n-1}$. The constant L_K is called isotropic constant of K.

Hyperplane Conjecture

Given K we consider a random vector X uniformly distributed in K and, for every $\theta \in S^{n-1}$ the real random variable $\langle x, \theta \rangle$ with density $f_\theta(t) = |K \cap \theta^\perp + t\theta|$



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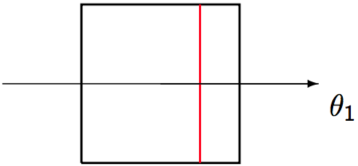

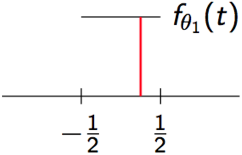

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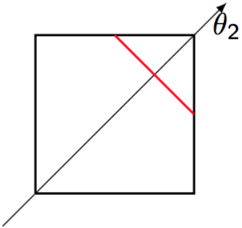

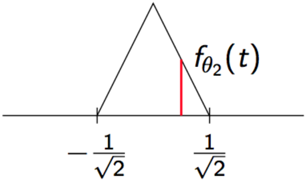

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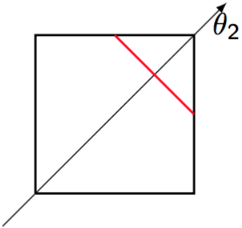

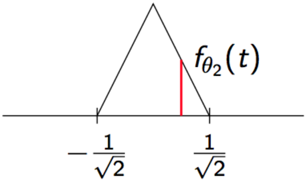

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The previous conjecture is equivalent with the following.
There exists an absolute constant $c > 0$ with the following property: for every $n \geq 1$ and every centered convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^{\perp}| \geq c.$$

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Klartag and Kozma proved that if $N > n$ and if $G_1, ..., G_N$ are independent standard Gaussian random vectors in R^n, then the isotropic constant of the random polytopes $K_N := conv\{\pm G_1, ..., \pm G_N\}$ and $C_N := conv\{G_1, ..., G_N\}$ is bounded by an absolute constant $C > 0$ with probability greater than $1 - Ce^{-cn}$

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Other examples are when the vertices are uniformly distributed on the cube $[-1/2, 1/2]^n$ or on the Euclidean sphere S^{n-1}.

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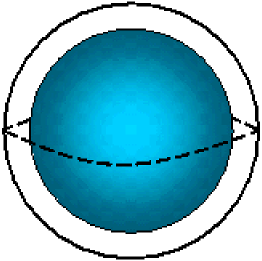

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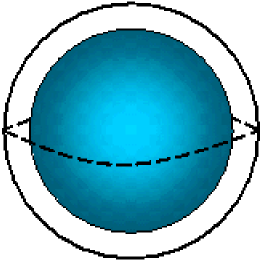

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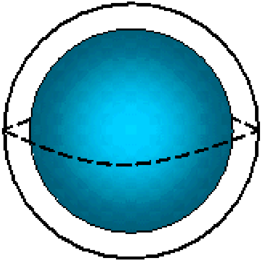

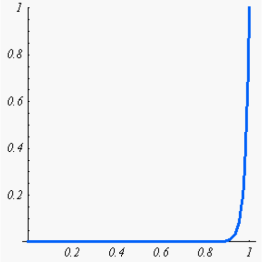

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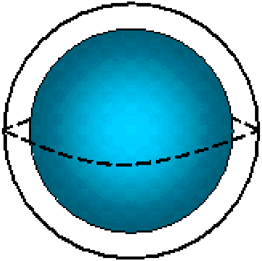

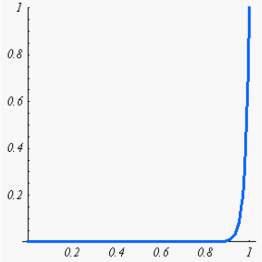

Almost all near the surface!

The Euclidean Ball

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To compute the volume of Euclidean unit ball B_2^n we use polar coordinates. We can write the integral of a function on $\mathbb{R}^n$ as

$$\int_{\mathbb{R}^n} f = n\omega_n \int_{r=0}^{\infty} \int_{S^{n-1}} f(r\theta) r^{n-1} d\sigma(\theta) dr,$$

where in the previous integral we are normalizing by pulling the surface area on the sphere $n\omega_n$ so that $\sigma = \sigma_{n-1}$ is the rotation-invariant measure on S^{n-1} of total mass 1.

The Euclidean Ball

To find ω_n we integrate the function

$$x \mapsto \exp(-\frac{1}{2}\sum_1^n x^2)$$

with both sides of the equality.

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} \prod_1^n e^{-xi^2/2} dx = \prod_1^n \left(\int_{-\infty}^{\infty} e^{-xi^2/2} dx_i \right) = (\sqrt{2\pi})^n$$

and

$$n\omega_n \int_0^\infty \int_{S^{n-1}} e^{-r^2/2} r^{n-1} d\sigma dr = \omega_n 2^{n/2} \Gamma\left(\frac{n}{2}+1\right).$$

Hence

$$\omega_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

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$$\Gamma\left(\frac{n}{2}+1\right) \sim \sqrt{2\pi}e^{-n/2}\left(\frac{n}{2}\right)^{(n+1)/2}$$

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So that ω_n is roughly $\left(\sqrt{\frac{2\pi e}{n}}\right)^n$,or equivalently, the Euclidean ball of volume 1 has radius about

$$\sqrt{\frac{n}{2\pi e}},$$

which is pretty big.

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The slice is an $(n-1)$-dimensional ball of this radius, so its volume is

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The slice at a distance x from the center has volume

$$\sqrt{e} \left(1 - \frac{x^2}{r^2} \right)^{(n-1)/2}$$

Since r is roughly $\sqrt{n/(2\pi e)}$, we get that

$$\sqrt{e} \left(1 - \frac{2\pi e x^2}{n} \right)^{(n-1)/2} \approx \sqrt{e} \exp(-\pi e x^2).$$

The Euclidean ball

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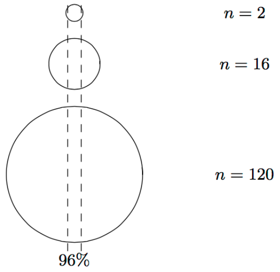

Figure 6. Balls in various dimensions, and the slab that contains about 96% of each of them.

Picture taken from book flavors in geometry by K. Ball

For the n-dimensional cube

Let $K \subset \mathbb{R}^n$ a convex body with $0 \in int(K)$. If we denote by $r(\theta)$ the radius of K in the direction θ then the volume of K is

$$n\omega_n \int_{S^{n-1}} \int_0^{r(\theta)} s^{n-1} ds d\sigma = \omega_n \int_{S^{n-1}} r(\theta)^n d\sigma(\theta).$$

For the cube $K = [-1, 1]^n$ that has volume 2^n the equality above says that

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For the ball of ℓ_1 (B_1^n)

$$\int_{S^{n-1}} r(\theta)^n = \frac{2^n}{n!\omega_n} \leq \left(\frac{2}{\sqrt{n}}\right)^n.$$

Volume distribution

The volume on high dimensional convex bodies concentrates in places our low dimensional intuition considers small!

Central Limit Theorem

[Klartag] Uniform distribution on high dimensional convex bodies has marginals that are approximately gaussian. Indeed, for an isotropic convex body $K \subset \mathbb{R}^n$ and a X uniformly distributed in K with some mild additional assumptions ($k \leq cn^{\kappa}$) we have that there exists $\mathcal{E} \subset G_{n,k}$ with $\sigma_{n,k}(\mathcal{E}) \geq 1 - e^{-c\sqrt{n}}$ such that for any $E \in \mathcal{E}$

$$\sup_{A \subseteq E} \left| Prob\{Proj_E(X) \in A\} - \frac{1}{(2\pi)^{k/2}} \int_A e^{\frac{-|x|}{2}} dx \right| \leq \frac{1}{n^{\kappa}},$$

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(Paouris) For vector a X as before it is also known that

$$Prob\{|X| \geq C\sqrt{n}\} \leq e^{-\sqrt{n}}$$

Volumetric shape of Convex bodies

A valid question seems to be: How do convex bodies in high dimension "look" like?

V. Milman's picture of high dimensional convex body

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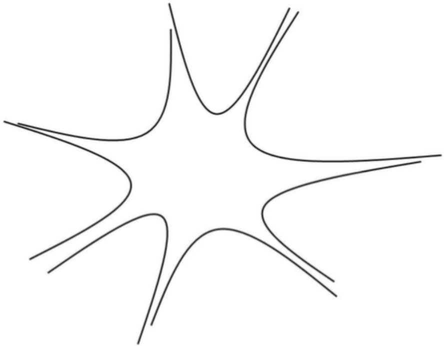

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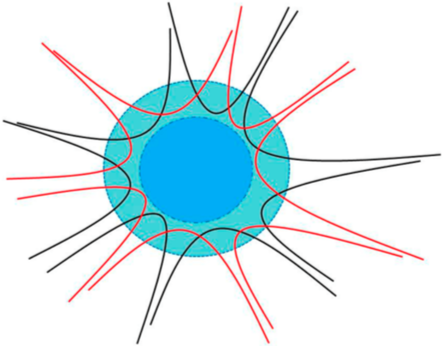

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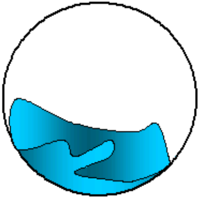
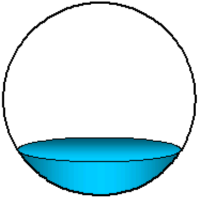


B spherical cap with $\mu_{n+1}(B) = \mu_{n+1}(A)$



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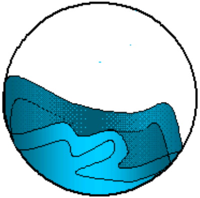

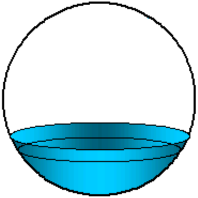

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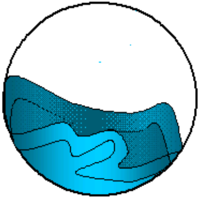

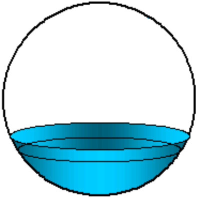

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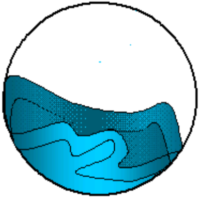

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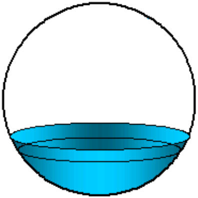

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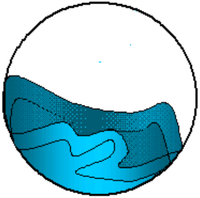

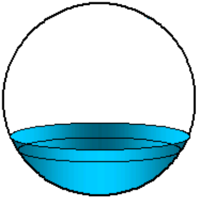

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General Problem

(Ω, d, μ) metric probability space. For $A \subset \Omega$ and $\epsilon > 0$, the *ϵ-expansion of A* is defined by

$$A_\epsilon^d = \{x \in \Omega : d(x, a) \leq \epsilon \text{ for some } a \in A\}.$$

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f L-Lipschitz then

$$\mu\{|f - M_f| \geq \varepsilon\} \leq 2\alpha(\varepsilon/L),$$

where M_f denotes a median of f.

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Result by Arias de Reyna, Ball, and Villa

- X n-dimensional uniformly convex normed space

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Proof using B-M by A-B-V

Geometric Brunn-Minkowski inequality

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Dvoretzky's Theorem

Theorem

Let $X = \mathbb{R}^n, \|\cdot\|$ *be an n-dimensional normed space with unit ball* K*. Consider* $M = \int_{S^{n-1}} \|x\| d\mu$, $b = \sup_{x \in S^{n-1}} \|x\|$. *Then there exists a subspace* E *of dimension* $k \approx c(\varepsilon) n \left(\frac{M}{b}\right)^2$ *such that*

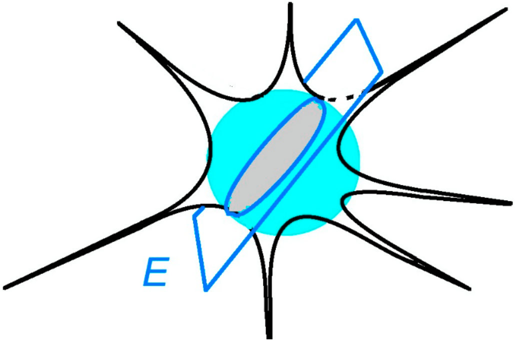
$$(1-\varepsilon)M|x| \leq \|x\| \leq (1+\varepsilon)M|x|$$

for all $x \in E$ *or equivalently*

$$\frac{1}{(1+\varepsilon)M}(B_2^n \cap E) \subseteq K \cap E \subseteq \frac{1}{(1-\varepsilon)M}(B_2^n \cap E)$$

which implies $d_{BM}(E, \ell_2^k) \leq \frac{1+\varepsilon}{1-\varepsilon}$.

Dvoretzky's Theorem



V.

Milman's picture of high dimensional convex body adapted by R. Vershynin

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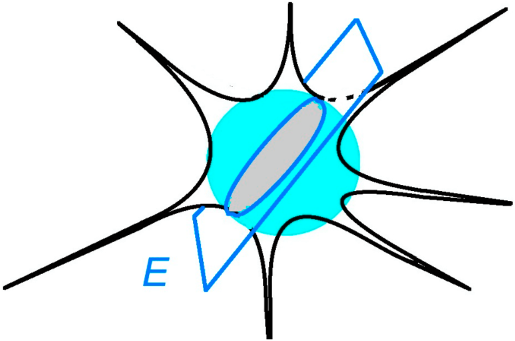

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Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

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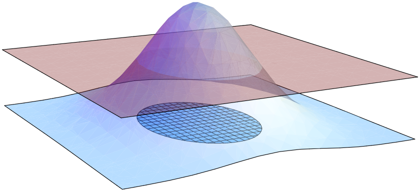

Prékopa-Leindler inequality

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For any $0 \leq \lambda \leq 1$

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Prékopa-Leindler inequality (1971)

For any three integrable functions $f, g, h : \mathbb{R}^n \to \mathbb{R}_+$ and $\lambda \in [0,1]$ such that for any $x, y \in \mathbb{R}^n$

$$h(\lambda x + (1-\lambda)y) \geq f^{\lambda}(x) g^{1-\lambda}(y)$$

we have

$$\int h(z)dz \geq \left(\int f(x)dx\right)^{\lambda} \left(\int g(y)dy\right)^{1-\lambda}.$$

Prékopa-Leindler inequality

If K, L convex bodies, taking $f = \chi_K$, $g = \chi_L$, $h = \chi_{\lambda K+(1-\lambda)L}$ we obtain

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Multiplicative Brunn-Minkowski inequality

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Thank you!