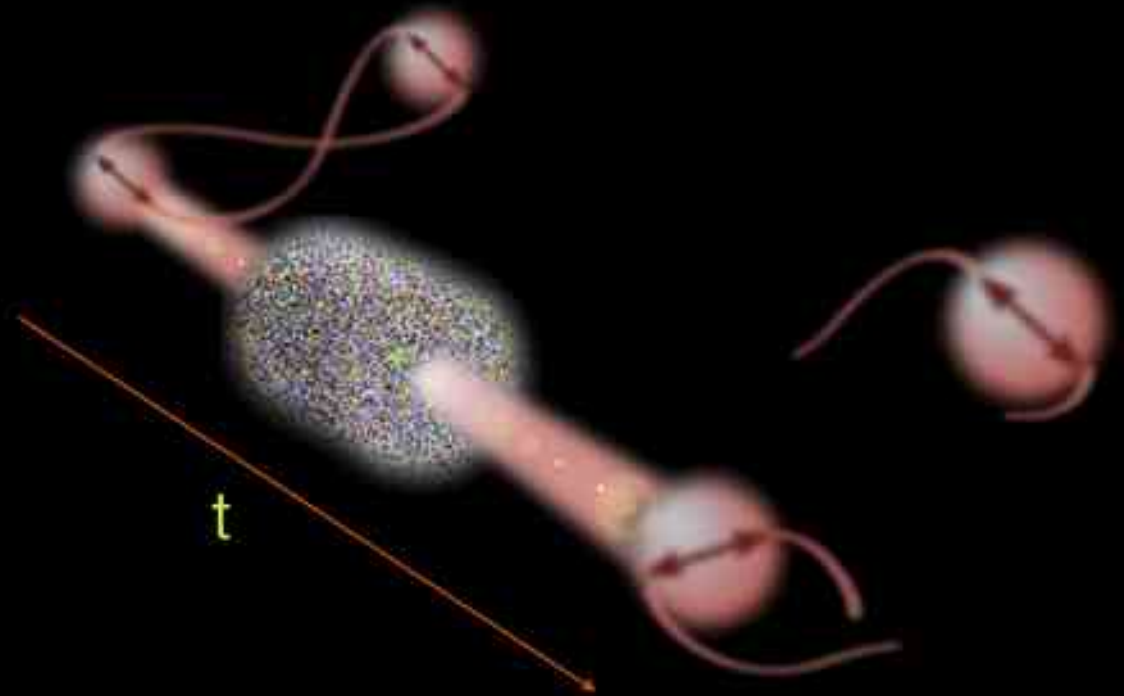




CBPF

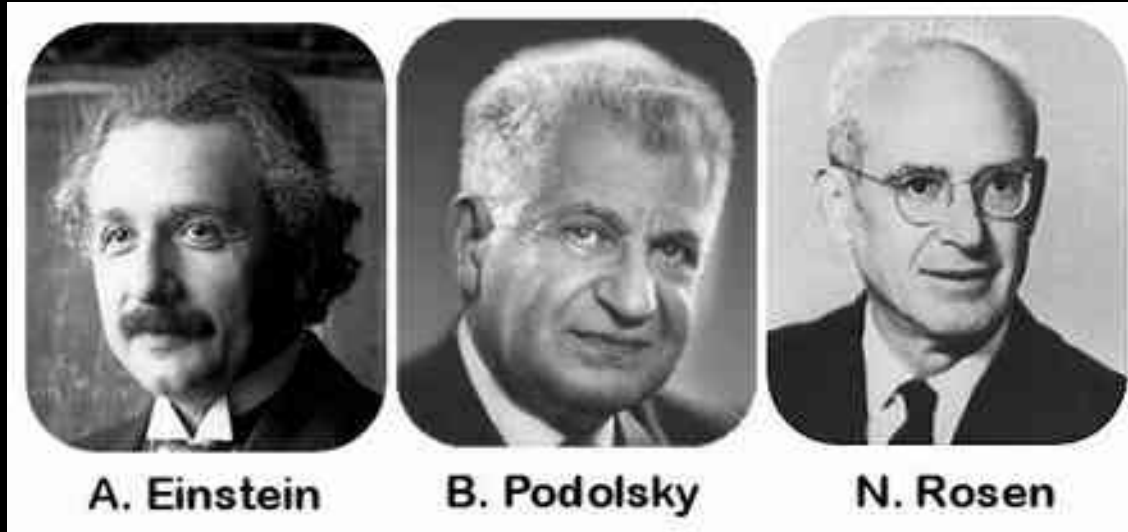
Centro Brasileiro de
Pesquisas Físicas

Open system dynamics of entanglement



Fernando de Melo
qig@CBPF

Entanglement...



Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777 (1935)

... the way of “paradoxes”

Best possible knowledge of a whole
does not include best possible
knowledge of its parts — and that is
what keeps coming back to haunt us

Erwin Schrödinger (1935)

Entanglement...



“On the Einstein-Podolsky-Rosen Paradox”
Physics 1, 195 (1964)

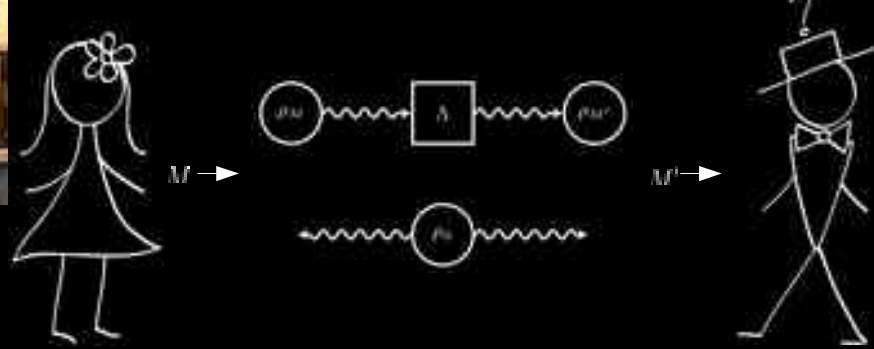
... correlation with physical consequences

Entanglement...



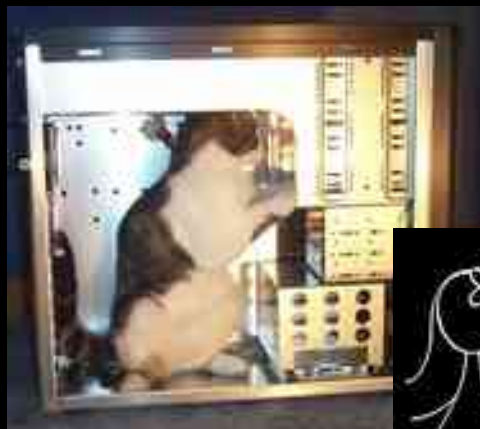
... a resource for quantum information

Entanglement...

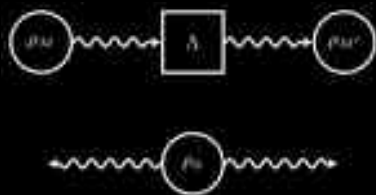


... a resource for quantum information

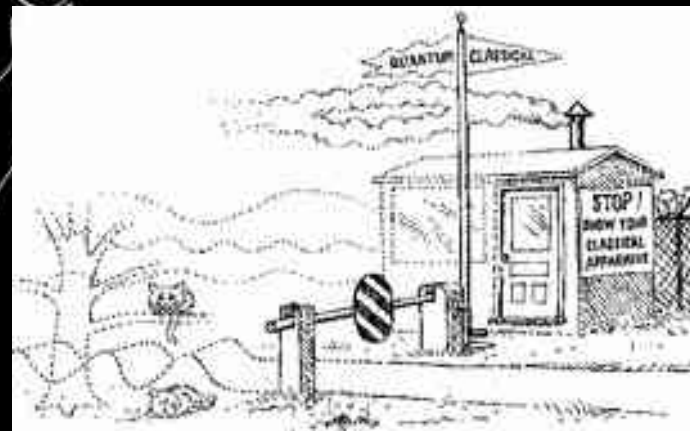
Entanglement...



$M \rightarrow$

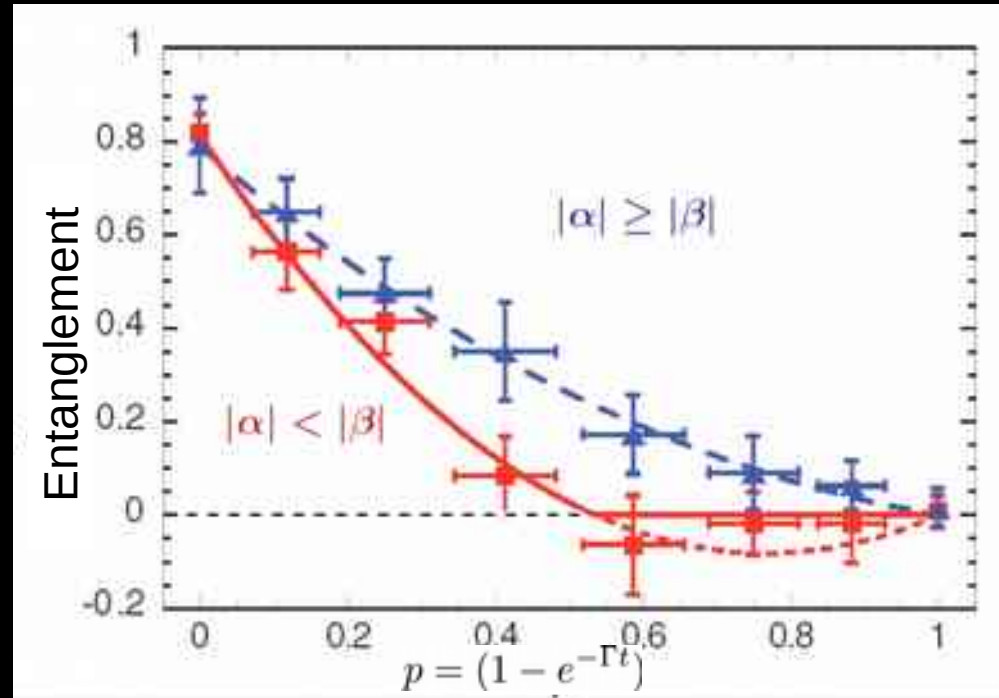


$M' \rightarrow$



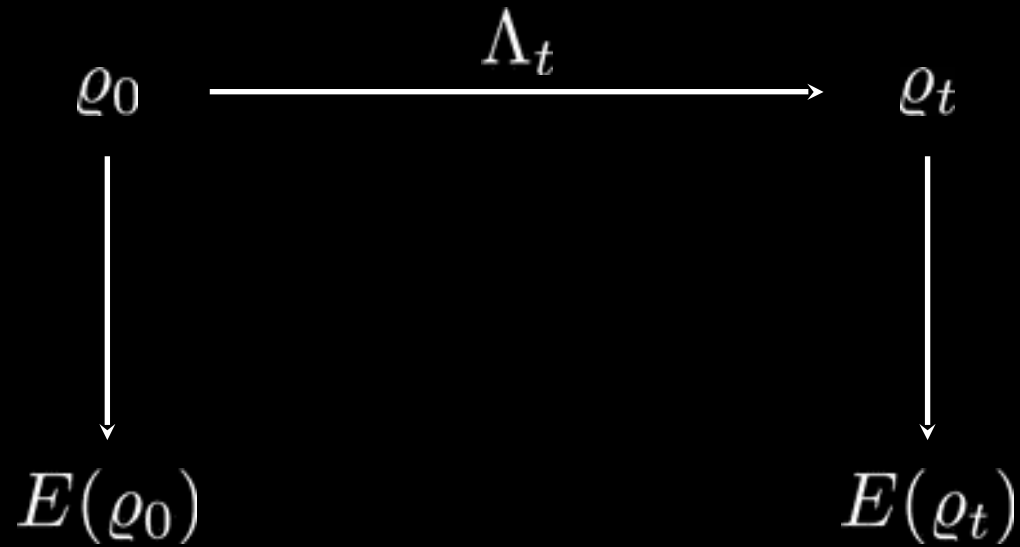
... a resource for quantum information

Caveat: Entanglement is fragile

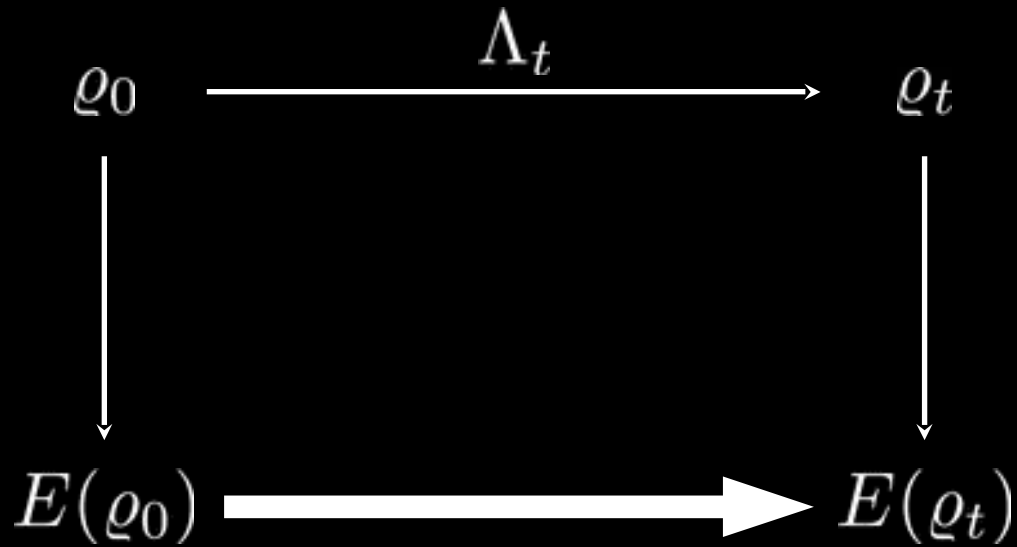


M.P Almeida, FdM, M. Hor-Meyll, A. Salles,
S.P. Walborn, P.H. Ribeiro, L. Davidovich
Science 316, 579 (2007)

Entanglement open system dynamics



Entanglement open system dynamics



Outline

- Quantum mechanics in a nutshell
- Entanglement: definition and quantification
- Entanglement dynamics
 - Deterministic equation of motion
 - Statistical approach: a universal behavior
- Conclusions and open questions



Quantum mechanics in a nutshell

- States
- State space
- Dynamics

1st postulate: To every physical system is associated a Hilbert space \mathcal{H} . The state of a quantum system is described by a unit vector $|\psi\rangle \in \mathcal{H}$.

Given an orthonormal basis $\{|e_i\rangle\}_{i=0}^{d-1} \in \mathbb{C}^d$
then

$$|\psi\rangle = \sum_{i=0}^{d-1} c_i |e_i\rangle$$

Constraints:

Normalization:
$$\sum_{i=0}^{d-1} |c_i|^2 = 1$$

Modulo global phase: $c_0 \in \mathbb{R}$

State space of $|\psi\rangle$'s : $2d-2$ sphere



Consider the case where the state of the system is not exactly know, but it is in the state $|\psi_i\rangle$ with probability p_i .

The expectation value of an observable \mathcal{A} (Hermitian linear operator), is given by:

$$\langle \mathcal{A} \rangle = \sum_i p_i \langle \psi_i | (\mathcal{A} | \psi_i \rangle)$$

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$$\begin{aligned}\langle \mathcal{A} \rangle &= \sum_i p_i \langle \psi_i | (\mathcal{A} | \psi_i \rangle) \\ &= \text{Tr} \left[\underbrace{\left(\sum_i p_i | \psi_i \rangle \langle \psi_i | \right)}_{\rho} \mathcal{A} \right]\end{aligned}$$

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Properties of the density matrix:

- linear operator

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$$

- trace one

$$\text{Tr } \rho = \sum_i p_i \text{Tr } |\psi_i\rangle \langle \psi_i| = \sum_i p_i = 1$$

- Hermitian

$$\rho^\dagger = \sum_i p_i^* (|\psi\rangle \langle \psi|)^\dagger = \rho$$

- positive semi-definite $\forall |\chi\rangle \in \mathcal{H} \quad \langle \chi | \rho | \chi \rangle \geq 0$

1st postulate reloaded: To every physical system is associated a Hilbert space \mathcal{H} . The state of a quantum system is described by a density matrix $\rho \in \mathcal{D}(\mathcal{H})$

The previous postulate is then a special case with the identification $\rho_\psi = |\psi\rangle\langle\psi| \rightarrow |\psi\rangle$

State space of ρ 's : convex set

If ρ_1 and ρ_2 are density matrices, then

$$\lambda\rho_1 + (1 - \lambda)\rho_2$$

with $\lambda \in [0, 1]$, is also a density matrix.

State space of ρ 's : convex set

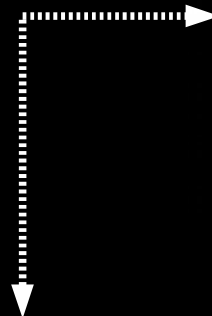
Inside: full rank states



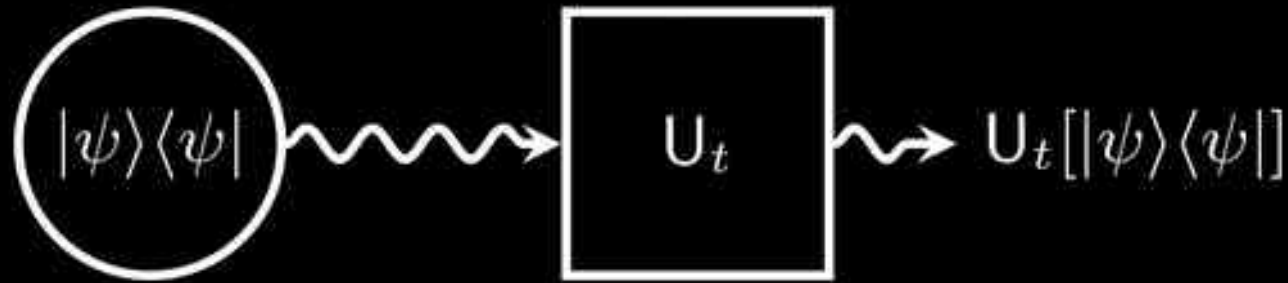
Extreme points: pure states $|\psi\rangle\langle\psi|$



At the facets: non-full rank states



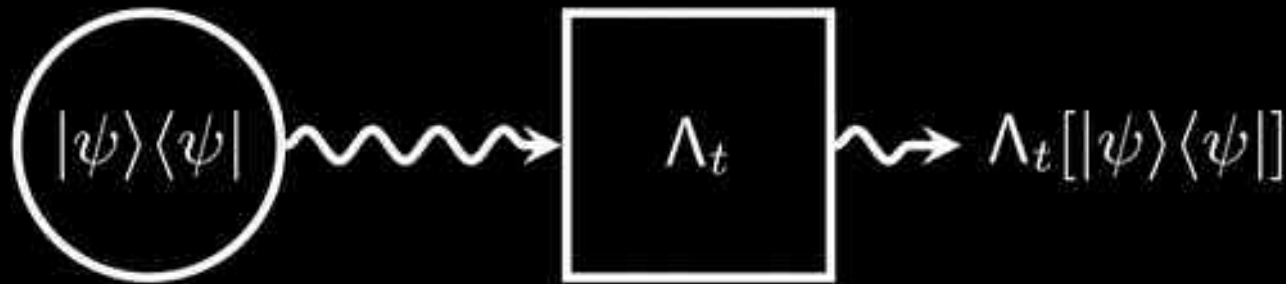
Dynamics: Closed system / Noiseless



Where $U_t[|\psi\rangle\langle\psi|] := U_t|\psi\rangle\langle\psi|U_t^\dagger$, and $U_t^\dagger U_t = \mathbb{1}$

Pure states remain pure – no information is lost

Dynamics: Open system / Noisy



Where $\Lambda_t[|\psi\rangle\langle\psi|] := \sum_i K_t^{(i)} |\psi\rangle\langle\psi| K_t^{(i)\dagger}$ and $\sum_i K_t^{(i)\dagger} K_t^{(i)} = \mathbb{1}$

Pure states might become mixed (not rank 1) –
information might be lost

Dynamics: Open system / Noisy

The linear map $\Lambda_t : \mathcal{D}(\mathcal{H}) \mapsto \mathcal{D}(\mathcal{H})$

- preserves the trace
- preserves Hermiticity
- preserves positive semi-definiteness
(more than that, it is completely positive)

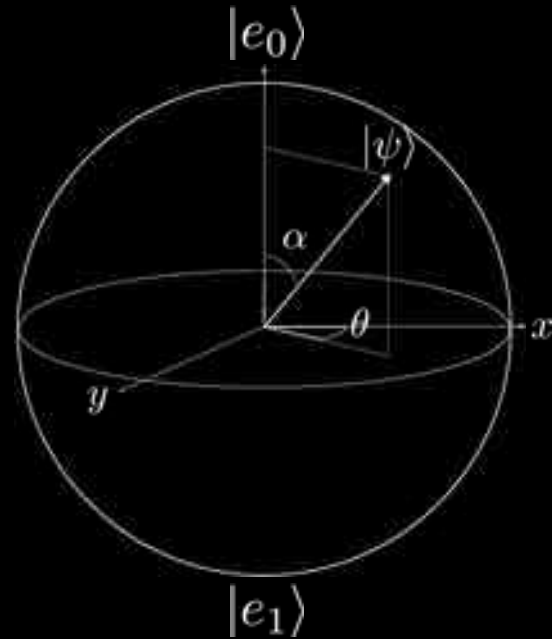
Example: 1 qubit dephasing

Consider the state,

$$|\psi\rangle = \cos \frac{\alpha}{2} |e_0\rangle + e^{i\theta} \sin \frac{\alpha}{2} |e_1\rangle$$

Its density matrix reads

$$\rho_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} \cos^2 \frac{\alpha}{2} & e^{-i\theta} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \\ e^{i\theta} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & \sin^2 \frac{\alpha}{2} \end{pmatrix}$$



Example: 1 qubit dephasing

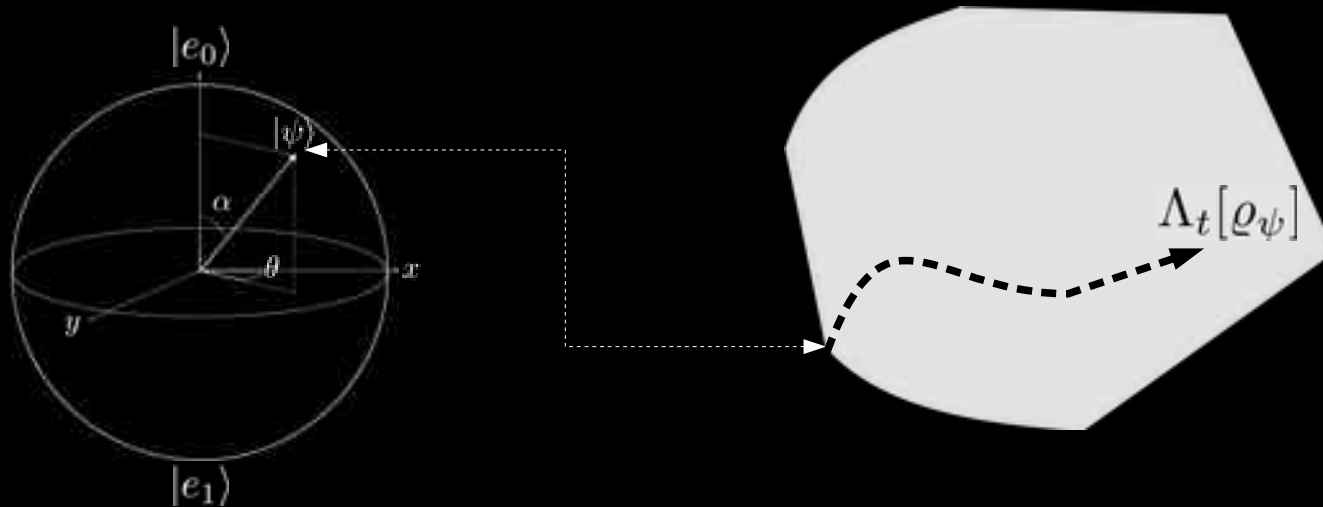
Now consider the map Λ_t with

$$K_t^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\Gamma t/2} \end{pmatrix} \quad K_t^{(1)} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - e^{-\Gamma t/2}} \end{pmatrix}$$

with $\Gamma > 0$ is a decay rate

The dynamics is then

$$\Lambda_t[\varrho_\psi] = \sum_i K_t^{(i)} |\psi\rangle\langle\psi| K_t^{(i)\dagger} = \begin{pmatrix} \cos^2 \frac{\alpha}{2} & e^{-\Gamma t/2} e^{-i\theta} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \\ e^{-\Gamma t/2} e^{i\theta} \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} & \sin^2 \frac{\alpha}{2} \end{pmatrix}$$



Entanglement

An abstract visualization of quantum entanglement. Two large, glowing spheres are positioned on the left and right. The left sphere is primarily green and yellow, while the right sphere is primarily blue and cyan. They are connected by a dense network of thin, multi-colored lines (green, blue, red, yellow) that radiate outwards and connect the two spheres. In the center, there is a bright, multi-colored point of light. The overall background is dark with a complex, web-like pattern of these same colors.

- Definition
- Measures

Consider now a system composed of two subsystems, say A and B. It is assigned to it a composed Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

But there exist states

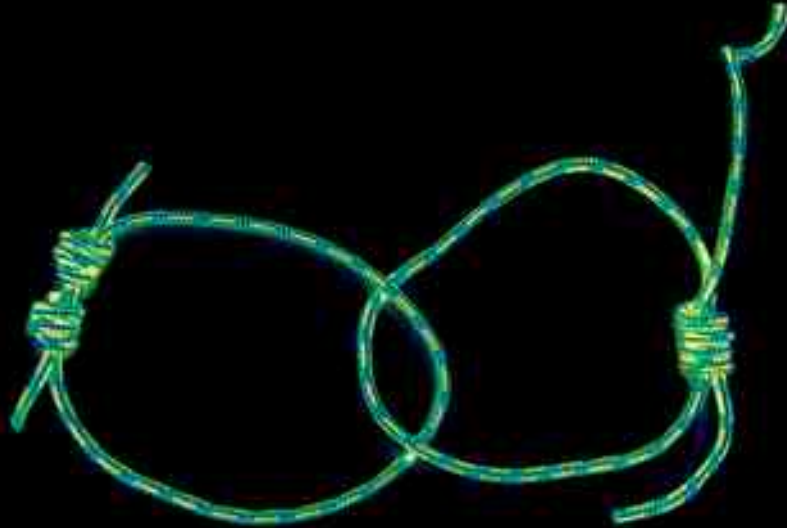
$$|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

for which there exist **NO** $|\phi\rangle \in \mathcal{H}_A$
and $|\chi\rangle \in \mathcal{H}_B$, such that

$$|\psi\rangle \stackrel{!}{=} |\phi\rangle \otimes |\chi\rangle$$

These are the entangled (pure) states

Bipartite entangled state



$$|\psi\rangle = \frac{1}{\sqrt{2}}(|e_0\rangle \otimes |e_0\rangle + |e_1\rangle \otimes |e_1\rangle)$$

Tripartite entangled states

Cluster state



$$\frac{1}{2} (|e_0e_0e_0\rangle + |e_0e_0e_1\rangle + |e_0e_1e_0\rangle - |e_0e_1e_1\rangle + |e_1e_0e_0\rangle + |e_1e_0e_1\rangle - |e_1e_1e_0\rangle + |e_1e_1e_1\rangle)$$

GHZ state

$$\frac{1}{\sqrt{2}} (|e_0e_0e_0\rangle + |e_1e_1e_1\rangle)$$



Entanglement definition

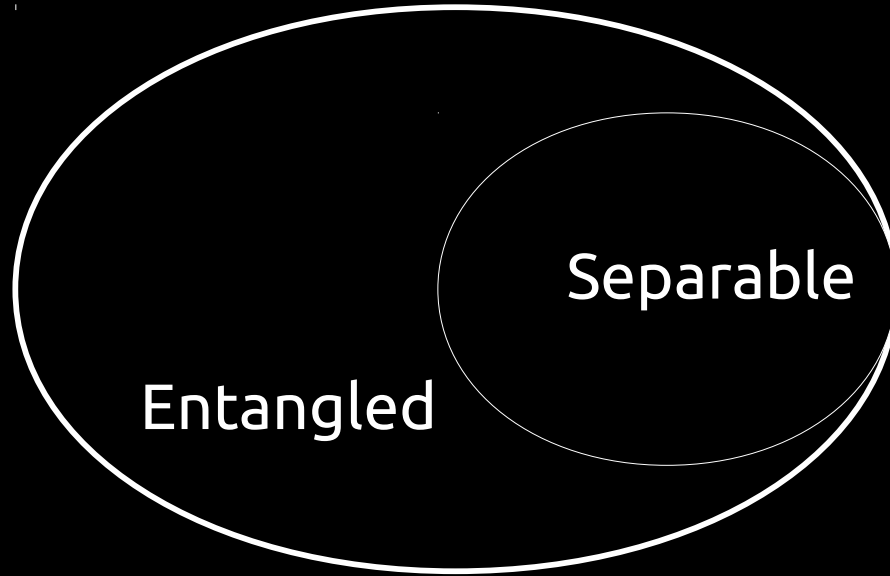
A multipartite state $\rho \in \mathcal{D}(\mathcal{H}_A) \otimes \mathcal{D}(\mathcal{H}_B) \otimes \cdots \otimes \mathcal{D}(\mathcal{H}_N)$ is **separable** if it can be written as

$$\rho = \sum_i p_i |\psi_A^{(i)}\rangle\langle\psi_A^{(i)}| \otimes |\psi_B^{(i)}\rangle\langle\psi_B^{(i)}| \otimes \cdots \otimes |\psi_N^{(i)}\rangle\langle\psi_N^{(i)}|$$

with $p_i \geq 0$ and $\sum_i p_i = 1$

The state is **entangled** otherwise

Entanglement in the space of states



Watch out: This is not a Venn diagram!

Entanglement measures

Let $E : \mathcal{D}(\mathcal{H}_A) \otimes \cdots \otimes \mathcal{D}(\mathcal{H}_N) \mapsto [0, 1]$ be a tentative entanglement measure. It then must satisfy:

- $E(\varrho) = 0$ iff ϱ is separable
- $E(\lambda\varrho + (1 - \lambda)\varrho') \leq \lambda E(\varrho) + (1 - \lambda)E(\varrho')$, $\forall \lambda \in [0, 1]$
- $|E(\varrho) - E(\varrho')| \xrightarrow{\|\varrho - \varrho'\| \rightarrow 0} 0$
- $E(\Lambda_{\text{LOCC}}[\varrho]) \leq E(\varrho)$

Entanglement measures

Recipe to construct an entanglement measure:

1. Take a valid measure over pure states
2. Extend it over density matrices. How?

Entanglement measures

Watch out: Entanglement measures cannot be linear!

Consider a purported measure E and the entangled states:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|e_0e_0\rangle + |e_1e_1\rangle) \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|e_0e_0\rangle - |e_1e_1\rangle)$$

s.t. $E(|\Phi^+\rangle\langle\Phi^+|) = E_+ > 0$ and $E(|\Phi^-\rangle\langle\Phi^-|) = E_- > 0$

Entanglement measures

Watch out: Entanglement measures cannot be linear!

If linear, then

$$E\left(\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-|\right) = \frac{1}{2}(E_+ + E_-) > 0,$$

Despite of the fact that

$$\frac{1}{2}|\Phi^+\rangle\langle\Phi^+| + \frac{1}{2}|\Phi^-\rangle\langle\Phi^-| = \frac{1}{2}\left(|e_0\rangle\langle e_0| \otimes |e_0\rangle\langle e_0| + |e_1\rangle\langle e_1| \otimes |e_1\rangle\langle e_1|\right)$$

is separable!

Entanglement measures



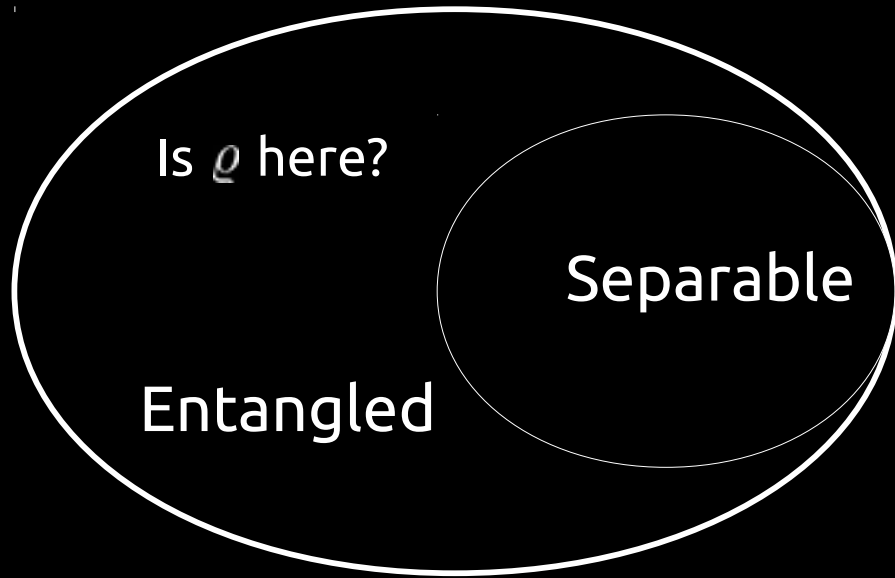
Given the state

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

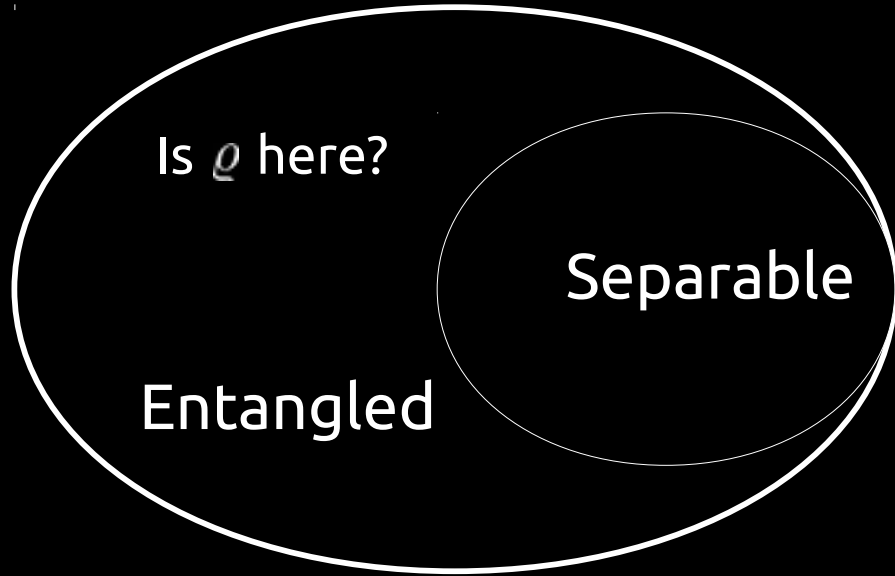
then

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

Finally: given ρ , is it entangled and how strongly?



Finally: given ρ , is it entangled and how strongly?



It's NP-Hard to discover that!

Only known cases:
2x2 and 2x3

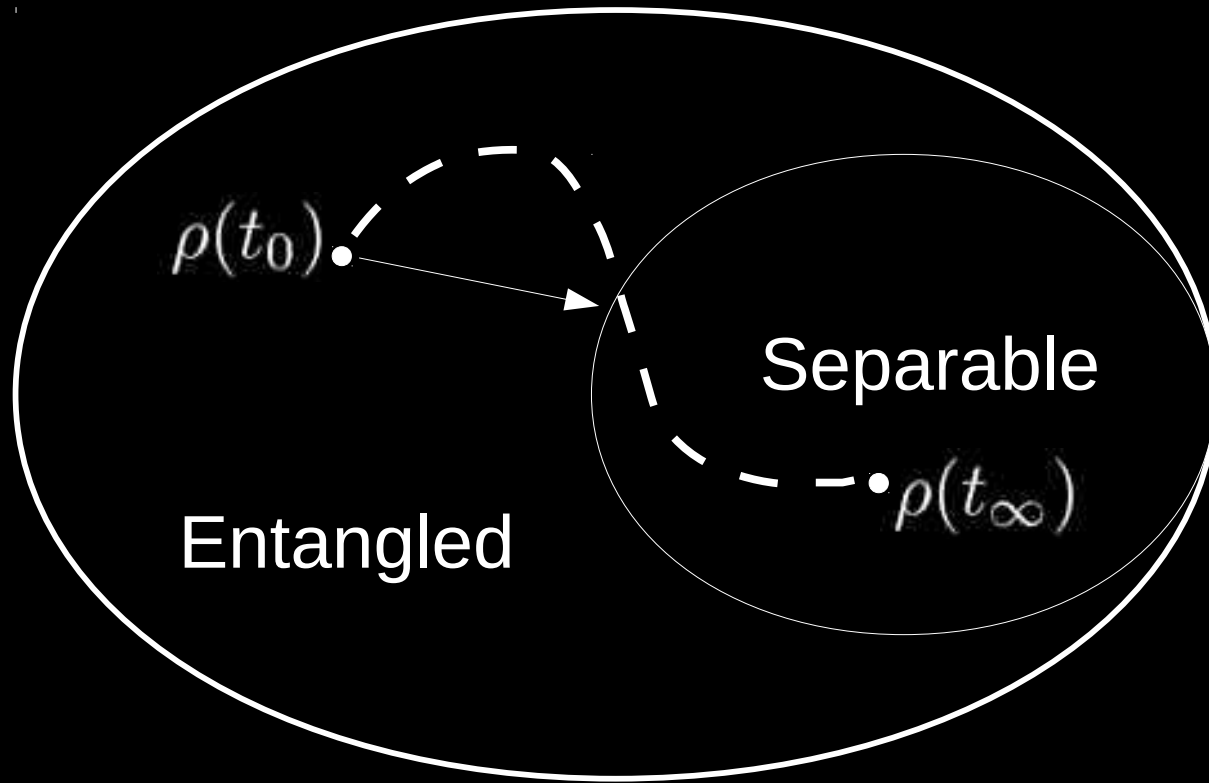
L. Gurvits,
In Proceedings of the thirty-fifth ACM symposium on
Theory of computing, pages 10 –19, (2003), ACM Press

The background features a complex, abstract visualization of particle tracks and molecular structures. The tracks are composed of numerous small, glowing yellow and white dots connected by thin lines, creating a sense of movement and interaction. The overall color palette is dominated by deep purples and blues, with bright highlights from the glowing particles. The tracks appear to be intertwined and branching, suggesting a complex, interconnected system.

Entanglement dynamics

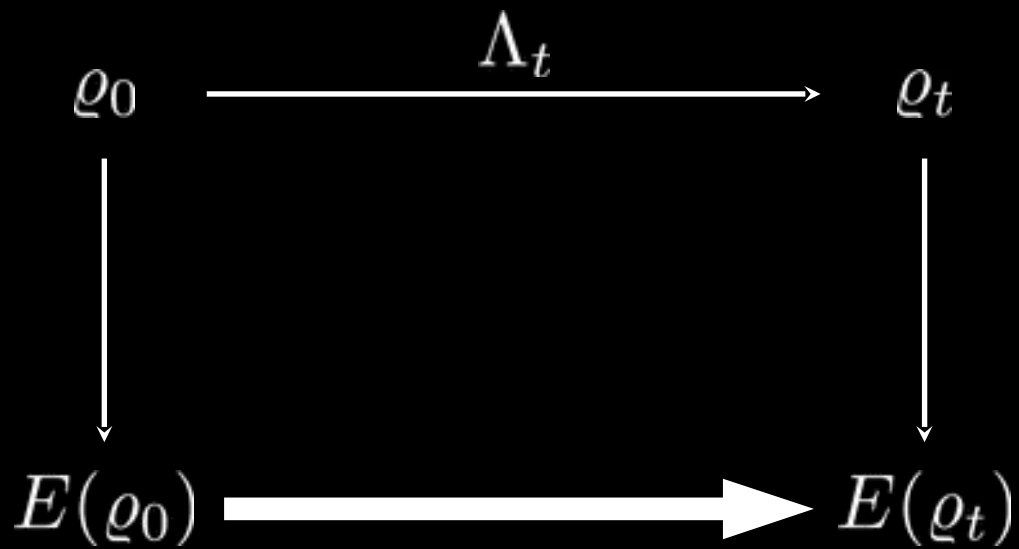
- Equation of motion
- Statistical approach

Where are we?



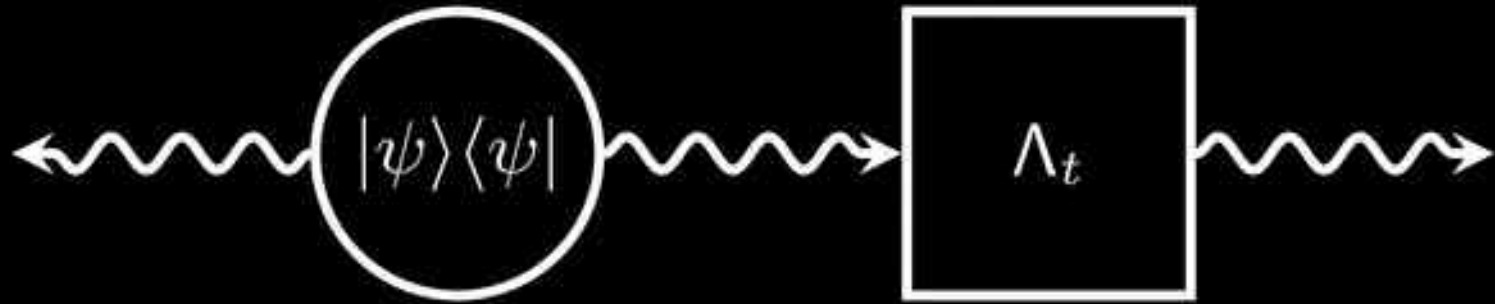
In trouble, definitely.

Dynamical approach:



Deterministic equation of motion

Bipartite states (dxd)



$$C_d\left(\mathbb{1} \otimes \Lambda_t[|\psi\rangle\langle\psi|]\right) = C_d\left(|\psi\rangle\langle\psi|\right) C_d\left(\mathbb{1} \otimes \Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\right)$$

where $|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |e_i e_i\rangle$

T. Konrad, FdM, M. Tiersch, C. Kasztelan, A. Aragão, and A. Buchleitner,
 Nat. Phys. 4, 99 (2008)

M. Tiersch, FdM, and A. Buchleitner,
 Phys. Rev. Lett. 101, 170502, (2008)

Proof ingredients

- (G-)Concurrence: determinant structure

$$|\psi\rangle = \sum_{i,j=0}^{d-1} \psi_{ij} |e_i e_j\rangle \Rightarrow C_d(|\psi\rangle) = d |\det \psi|^{2/d}$$

- Jamiołkowski isomorphism

$$|\psi\rangle = M_\psi \otimes \mathbb{1} |\Phi_d^+\rangle \quad \text{with} \quad M_\psi = \sqrt{2} \sum_{i,j} \psi_{i,j} |e_i\rangle \langle e_j|$$

A. Jamiołkowski, Rep. Math. Phys. **3**, 275 (1972)

Further results:

Initially mixed states:



$$C_d\left(\mathbb{1} \otimes \Lambda_t[\rho_0]\right) \leq C_d(\rho_0) C_d\left(\mathbb{1} \otimes \Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\right)$$

Two-sided channels:



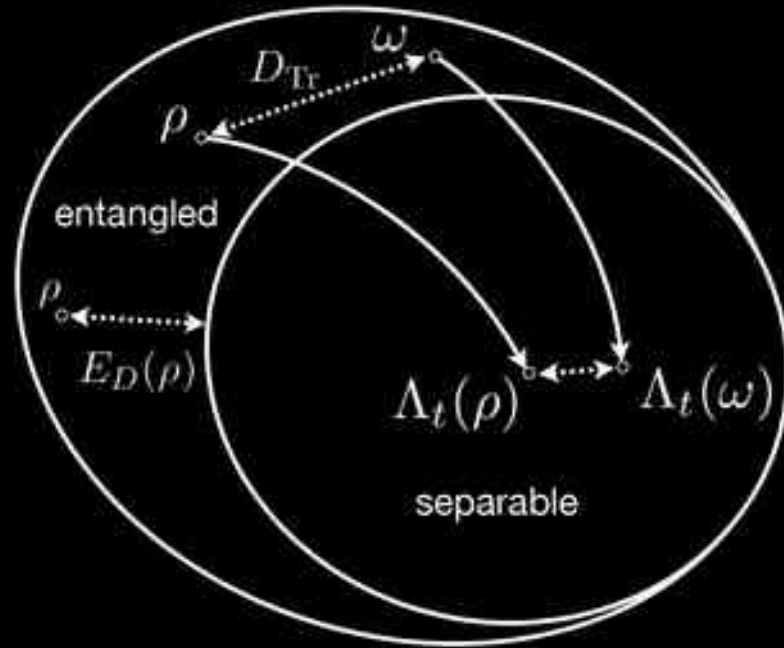
$$C_d\left(\Phi_t \otimes \Lambda_t[\rho_0]\right) \leq C_d(\rho_0) C_d\left(\mathbb{1} \otimes \Lambda_t[|\Phi_d^+\rangle\langle\Phi_d^+|]\right) C_d\left(\Phi_t \otimes \mathbb{1}[|\Phi_d^+\rangle\langle\Phi_d^+|]\right)$$

Statistical approach: a universal behavior

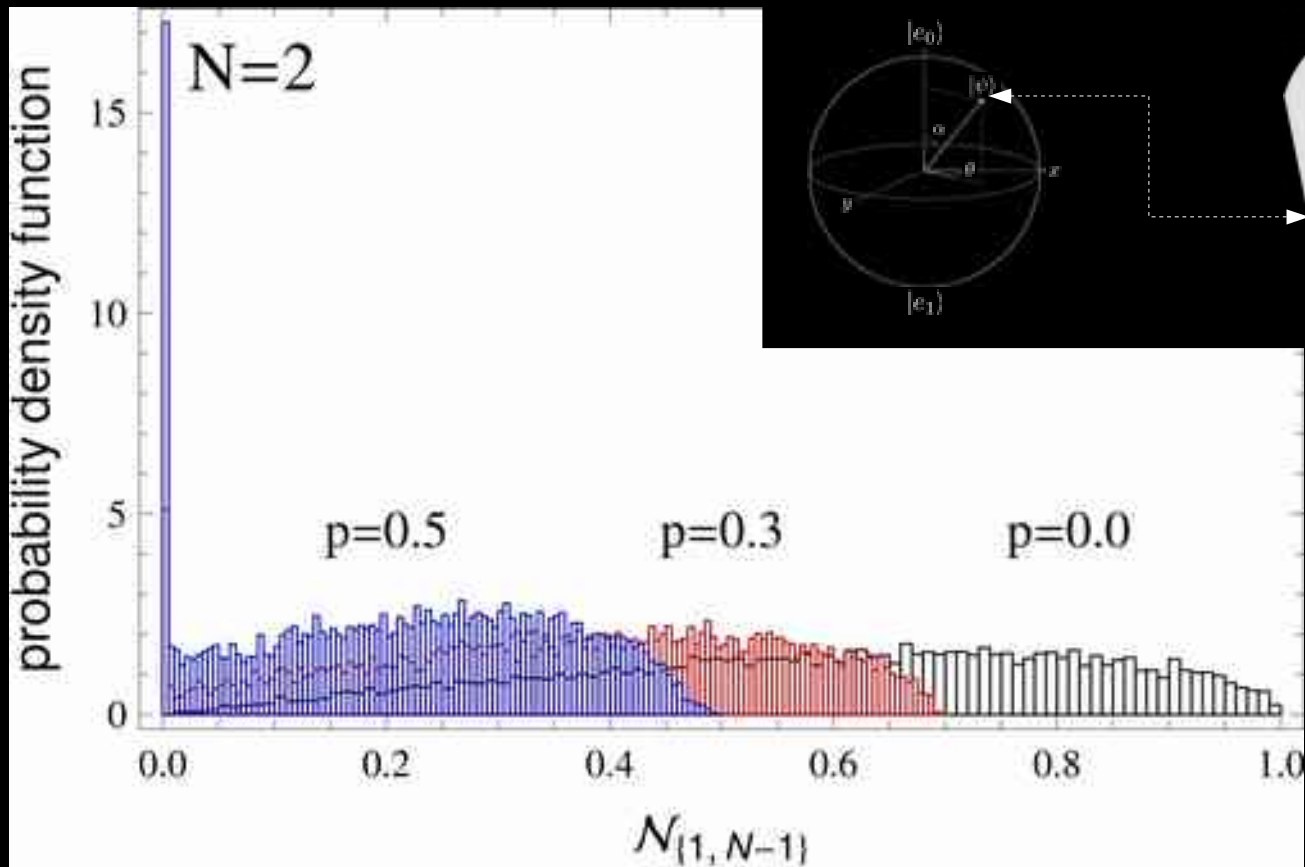
Multipartite states

Concentration of measure

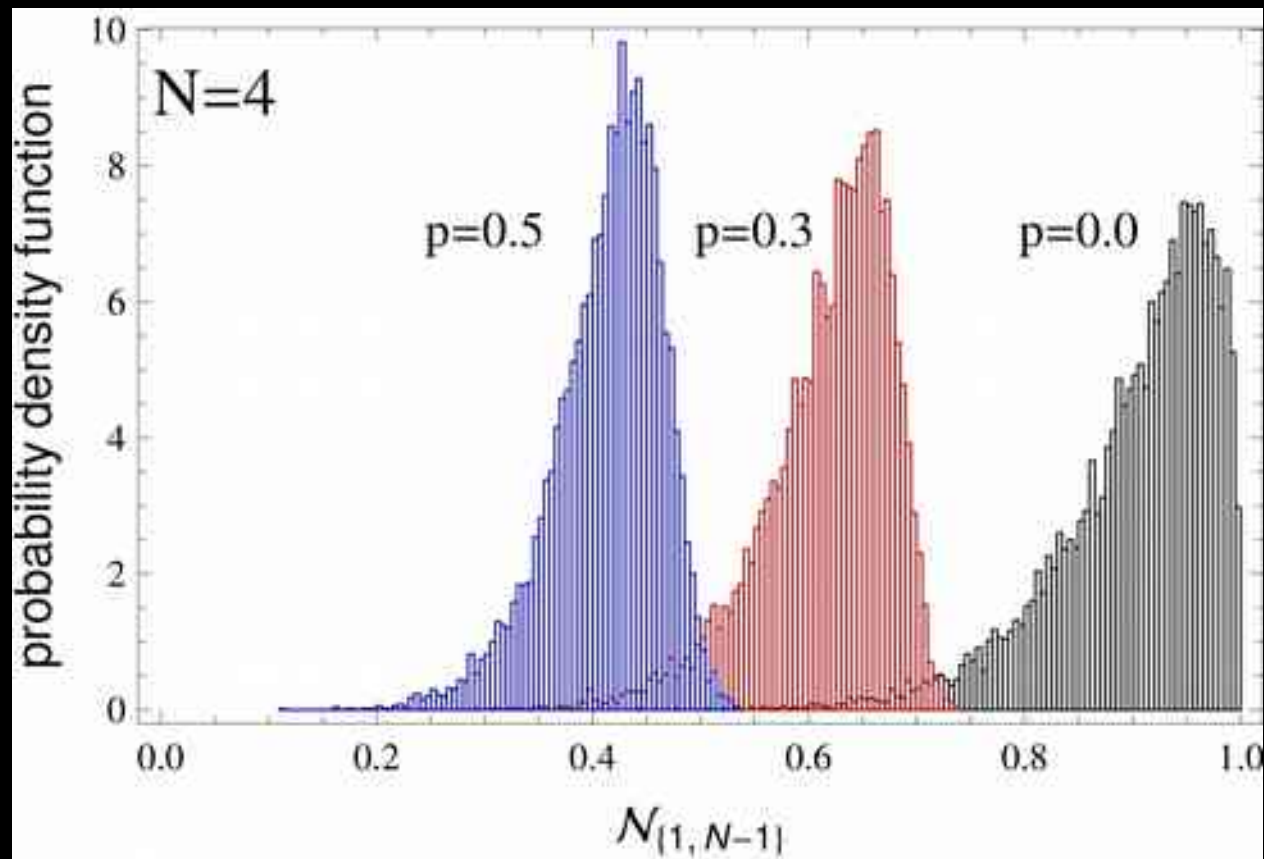
Consider a system in $\mathcal{H} = \mathcal{H}_A \otimes \cdots \otimes \mathcal{H}_N$.
If one uniformly samples pure states $|\psi\rangle \in \mathcal{H}$,
the chance that it is maximally entangled
approaches 1 exponentially fast with the total
dimension $d = d_A \cdot d_B \cdots d_N$



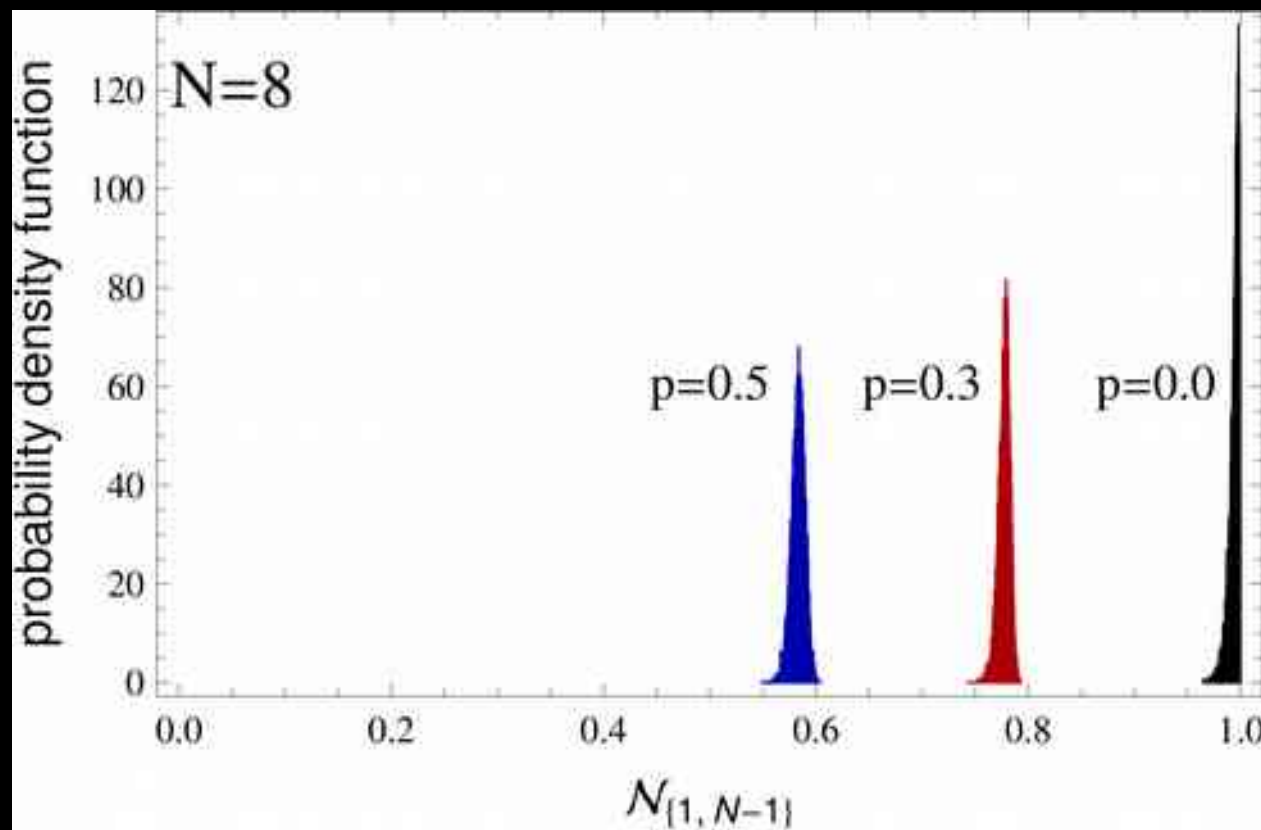
$$\|\Lambda_t(\rho) - \Lambda_t(\omega)\|_1 \leq \eta_{\Lambda_t} \|\rho - \omega\|_1 ; \eta_{\Lambda_t} \leq 1$$



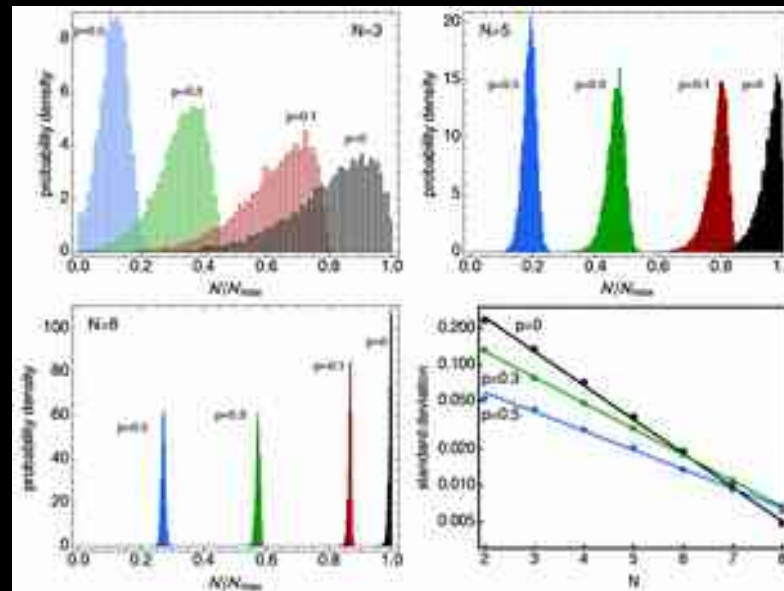
Dephasing ($p = 1 - e^{\Gamma t/2}$)



Dephasing ($p = 1 - e^{\Gamma t/2}$)



Dephasing ($p = 1 - e^{\Gamma t/2}$)



$$\Pr (|E[\Lambda_t(|\psi\rangle\langle\psi|)] - E(t)| > \epsilon) \leq 4 \exp \left(-C \frac{2d-1}{4\eta_E^2 \eta_{\Lambda_t}^2} \epsilon^2 \right)$$

with $C = (24\pi^2)^{-1}$ and $E(t) := \int d\chi E[\Lambda_t(|\chi\rangle\langle\chi|)]$

Proof ingredients

- Lipschitz continuous entanglement measure

$$|E(\varrho) - E(\omega)| \leq \eta_E \|\varrho - \omega\|_1$$

- Levy's lemma:

Let $f : \mathbb{S}^n \rightarrow \mathbb{R}$, with $|f(X) - f(Y)| \leq \eta \|X - Y\| \forall X, Y \in \mathbb{S}^n$

$$\Pr (|f(X) - \langle f \rangle| > \epsilon) \leq 4 \exp \left(-C \frac{n+1}{\eta^2} \epsilon^2 \right)$$

for a random $X \in \mathbb{S}^n$, and $C > 0$.

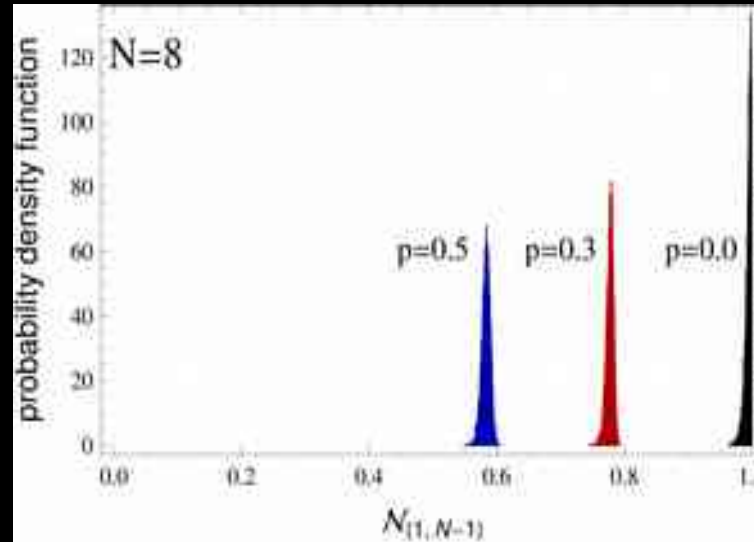
A close-up photograph of a hippopotamus in a river, yawning widely. Its mouth is open, showing its large, dark, sharp teeth and pink tongue. A small bird with a red beak is perched on the hippo's back. The background shows a calm river with a rock in the distance.

Conclusions and open questions

Conclusions

- Entanglement theory is rich and interesting
- Determining entanglement is a hard task
- The complexity seems to be reduced when introducing dynamical aspects

Open questions



- Finding an equation of motion for the mean entanglement
- Use an efficient sampling distribution

Interested?



To appear in Rep. Prog. Phys.

78 pages
560 references
33 figures

Also available at
arXiv:1402.3713

quantum information group @ CBPF
www.cbpf.br/~qig
twitter: @qig_CBPF
quantumrio.wordpress.com



O POTE DE OURO DE BELL



50 ANOS DAS DESIGUALDADES DE BELL

PALESTRANTES

ERNESTO DALVÃO (UFF)
Não-localidade além da mecânica quântica

IVAN S. OLIVEIRA (CBPF)
Emaranhamento e desigualdades de Bell em sistemas multipartiticos

LUISAS C. CÉLERI (UFPA)
Dados experimentais das desigualdades de Bell

NELSON PINTO (CBPF)
A perspectiva de de Broglie-Bohm, teoria realista não-local

RAFAEL BARCELÓ (UFMG)
Mecânica quântica fora do contexto?

BUYNET H. FOLMO (UFFLJ)
Algumas considerações sobre a realidade da função de onda

LOCAL: Centro Brasileiro de Pesquisas Físicas

DATA: 05 de dezembro

INSCRIÇÕES E INFORMAÇÕES: <http://qigcbpf.wordpress.com/bell50anos/>

EVENTO ABERTO E FREQUÊNCIA AO PÚBLICO

ORGANIZAÇÃO: Alessandra M. Tassi (CBPF), César Leite Faria (ICB),
Fernando de Melo (CBPF), Roberto S. Sanchez (CBPF)

SUPOSTO:

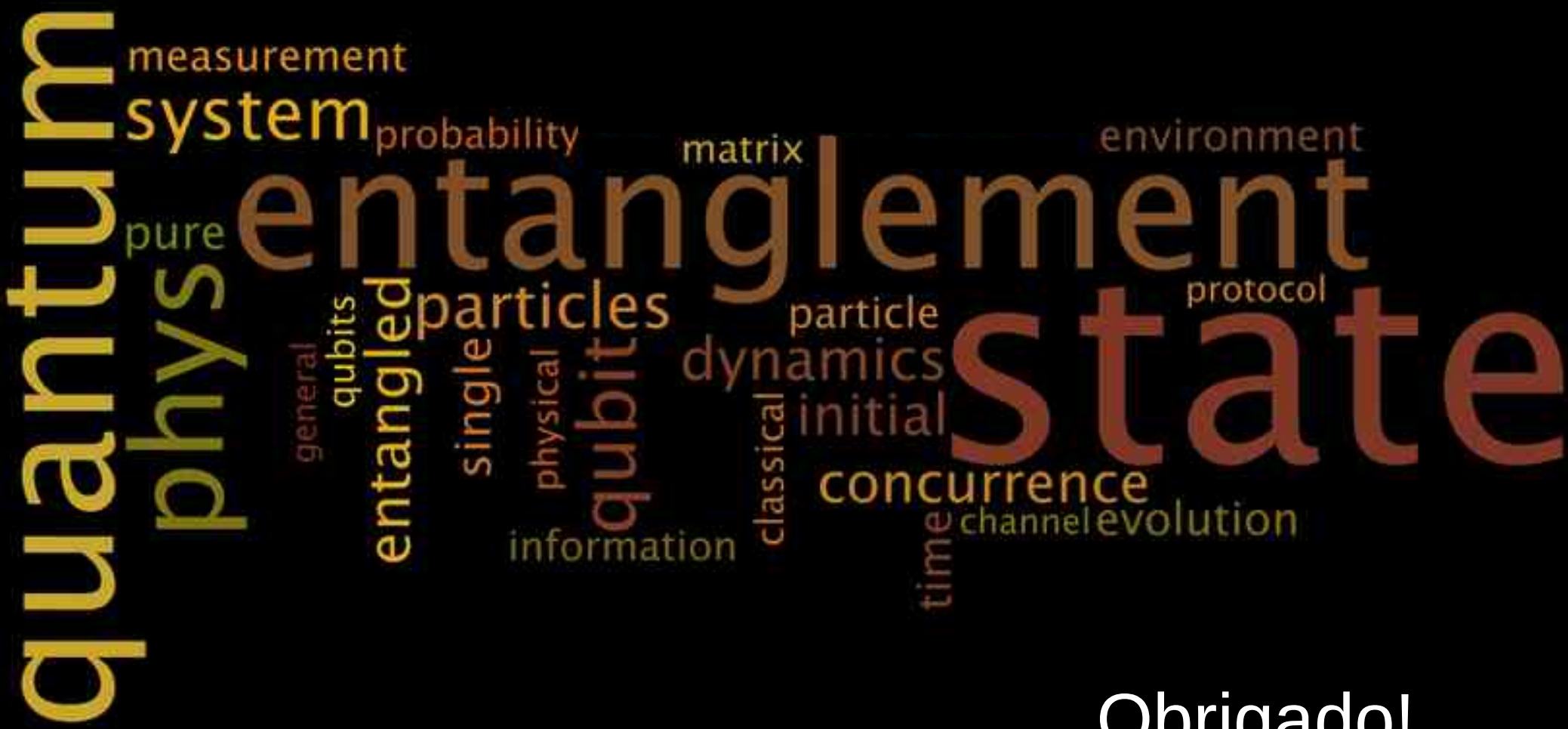


Coordenadas: 05.12, 13h CBPF

Inscrições e informações:

<http://qigcbpf.wordpress.com/bell50anos/>

Grátis!



Obrigado!