Revisiting the Contact Process

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**The classical contact process**

- $G = (\mathcal{V}, \mathcal{E})$ graph, locally finite. Most classical example $G = \mathbb{Z}^d$.

- A Markov process $\{\xi_t\}_{t \geq 0}$ with values on $\{0, 1\}^\mathcal{V}$:

  $\xi_t(x) = 1$ means $x$ is infected at time $t$

  $\xi_t(x) = 0$ means $x$ is healthy at time $t$

- Time evolution:

  An infected individual transmits the infection with rate $\lambda > 0$ to each of its healthy neighbors, and heals with rate $1$.

  Identify $\xi_t$ with $\{x : \xi_t(x) = 1\}$ (set of infected individuals at time $t$)

Model introduced by T. Harris in 1974.
The classical contact process
Dynamical phase transition

There exists $\lambda_c \in (0, +\infty)$ so that

- If $\lambda < \lambda_c$ then $P(\xi_t^0 = 0 \text{ for some } t) = 1$ (subcritical)
- If $\lambda > \lambda_c$ then $P(\xi_t^0 \neq 0 \text{ for all } t) > 0$ (supercritical)
- $\lambda > \lambda_c \implies$ positive probability that the infection remains forever
**Dynamical phase transition**

There exists \( \lambda_c \in (0, +\infty) \) so that

- If \( \lambda < \lambda_c \) then \( P(\xi_t^{\{0\}} = 0 \text{ for some } t) = 1 \) (subcritical)
- If \( \lambda > \lambda_c \) then \( P(\xi_t^{\{0\}} \neq 0 \text{ for all } t) > 0 \) (supercritical)
- \( \lambda > \lambda_c \Rightarrow \) positive probability that the infection remains forever

For more general graphs than \( \mathbb{Z}^d \) the supercritical regime splits into at least two:
\( 0 < \lambda_{1,c} < \lambda_{2,c} < \infty \) (Pemantle (1992), homogeneous tree)

- **Weak survival** \( \lambda \in (\lambda_{1,c}, \lambda_{2,c}) \)
- **Strong survival** \( \lambda > \lambda_{2,c} \)

- For \( G = \mathbb{Z}^d \) these two critical values coincide.
  \( \lambda > \lambda_c \Rightarrow \) two extremal invariant measures: \( \nu_\lambda, \delta_0 \).

**Remark:** There is a huge literature. Not all detailed credits in this too quick review. (See the related monograph by T. Liggett)
Dynamical phase transition

For $G = \mathbb{Z}^d$, the process dies out at criticality (Bezuidenhout and Grimmett (1990)).

Simulation by Stefanos van Dijk $d = 1$, $\lambda$ close to $\lambda_c$. 
Metastability

If $G = \mathbb{Z}^d$ and $\lambda > \lambda_c$ the model exhibits metastability. (Valid also for more general graphs if $\lambda$ large enough).

Metaestability: frequent phenomenon in thermodynamic systems close of a first order phase transition.
Metastability

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Metaestability: frequent phenomenon in thermodynamic systems close of a first order phase transition.

Plenty of examples in nature, in physical systems.

- supercooled liquids, super-saturated vapors;
- ferromagnets, finds many applications.
Metastability for the contact process

Process restricted to a large finite box \( \Lambda_N = \{ x : \|x\|_\infty \leq N \} \) in \( \mathbb{Z}^d \).

\( \lambda > \lambda_c \) and large initial configuration.

- Extinction time \( \tau_N \) is finite but exponentially large in \( |\Lambda_N| \), and loses memory as \( N \to \infty \).

\[
\frac{\tau_N}{E(\tau_N)} \to \text{EXP (1)}
\]

- Process behaves as if in equilibrium with the largest invariant measure before collapsing.


\( d \geq 2 \) Mountford (1993, 1999)

Trees and more general graphs Mountford, Mourrat, Schapira, Valesin, Yao (2016)

A contact process with two species. Mariela P. Machado (2018) - preprint
Metastability for the contact process

Simulations by Daniel Valesin
The renewal contact process

Same construction except that the recovery times are not anymore given by Poisson processes.

For each ordered pair \((x, y)\) of neighbouring points in \(\mathbb{Z}^d\) a Poisson process \(N_{x,y}\) of rate \(\lambda\). (The arrows)

Take independent renewal processes \(\mathcal{R}_x\) for \(x \in \mathbb{Z}^d\). (The crosses)

Parameters: \(\lambda\) and \(\mu\) (the law of the times between two consecutive crosses, assumed i.i.d.)
**The renewal contact process**

Our process is then constructed via *paths* as before.

The contact renewal process starting at $A \subseteq \mathbb{Z}^d$, $\xi^A_t$

$$\xi^A_t = \{ y : \exists \text{ a path from } (x, 0) \text{ to } (y, t) \text{ for some } x \in A \}.$$ 

- We no longer have a Markov process on $\{0, 1\}^{\mathbb{Z}^d}$.
- The questions regarding percolation continue to make sense:

$$\lambda_c = \inf \{ \lambda : P(\xi^\{0\}_t \neq \emptyset \forall t) > 0 \}$$

- May we have $\lambda_c = 0$?
Theorem 1. (Fontes, Marchetti, Mountford, V)
If \( \mu(t, +\infty) \geq t^{-\alpha} \) for some \( \alpha < 1 \) (all \( t \) large) plus some regularity conditions, then \( \lambda_c = 0 \).

Theorem 2. (Fontes, Mountford, V, 2018)
If \( \int t^2 \mu(dt) < \infty \) then \( \lambda_c > 0 \) for any \( d \geq 1 \).

(Robust argument; branching)

How to improve this?
Hypothesis A: $\mu$ has a density $f$ and the hazard rate $h(t) = \frac{f(t)}{\mu(t, +\infty)}$ is decreasing in $t$.

Theorem 3. (Fontes, Mountford, V, 2018)
Let $d = 1$. If $\mu$ satisfies Hypothesis A and $\int t^\alpha \mu(dt) < \infty$ for some $\alpha > 1$, then $\lambda_c > 0$.

Our arguments rely on putting together distinct crossing paths. They require $d = 1$. 
Hypothesis A and FKG inequalities

A very convenient construction:

- $h$ the hazard rate function

$\eta$ be a P.p.p. on $\mathbb{R} \times (0, +\infty)$ with rate 1.

To construct a renewal process starting at some point $t_0 \in \mathbb{R}$, consider all points of $\eta$ in $(t_0, +\infty) \times (0, +\infty)$ that are under the graph of the function $t \mapsto h(t - t_0)$.

- Take the point with the smallest first coordinate, say $(t_1, u_1)$;

$P(t_1 - t_0 > s) = e^{-\int_0^s h(v)dv} = \mu(s, \infty)$ i.e. $t_1 - t_0$ distributed according to $\mu$.

- Having obtained $t_1$ we repeat the procedure replacing $t_0$ by $t_1$.

- The properties of the P.p.p. $\Rightarrow t_1 < t_2 < \ldots$ so that $t_i - t_{i-1}, i \geq 1$ are i.i.d. with density $f$.

Useful consequence:

- If $h$ is decreasing, the renewal process is an increasing function of the points in the P.p.p.
Hypothesis A and FKG inequalities
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(\mathbf{h}(t-t_1), (t_2, y_2))

\mathbf{h}(t-t_1)
Hypothesis A and FKG inequalities
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\[ h(t-t_1) \]

\[ (t_1, y_1) \]
Hypothesis A and FKG inequalities

$$(t_1, y_1) \quad \quad \quad \quad \quad (t_2, y_2)$$

$h(t-t_1)$
Hypothesis A and FKG inequalities

\[
h(t-t_2)
\]

\[
(h(t_1, y_1), (t_2, y_2))
\]
Hypothesis A and FKG inequalities

\[ h(t-t_2) \]

\[ (t_1, y_1) \]

\[ (t_2, y_2) \]

\[ (t_3, y_3) \]
Hypothesis A and FKG inequalities

\[ h(t-t_3)(t_1, y_1)(t_2, y_2) \]

\[ \text{h}(t-t_3) \]
Hypothesis A and FKG inequalities
Hypothesis A and FKG inequalities

\[
\begin{align*}
    h(t-t_4) & (t_1, y_1) \\
    (t_2, y_2) & (t_3, y_3) \\
    (t_4, y_4) &
\end{align*}
\]
Hypothesis A and FKG inequalities

\[ h(t-t_4) \]

\[ (t_1, y_1) \]

\[ (t_2, y_2) \]

\[ (t_3, y_3) \]

\[ (t_4, y_4) \]
**Hypothesis A and FKG inequalities**

**Definition:** We say that an event depending on renewal and $\lambda$ Poisson process points in a finite space time rectangle is *increasing* if it is increasing with respect to the $\lambda$ Poisson processes of arrows, and decreasing with the renewal processes.

As a consequence:

**Proposition** Assume hypothesis A. Let events $A_1, A_2, \ldots, A_n$ be *increasing* events on a finite space time rectangle. Then

$$P(\cap_{i=1}^n A_i) \geq \prod_{i=1}^n P(A_i).$$

- Allows to use arguments that show similarity with known RSW estimates in percolation.
\[ A_0 = \{ \text{exists crossing from } \{0\} \times [0, \epsilon) \text{ to } \{L\} \times [\frac{2}{3}T, \frac{2}{3}T + \epsilon) \text{ in } [0, L] \times [0, \infty) \}. \]
$P(\text{temporal crossing of } [0, L] \times [\epsilon, \epsilon + mT]) \geq P(A_0)^{\frac{8m}{3}+2}$. 
Sketch of the proof of Theorem 3

Let $0 < \beta < \alpha - 1$.

- $P_r$: the supremum over the probabilities for the space-time rectangle $[0, 2^{\beta r}] \times [0, 2^r]$ of either a spatial or a temporal crossing.

The supremum is over all product renewal measures with inter-arrival distribution $\mu$, for the death points starting at times points strictly less than $0$. (The starting points or times need not be the same.)

The main ingredient is the following

**Proposition** Assume $\beta \in (0, \alpha - 1)$, with $\alpha$ as in the statement of main theorem. There exists $\lambda_0 > 0$ so that for $0 \leq \lambda < \lambda_0$

$$P_r \xrightarrow{r \to \infty} 0.$$
The theorem follows (quite easily) from the Proposition.

\[ \{ \xi_{2r}^{(0)} \neq \emptyset \} \subseteq (I) \cup (II) \cup (III), \]

where, letting \( R = [-2^{r\beta}/2, 2^{r\beta}/2] \times [0, 2^r] \):

(I) \{ \exists \text{ path from } (0, 0) \text{ to } \mathbb{Z} \times \{2^r\} \text{ in } R \} \\
(II) \{ \exists \text{ path from } (0, 0) \text{ to } \{2^{r\beta}/2\} \times [0, 2^r] \} \\
(III) \{ \exists \text{ path from } (0, 0) \text{ to } \{-2^{r\beta}/2\} \times [0, 2^r] \}.

• \( P((I)) \leq P_r. \)

Using the previous FKG type estimate

• \( P(II) = P(III) \leq K^2(P_r)^{1/(2^{\lceil 1/\beta \rceil}+1)} + KP_r^{-\lceil 1/\beta \rceil}, \)

where \( K \) is suitably large (depending on \( \beta \) but not on \( r \)).
(Key estimate) Control $P_r$ through an iterative procedure.

Consider the probability of temporal crossing $(X(s))_{0 \leq s \leq 2^n}$ of $[0, 2^{n\beta}] \times [0, 2^n]$.

- Take $k$ suitably large (but not depending on $n$) and consider the restriction each of the $2^{k-1}$ (even) rectangles $[0, 2^{\beta n}] \times [2i2^{n-k}, (2i + 1)2^{n-k}]$

- Show there must be a crossing (space or time) of smaller but very "similar" scales $2^{\beta(n-k-i)} \times 2^{n-k-i}$ ($i \leq 4$)

- Conditioning on the existence of a renewal (cross mark) for each $x \in [0, 2^{\beta n}]$ in the previous time interval $[(2i - 1)2^{n-k}, 2i2^{n-k}]$, we have that the probability of the vertical crossing is bounded

$$C(k)(\sup_{n-k-4 \leq r \leq n-k} P_r)^{1/10}$$

- $0 < \beta < \alpha - 1$ guarantees that the probability of not having a cross for at least one $x$ and at least one of the odd time intervals is small. The renewals bring some sort of 'independence'. We can combine all time intervals to beat power $1/10$. 
Taking $2^{k-1} > 20$ and arguing similarly for the space crossings (easier) one gets if $n - k$ is large

$$P_n \leq 2^{-n\epsilon_0^2} + C(k)\left(\sup_{n-k-4 \leq n-k} P_r\right)^2$$

Out of this it is simple to conclude the result.

Indeed, if we have

$$P_r \leq 2^{-r\epsilon_0^2/5} \quad \text{for each } n - k - 4 \leq r \leq n - k \quad (\ast)$$

then

$$P_n \leq 2^{-n\epsilon_0^2/2} + C'' 2^{-2n\epsilon_0/5} 2^{2(k+4)\epsilon_0/5} \leq 2^{-n\epsilon_0^2/5}$$

for all $n$ large.

Choose $\lambda_0$ small so that $(\ast)$ holds for $n = n_0$ and $\lambda \in (0, \lambda_0)$. 
THANKS