STABLE CONSTANT MEAN CURVATURE HYPERSURFACES

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Abstract. Let \( N^{n+1} \) be a Riemannian manifold with sectional curvatures uniformly bounded from below. When \( n = 3, 4 \), we prove that there are no complete (strongly) stable \( H \)-hypersurfaces, without boundary, provided \( |H| \) is large enough. In particular we prove that there are no complete strongly stable \( H \)-hypersurfaces in \( \mathbb{R}^{n+1} \) without boundary, \( H \neq 0 \).

1. Introduction

Consider a Riemannian manifold \( N \) of dimension \( n+1 \) with sectional curvatures uniformly bounded from below; denote by \( \text{sec}(N) \) the infimum of the sectional curvatures of \( N \). Let \( M \) be an immersed submanifold of codimension one and let \( H \) be the mean curvature of \( M \) in the metric induced by the immersion. If \( H \) is constant, we call \( M \) an \( H \)-hypersurface. We prove the following diameter estimate.

**Theorem 1** Let \( M^n \subset N^{n+1} \) be a stable complete \( H \)-submanifold, \( n = 3, 4 \). There exists a constant \( c = c(n, H, \text{sec}(N)) \) such that for any \( p \in M \) one has: \( \text{dist}_M(p, \partial M) \leq c \) whenever \( |H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|} \).

For the definition of stability see Section 2. Particular cases of the previous Theorem in \( \mathbb{R}^3, \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{R} \) and any homogeneously regular three manifold are proved in [9], [5], [7], [8] respectively.

We wonder if Theorem 1 holds in all dimensions.

**Corollary 1** Let \( M^n \) be a complete stable \( H \)-hypersurface of \( N^{n+1} \). If \( n = 3, 4 \) and \( |H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|} \), then \( \partial M \neq 0 \).

In [12] it is proved that an \( H \)-hypersurface in \( \mathbb{R}^{n+1} \), with finite total curvature, is minimal, so, if it is stable, it is a hyperplane (cf. [4]). For \( n = 3, 4 \), we are able to generalize this result in the following sense. We do not need the finite total curvature hypothesis on \( M \) and the ambient space can be any manifold with uniformly bounded sectional curvature, provided the mean curvature \( |H| \) is large enough (See Corollary 1).

As a consequence of the diameter estimate in Theorem 1, we have the Maximum Principle at Infinity.

**Theorem 2** Let \( N^{n+1} \) have uniformly bounded sectional curvature, \( n = 3, 4 \). If \( |H| > 2\sqrt{|\min\{0, \text{sec}(N)\}|} \) and \( M_1, M_2 \) are properly embedded \( H \)-hypersurfaces in \( N^{n+1} \), which bound a connected domain \( W \), then the mean curvature vector points out of \( W \) along the boundary of \( W \).

The proof of Theorem 2 is the same as in [8], where the result is proved for \( n = 2 \).

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2. Proofs

Let \( M \) be a \( H \)-hypersurface in a manifold \( \mathcal{N} \) and let \( N \) be a unit vector field normal to \( M \) in \( \mathcal{N} \). The stability operator of \( M \) is \( L = \Delta + |A|^2 + \text{Ric}(N) \) where \( \text{Ric}(N) \) is the Ricci curvature of the ambient manifold \( \mathcal{N} \) in the direction of \( N \) and \( A \) is the shape operator of the immersion. We say that \( M \) is stable if

\[
- \int_M uLu \geq 0,
\]

for any smooth function \( u \) with compact support on \( M \). Our definition of stability is usually known as strong stability. The usual definition of stability (weak stability) also requires the test function \( u \) to satisfy \( \int_M u = 0 \). Geodesic spheres in a space form are weakly stable but they are not stable (cf. [2]). We remark that the solutions of the Plateau problem are stable hypersurfaces in our sense as well as any \( H \)-hypersurface transverse to some Killing vector field of the ambient manifold. The proof of the latter is standard (cf. for example [7]). For further relations between the two notions of stability see [1] and [2].

Proof of Theorem 1. Consider the traceless operator \( \Phi = A - H I \). One can write the stability operator of \( M \) in terms of \( \Phi \), namely, \( L = \Delta + |\Phi|^2 + nH^2 + \text{Ric}(N) \). Since \( M \) is stable, there exists a function \( u > 0 \) on \( M \) such that \( Lu = 0 \) on \( M \) (cf. [6]).

Denote by \( ds^2 \) the metric on \( M \) induced by the immersion in \( \mathcal{N} \) and let \( d\tilde{s}^2 = u^{2k} ds^2 \), with \( \frac{5(n-1)}{4n} \leq k < \frac{4}{n-1} \). This choice of \( k \) will be justified later. Notice that, in order to have some \( k \) satisfying the previous inequality, one needs \( n = 3, 4 \).

Consider \( p \in M \) and let \( r > 0 \) be such that the intrinsic ball \( B_r \) of \( M \), centered at \( p \) of \( ds \)-radius \( r \), is contained in the interior of \( M \). Let \( \gamma \) be a \( d\tilde{s} \)-minimizing geodesic in \( B_r \) joining \( p \) to \( \partial B_r \). Let \( a \) be the \( d\tilde{s} \)-length of \( \gamma \). Then \( a \geq r \) and it is enough to prove that there exists a constant \( c(n, H, \text{sec}(\mathcal{N})) \) such that \( a \leq c \).

Let \( R \) and \( \tilde{R} \) be the curvature tensor of \( M \) in the metric \( ds \) and \( d\tilde{s} \), respectively. Choose a basis \( \{ \tilde{e}_1 = \frac{\partial}{\partial s}, \tilde{e}_2, \ldots, \tilde{e}_n \} \) orthonormal for the metric \( d\tilde{s} \), such that \( \tilde{e}_2, \ldots, \tilde{e}_n \) are parallel along \( \gamma \) and let \( \tilde{e}_{n+1} = N \). The basis \( \{ e_1 = \frac{\partial}{\partial s} = u^{k} \tilde{e}_1, e_2 = u^{k} \tilde{e}_2, \ldots, e_n = u^{k} \tilde{e}_n \} \) is orthonormal for the metric \( ds \). Denote by \( R_{11} \) and \( \tilde{R}_{11} \) the Ricci curvature in the direction of \( e_1 \) for the metric \( ds \) and \( d\tilde{s} \) respectively. Let \( \tilde{R} \) be the curvature tensor of the ambient manifold \( \mathcal{N} \) and write \( \text{Ric}(N) = \tilde{R}_{n+1, n+1} \).

Let \( \tilde{r} \) be the length of \( \gamma \) in the \( d\tilde{s} \) metric. Since \( \gamma \) is \( d\tilde{s} \) minimizing, by the second variation formula, we have

\[
\int_0^{\tilde{r}} \left[ (n-1) \left( \frac{d\varphi}{d\tilde{s}} \right)^2 - \tilde{R}_{11} \varphi^2 \right] d\tilde{s} \geq 0,
\]

for any smooth function \( \varphi \) such that \( \varphi(0) = \varphi(\tilde{r}) = 0 \). As it is proved in the Appendix

\[
\tilde{R}_{11} = u^{-2k} \left\{ R_{11} - k(n-2)(\ln u)_{ss} - k \frac{\Delta u}{u} + k \frac{\nabla u \cdot \nabla u}{u^2} \right\}.
\]

Now use that \( Lu \) = \( (\Delta + |\Phi|^2 + nH^2 + \tilde{R}_{n+1, n+1})u \) to obtain
\( \tilde{R}_{11} = u^{-2k} \left\{ R_{11} - k(n-2)(\ln u)_{ss} + k(|\Phi|^2 + nH^2 + \tilde{R}_{n+1,n+1}) + k\frac{\nabla u^2}{u^2} \right\} . \)

From the Gauss equation one has

\[ R_{ijij} = \hat{R}_{ijij} + h_{ii}h_{jj} - h_{ij}^2, \]

which can be rewritten as

\[ R_{ijij} = \hat{R}_{ijij} + (\Phi_{ii} + H)(\Phi_{jj} + H) - (\Phi_{ij} + H\delta_{ij})^2. \]

Taking \( i = 1 \) and summing up in \( j = 2, \ldots, n \) we obtain

\[ R_{11} = \sum_{j=2}^{n} \hat{R}_{1jjj} + \sum_{j=2}^{n} \Phi_{11}\Phi_{jj} + (n-2)H\Phi_{11} + (n-1)H^2 - \sum_{j=2}^{n} \Phi_{1j}^2. \]

Since \( \sum_{j=1}^{n} \Phi_{jj} = 0 \), we have

\[ R_{11} = \sum_{j=2}^{n} \hat{R}_{1jjj} - \Phi_{11}^2 + (n-2)H\Phi_{11} + (n-1)H^2 - \sum_{j=2}^{n} \Phi_{1j}^2. \]

Replacing the last relation in equation (3), yields

\[ \tilde{R}_{11} = u^{-2k} \left[ \sum_{j=2}^{n} \hat{R}_{1jjj} + k\hat{R}_{n+1,n+1} + (kn + n-1)H^2 + (n-2)H\Phi_{11} \right] \]

\[ + u^{-2k} \left[ k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^{n} \Phi_{1j}^2 - k(n-2)(\ln u)_{ss} + k\frac{\nabla u^2}{u^2} \right]. \]

Combining the last equation with inequality (1) gives (by abuse of notation we denote again by \( \varphi \) the composition \( \varphi \circ \tilde{s} \), hence \( \varphi(0) = \varphi(a) = 0 \))

\[ (n-1) \int_{0}^{a} (\varphi_s)^2 u^{-k}ds \geq \int_{0}^{a} \varphi^2 u^{-k} \left[ \sum_{j=2}^{n} \hat{R}_{1jjj} + k\hat{R}_{n+1,n+1} \right] ds \]

\[ + \int_{0}^{a} \varphi^2 u^{-k} \left[ (kn + n-1)H^2 + (n-2)H\Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^{n} \Phi_{1j}^2 \right] ds \]

\[ - \int_{0}^{a} \varphi^2 u^{-k} \left[ k(n-2)(\ln u)_{ss} + k\frac{\nabla u^2}{u^2} \right] ds. \]

Replace \( \varphi \) by \( \varphi u^{\frac{k}{2}} \) to get rid of \( u^k \) in the denominator. The last relation becomes
\[(n - 1) \int_0^a (\varphi_s)^2 \, ds + k(n - 1) \int_0^a \varphi \varphi_s u_s u^{-1} \, ds + \frac{k^2(n - 1)}{4} \int_0^a \varphi_s^2 u_s^2 u^{-2} \, ds \geq \int_0^a \varphi^2 \left[ \sum_{j=2}^n \hat{R}_{1jj} + k \hat{R}_{n+1,n+1} \right] \, ds \]

\[+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] \, ds \]

Integration by parts gives

\[\int \varphi^2 (\ln u)_{ss} \, ds = -2 \int \varphi \varphi_s \frac{u_s}{u} \, ds.\]

Then, replacing in inequality (5), we obtain

\[(n - 1) \int_0^a (\varphi_s)^2 \, ds \geq k(n - 3) \int_0^a \varphi \varphi_s u_s u^{-1} \, ds - \frac{(n - 1)}{4} \int_0^a \varphi^2 (\ln u)^2 \, ds \]

\[+ k \int_0^a \varphi^2 \left| \nabla u \right|^2 \, ds + \int_0^a \varphi^2 \left[ k \hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1jj} \right] \, ds \]

\[+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] \, ds,\]

that is

\[(n - 1) \int_0^a (\varphi_s)^2 \, ds \geq k(n - 3) \int_0^a \varphi \varphi_s u_s u^{-1} \, ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u)^2 \, ds \]

\[+ \int_0^a \varphi^2 \left[ k \hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1jj} \right] \, ds \]

\[+ \int_0^a \varphi^2 \left[ (kn + n - 1)H^2 + (n - 2)H \Phi_{11} + k|\Phi|^2 - \Phi_{11}^2 - \sum_{j=2}^n \Phi_{1j}^2 \right] \, ds.\]

We now use that \(a^2 + b^2 \geq -2ab\) with \(a = (n - 2)H\) and \(b = \frac{\Phi_{11}}{2}\), to obtain

\[\frac{(n - 2)^2 H^2 + \Phi_{11}^2}{4} \geq -(n - 2)H \Phi_{11}.\]

Replacing in inequality (6) yields

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\[(n - 1) \int_0^a (\varphi_s)^2ds \geq k(n - 3) \int_0^a \varphi \varphi_s u_s^{-1}ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u_k)^2 ds \]
\[+ \int_0^a \varphi^2 \left[ k\widehat{R}_{n+1,n+1} + \sum_{j=2}^n \widehat{R}_{1j} + (kn - n^2 + 5n - 5)H^2 + \right] ds \]
\[= \int_0^a \varphi^2 \left[ -\sum_{j=2}^n \Phi_j^2 + \sum_{i=2}^n \left( \frac{n - 5}{2} \right) \right] ds. \quad (7)\]

We will now prove that the last term in inequality (7) is greater or equal than zero. We know that
\[|\Phi|^2 \geq \Phi_{11}^2 + \Phi_{22}^2 + \cdots + \Phi_{nn}^2 + 2 \sum_{j=2}^n \Phi_{ij}^2, \quad (8)\]
and since \(\sum_{j=1}^n \Phi_{jj} = 0\), we have

\[|\Phi|^2 \geq \frac{n}{n-1} \Phi_{11}^2 + 2 \sum_{j=2}^n \Phi_{ij}^2. \quad (9)\]

Since \(k \geq \frac{5(n-1)}{4n}\), using inequality (8), we obtain

\[k|\Phi|^2 - \frac{5}{4} \Phi_{11}^2 - \sum_{j=2}^n \Phi_{ij}^2 \geq 0. \quad (7)\]

Then, inequality (7) yields

\[(n - 1) \int_0^a (\varphi_s)^2ds \geq (n - 3) \int_0^a \varphi \varphi_s (\ln u_k)_s ds + \left[ \frac{1}{k} - \frac{(n - 1)}{4} \right] \int_0^a \varphi^2 (\ln u_k)^2 ds \]
\[+ \int_0^a \varphi^2 \left[ k\widehat{R}_{n+1,n+1} + \sum_{j=2}^n \widehat{R}_{1j} + (kn - n^2 + 5n - 5)H^2 + \right] ds. \quad (9)\]

We now use that \(a^2 + b^2 \geq -2ab\) with \(a = \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^\frac{1}{2} \varphi_s \) and
\(b = \frac{(n-3)}{2} \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{-\frac{1}{2}} \varphi_s\), to obtain

\[\left( \frac{1}{k} - \frac{(n-1)}{4} \right) \varphi^2 (\ln u_k)_s + \frac{(n-3)^2}{4} \left( \frac{1}{k} - \frac{(n-1)}{4} \right)^{-1} \varphi^2 \geq -(n-3)\varphi \varphi_s (\ln u_k)_s. \]

The last inequality together with inequality (9) gives
\[(n - 1) \int_0^a (\varphi_s)^2 ds \geq -\frac{(n - 3)^2}{4} \left( \frac{1}{k} - \frac{(n - 1)}{4} \right)^{-1} \int_0^a (\varphi_s)^2 ds + \int_0^a \varphi^2 \left[ (kn - n^2 + 5n - 5)H^2 + k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1jjj} \right] ds.\]

Then setting \(A = \frac{4(k(2 - n) + (n - 1))}{4 - k(n - 1)}\) and making a suitable choice of a positive constant \(B\), we can rewrite the last inequality as

\[(10) \quad A \int_0^a (\varphi_s)^2 ds \geq B \int_0^a \varphi^2 ds.\]

We remark that \(A\) is positive as soon as \(k < \frac{4}{n - 1} \leq \frac{n - 1}{n - 2}\). We now want to choose \(B\) such that

\[0 < B \leq (kn - n^2 + 5n - 5)H^2 + \left( k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1jjj} \right).\]

When the curvature of the ambient manifold is non-negative, we set \(B = (kn - n^2 + 5n - 5)H^2\), which is positive if \(H \neq 0\) (remember that \(k > \frac{5(n-1)}{4n}\) and that \(n = 3, 4\)). In this case we can set \(c_1 = 0\).

Otherwise, we proceed as follows. By a straightforward computation one has

\[k\hat{R}_{n+1,n+1} + \sum_{j=2}^n \hat{R}_{1jjj} \geq (kn + n - 1) \inf \\{\text{sectional curvatures of } \mathcal{N}\} = \sec(\mathcal{N}),\]

then we set \(B = (kn - n^2 + 5n - 5)H^2 + (kn + n - 1)\sec(\mathcal{N})\). If

\[(11) \quad H^2 > \frac{kn + n - 1}{kn - n^2 + 5n - 5} \sec(\mathcal{N}),\]

then \(B\) is positive. In this case, one can set \(c_1 = 2\sqrt{|\sec(\mathcal{N})|}\) (using the restrictions on \(k\) one can prove that \(\frac{kn + n - 1}{kn - n^2 + 5n - 5} < 4\)).

Integration by parts in inequality (10) yields

\[\int_0^a (\varphi_{ss}A + B\varphi)\varphi ds \leq 0.\]

Choosing \(\varphi = \sin(\pi sa^{-1})\), \(s \in [0, a]\) one has

\[\int_0^a \left[ B - \frac{A\pi^2}{a^2} \right] \sin^2(\pi sa^{-1}) ds \leq 0.\]

Finally

\[B - \frac{A\pi^2}{a^2} \leq 0,\]

and this gives the desired inequality, if we choose
\[ c = \frac{2\pi \sqrt{k(2-n)+(n-1)}}{\sqrt{(4-k(n-1))(kn-n^2+5n-5)H^2+(kn+n-1) \min\{0, \sec(N)\}}} \]

Proof of Corollary 1. Assume that such an \( M \) exists. In the proof of Theorem 1, we showed that the radius of an intrinsic disc of \( M \), that does not touch \( \partial M \), is at most \( c \). Hence, when \( \partial M = \emptyset \), the diameter of \( M \) is at most \( c \) and then \( M \) is compact. As \( M \) is stable, there exists a positive function \( f \) on \( M \) such that \( L(f) = 0 \) (cf. [6]). Let \( p \in M \) be a minimum of the function \( f \). At \( p \), one has:

\[ 0 \leq \Delta f(p) = -(|\Phi|^2(p) + nH^2 + \hat{R}_{n+1,n+1}(p))f(p). \]

By our choice of \( H \), the potential \( |\Phi|^2 + nH^2 + \hat{R}_{n+1,n+1} \) is strictly positive on \( M \), hence the previous inequality yields a contradiction.

\[ \square \]

3. Appendix

The transformation law of the curvature under the conformal change of the metric \( ds^2 = u^{2k}ds^2 \) is the following (cf. [10] page 184 and [11] formula (4))

\[ \tilde{R}_{11} = \tilde{Ric}(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}) = \left\{ \begin{array}{l} Ric(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}) - k(n-2)Hess(ln u) (\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}) \\ + k^2(n-2)|\frac{\partial \gamma}{\partial s}(ln u)|^2 - [k\Delta(ln u) + k^2(n-2)|\nabla ln u|^2]u^{-2k} \end{array} \right\}. \]

In order to simplify this equation we need to compute \( \nabla_\gamma \frac{\partial \gamma}{\partial s} \). Using the relation between the connections of conformal metrics we obtain

\[ \tilde{\nabla}_\gamma \frac{\partial \gamma}{\partial s} = \nabla_\gamma \frac{\partial \gamma}{\partial s} + 2k < \nabla ln u, \frac{\partial \gamma}{\partial s} > \frac{\partial \gamma}{\partial s} - k\nabla ln u. \]

Since \( \gamma \) is geodesic in the \( ds^2 \) metric we have that \( \tilde{\nabla}_\gamma \frac{\partial \gamma}{\partial s} = 0 \) and thus

\[ \tilde{\nabla}_\gamma \frac{\partial \gamma}{\partial s} = k < \nabla ln u, \frac{\partial \gamma}{\partial s} > \frac{\partial \gamma}{\partial s}. \]

The last two equations yield

\[ \nabla_\gamma \frac{\partial \gamma}{\partial s} = k(\nabla ln u)^\perp, \]

where \((\nabla ln u)^\perp\) means the component of \( \nabla ln u \) perpendicular to \( \frac{\partial \gamma}{\partial s} \). Now we observe that

\[ Hess(ln u)(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial s}) = u^{-2k}\left( (ln u)_{ss} - (\nabla_\gamma \frac{\partial \gamma}{\partial s}) ln u \right) \]

\[ = u^{-2k}\left( (ln u)_{ss} - k|\nabla ln u|^2 \right), \]
where in the last equality we use (13). Replacing this last equation in (12) one obtains
\[
\tilde{R}_{11} = u^{-2k}\left\{ R_{11} - k(n - 2)(\ln u)_{ss} + k^2(n - 2)|\nabla \ln u|^2 \right. \\
\left. + k^2(n - 2)(\ln u)_{s}^2 - \left[k\Delta (\ln u) + k^2(n - 2)|\nabla \ln u|^2\right]\right\},
\]
which can be rewritten as
\[
\tilde{R}_{11} = u^{-2k}\left\{ R_{11} - k(n - 2)(\ln u)_{ss} - k\Delta (\ln u)\right. \\
\left. - k\Delta u + k\frac{|\nabla u|^2}{u^2}\right\}.
\]

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