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# AUSTERE SUBMANIFOLDS OF DIMENSION FOUR: EXAMPLES AND MAXIMAL TYPES

MARIANTY IONEL AND THOMAS IVEY

ABSTRACT. Austere submanifolds in Euclidean space were introduced by Harvey and Lawson in connection with their study of calibrated geometries. The algebraic possibilities for second fundamental forms of 4-dimensional austere submanifolds were classified by Bryant, into three types which we label A, B and C. In this paper, we show that type A submanifolds correspond exactly to real Kähler submanifolds, we construct new examples of such submanifolds in $\mathbb{R}^6$ and $\mathbb{R}^{10}$, and we obtain classification results on submanifolds with second fundamental forms of maximal type.

## 1. Introduction

**Definitions and background.** Recall that for an immersed submanifold $M^n \subset \mathbb{R}^{n+r}$ with normal bundle $N(M)$, the second fundamental form $\mathrm{II} : TM \otimes TM \to N(M)$ is defined by

$$\mathrm{II}(X,Y) = \pi_N \nabla_X Y,$$

where $X, Y$ are tangent vectors to $M$, $\nabla$ is the Euclidean connection in $\mathbb{R}^{n+r}$, and $\pi_N$ is the orthogonal projection onto the normal bundle. Then $M$ is *austere* if, for any normal vector field $\nu$, the eigenvalues of the quadratic form $\mathrm{II}_\nu(X,Y) := \nu \cdot \mathrm{II}(X,Y)$ with respect to the metric are at each point symmetrically arranged around zero on the real line (equivalently, all odd degree symmetric polynomials in these eigenvalues vanish). When $n = 2$, this just means that $M$ is a minimal surface in $\mathbb{R}^{2+r}$. However, when $n > 2$ the austere condition is stronger than minimality, and leads to a highly overdetermined system of PDEs for the immersion. For example, because of nonlinearity of

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the higher degree symmetric polynomials, it does not suffice to impose the eigenvalue condition on $\mathrm{II}_\nu$ when $\nu$ runs over a basis $\{\nu_a\}$, $1 \le a \le r$, for the orthogonal complement of $T_pM$; rather, the eigenvalue condition applies to all quadratic forms in the space $|\,\mathrm{II}_p\,| \subset S^2T_p^*M$ spanned by the $\nu_a \cdot \mathrm{II}$.

The austere condition was introduced by Harvey and Lawson [6] in connection with special Lagrangian submanifolds. A special Lagrangian submanifold in $\mathbb{C}^n$ is a submanifold of real dimension $n$ that is both Lagrangian and minimal. The importance of special Lagrangian submanifolds lies mainly in the fact that they are area-minimizing. (These submanifolds have also received much recent attention because of their relation to mirror symmetry [11].) Harvey and Lawson showed that the conormal bundle of an immersed submanifold $M \subset \mathbb{R}^n$ is special Lagrangian in the cotangent bundle $T^*\mathbb{R}^n$, equipped with its canonical symplectic structure and metric, if and only if $M$ is an austere submanifold. This result was generalized by Karigiannis and Min-Oo [8] to submanifolds in $S^n$, but with $T^*S^n$ carrying the Stenzel metric and symplectic structure [10].

A systematic study and classification of austere submanifolds of dimension 3 in Euclidean space was first undertaken by Bryant [2], and generalized by Dajczer and Florit [4] to austere submanifolds of arbitrary dimension whose Gauss map has rank two. In studying the austere submanifolds of Euclidean space, we are led first to an algebraic problem. Taking $V = \mathbb{R}^n$ with the standard inner product, a linear subspace $\mathcal{S} \subset S^2V^*$ is called *austere* if any element of $\mathcal{S}$ has eigenvalues occurring in oppositely signed pairs. Since any linear subspace of an austere subspace is also austere, and any isometry of $V$ carries one austere subspace to another, to classify austere subspaces it suffices to find all *maximal* austere subspaces of $S^2V^*$ up to isometries. Bryant classified these spaces for $n=3$ and $n=4$. For $n=3$, he also described the austere 3-folds while the case $n=4$ was left open. We recall his results.

THEOREM 1 (Bryant). *Let $V = \mathbb{R}^3$ and let $\mathcal{S} \subset S^2V^*$ be a maximal austere subspace. Then $\mathcal{S}$ is $O(3)$-conjugate to one of the following*:

$$\text{(a)}\quad \mathcal{S}_A = \left\{ \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \,\middle|\, A \textit{ is a traceless symmetric } 2\times 2 \textit{ matrix} \right\},$$

$$\text{(b)}\quad \mathcal{S}_B = \left\{ \begin{bmatrix} 0 & b \\ {}^tb & 0 \end{bmatrix} \,\middle|\, b \textit{ is a } 1\times 2 \textit{ row vector} \right\}.$$

*Moreover, if $M^3 \subset \mathbb{R}^n$ is an austere submanifold such that for every $p$ in an open subset of $M$ the span $|\,\mathrm{II}_p\,|$ is two-dimensional and is $O(3)$-conjugate to $\mathcal{S}_A$, then $M$ is a product of a minimal surface in $\mathbb{R}^{n-1}$ with a line, an open subset of a cone or a* twisted cone *over a minimal surface in the sphere $S^{n-1}$. Likewise, if $|\,\mathrm{II}_p\,|$ is conjugate to $\mathcal{S}_B$, then $M$ is an open subset of a* generalized helicoid.

In this context, a generalized helicoid $M^n \subset \mathbb{R}^{2n-1}$ is the image of a parametrization

$$(x_0,\dots,x_{n-1}) \mapsto (x_0, x_1\cos(\lambda_1 x_0), x_1\sin(\lambda_1 x_0),\dots,\\ x_s\cos(\lambda_s x_0), x_s\sin(\lambda_s x_0), x_{s+1},\dots,x_{n-1}),$$

where $\lambda_1,\dots,\lambda_s$ are positive constants and $s<n$. For the construction of the twisted cone, see [2].

THEOREM 2 (Bryant). *Let $V=\mathbb{R}^4$ and let $\mathcal{S}\subset S^2V^*$ be a maximal austere subspace. Then $\mathcal{S}$ is $O(4)$-conjugate to one of the following*:

$$\text{(a)}\quad \mathcal{Q}_A = \left\{ \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \,\middle|\, A, B \textit{ are symmetric } 2\times 2 \textit{ matrices} \right\},$$

$$\text{(b)}\quad \mathcal{Q}_B = \left\{ \begin{bmatrix} mI & B \\ {}^tB & -mI \end{bmatrix} \,\middle|\, m\in\mathbb{R}, B \textit{ is a } 2\times 2 \textit{ matrix} \right\},$$

$$\text{(c)}\quad \mathcal{Q}_C = \left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & \lambda_3x_3 & \lambda_2x_2 \\ x_2 & \lambda_3x_3 & 0 & \lambda_1x_1 \\ x_3 & \lambda_2x_2 & \lambda_1x_1 & 0 \end{bmatrix} \,\middle|\, x_1,x_2,x_3\in\mathbb{R} \right\},$$

*where*, *in the last case*, *parameters $\lambda_1,\lambda_2,\lambda_3$ satisfy*

$$\lambda_1 \ge \lambda_2 \ge 0 \ge \lambda_3, \qquad \lambda_1\lambda_2\lambda_3+\lambda_1+\lambda_2+\lambda_3=0. \tag{1}$$

We will say that an austere submanifold $M^4$ is of *type A, B or C* respectively, if for every point $p\in M$ the space $|\,\mathrm{II}_p\,|$ is $O(4)$-conjugate to a subspace of the corresponding maximal austere subspace given in Theorem 2. (In the case of type C, we allow the parameters $\lambda_i$ to vary from point to point in $M$.) It is possible for $|\,\mathrm{II}_p\,|$ to be conjugate to a subspace of more than one maximal subspace (e.g., when $M$ is a hypersurface), but we will assume that there is one particular maximal subspace which applies at all points of $M$.

It is easy to give examples of austere 4-folds of types A and C. When the codimension $r$ is even, any holomorphic submanifold $M^4\subset\mathbb{C}^{2+(r/2)}$ is an austere 4-fold of type A. To see this, let $\mathsf{J}$ be the complex structure, and note that for any vector fields $X,Y$ tangent to $M$

$$\nu_a\cdot\mathrm{II}(X,\mathsf{J}Y)=\nu_a\cdot\nabla_X\mathsf{J}Y=\nu_a\cdot\mathsf{J}(\nabla_XY)=-(\mathsf{J}\nu_a)\cdot\nabla_XY, \tag{2}$$

and therefore

$$\nu_a\cdot\mathrm{II}(X,\mathsf{J}Y)=\nu_a\cdot\mathrm{II}(\mathsf{J}X,Y). \tag{3}$$

Dajczer and Gromoll [5] defined a submanifold $M$ to be *circular* if it carries a parallel complex structure such that (3) holds, and they observed that this condition implies that $M$ is austere.

It follows from (3) that II is represented by matrices in $\mathcal{Q}_A$ when we choose a moving frame $e_1,e_2,e_3,e_4$ along $M$ such that $\mathsf{J}e_1=e_3$ and $\mathsf{J}e_2=e_4$. As for austere 4-folds $M$ of type C, another result of Bryant (see [2], Theorem 3.1) implies that $M$ is a generalized helicoid in $\mathbb{R}^7$ if and only if the parameters $\lambda_1, \lambda_2, \lambda_3$ are identically zero. On the other hand, we do not know if other austere 4-folds of type C exist.

**Our approach.** Our goals in studying austere submanifolds are to obtain new examples and, where possible, to classify austere 4-folds of a given type. We employ the method of moving frames to generate exterior differential systems (EDS) whose solutions correspond to austere 4-folds of a given type and codimension. When such systems are involutive, Cartan–Kähler theory (see [3]) gives us a measure of the size of the solution space, in the form of what initial data may be chosen for a sequence of Cauchy problems that determine every possible local solution. Studying the structure of the exterior differential system can also enable us to establish global properties of solutions (see, e.g., Proposition 5 and Proposition 14 below).

One could organize a classification scheme for austere 4-folds in Euclidean space by type and the dimension $\delta$ of $|\,\mathrm{II}\,|$ (assumed constant over the submanifold).[1] However, we expect to obtain the strongest classification theorems when the austere condition is strongest, that is, when $\delta$ is as large as possible for a given type. Thus, like the earlier results of Bryant on 3-folds, the classification results in this paper are obtained assuming that $|\,\mathrm{II}\,|$ is conjugate to one of $\mathcal{Q}_A$, $\mathcal{Q}_B$ or $\mathcal{Q}_C$. (In this case, we say $M$ is of *maximal* type A, B, or C, respectively.) Classifying austere 4-folds of $M$ of nonmaximal type would involve parametrizing the possible subspaces of a given dimension within $\mathcal{Q}_A$, $\mathcal{Q}_B$ or $\mathcal{Q}_C$ and analyzing the associated EDS. In many instances, the many additional parameters involved make the EDS intractable, even with the assistance of computer algebra systems.

To obtain new examples, an often successful strategy is to assume additional conditions. In the last part of this paper, we obtain new examples of austere 4-folds of nonmaximal type A by assuming that $\delta = 2$ and the space $|\,\mathrm{II}\,|$ lies on a nonprincipal orbit of the action of the symmetry group of $\mathcal{Q}_A$ on the Grassmannian of two-dimensional subspaces of $\mathcal{Q}_A$. One can also carry out this approach for type B with $\delta = 2$, but this yields no new examples. The approach is not feasible for type C because in that case the symmetry group is discrete.

**Outline and summary of results.** In Section 2, we define the moving frames and associated geometric structures we will use in the rest of the paper. Because the exterior differential systems we use are tailored for submanifolds in a specific codimension, we prove a preliminary result in Section 2.1 to the effect that, when $|\,\mathrm{II}\,|$ satisfies certain algebraic criteria, then $\delta$ equals the

[1] By duality, this $\delta$ is also the rank of II as a linear map into the normal bundle; hence, we will often refer to it as the *normal rank* of $M$.

*effective codimension* of $M$ (i.e., the codimension of $M$ within the smallest totally geodesic submanifold containing it).

Section 3 is concerned with austere 4-folds of type A. Note that such submanifolds carry a well-defined almost complex structure $\mathsf{J}$ satisfying (3). In Section 3.1, we derive sufficient conditions on $|\,\mathrm{II}\,|$ for $\mathsf{J}$ to be parallel, implying that $M$ is Kähler. (This gives a converse to Dajczer and Gromoll's result.) We show that the Kählerness conditions apply whenever $|\,\mathrm{II}\,|$ has dimension at least two. The main result of Section 3.2 is the description of the generality of maximal type A austere 4-folds. We show that such submanifolds in $\mathbb{R}^{r+4}$ depend on a choice of $2(r-1)$ functions of 2 variables, in the sense of the Cartan–Kähler Theorem. We conclude that, generically, these austere submanifolds of type A are not holomorphic submanifolds.

Section 4 is concerned with classifying austere 4-folds of maximal types B and C. In Section 4.1, we show that austere 4-folds of maximal type B do not exist. In Section 4.2, we prove two results about maximal type C. First, if $|\,\mathrm{II}\,|$ is at each point conjugate to a fixed maximal austere subspace $\mathcal{Q}_C$ (i.e., the parameters $\lambda_i$ are assumed to be constant over $M$) then $M$ must be a generalized helicoid. Second, even without requiring the parameter values to be fixed, we show that there is only a finite-dimensional family of submanifolds of maximal type C. This follows from showing that, away from certain exceptional parameter values, the characteristic variety of the relevant EDS is empty; we also show that the parameters take value in the exceptional locus on (at most) the complement of an open dense subset of $M$.

In Section 5, we give some interesting examples of austere 4-folds of non-maximal type. As mentioned above, one approach is to assume that $|\,\mathrm{II}\,|$, as a point in the Grassmannian of the relevant maximal austere subspace, is nongeneric for the action of the symmetry group (i.e., it lies along a nonprincipal orbit). In Section 5, we use the symmetry group of $\mathcal{Q}_A$ to normalize 2-dimensional subspaces of $\mathcal{Q}_A$, and identify the nongeneric subspaces. We then classify the type A austere 4-folds for which $|\,\mathrm{II}\,|$ has dimension two and is of fixed nongeneric type, assuming that the Gauss map is nondegenerate. (One can show that if the Gauss map of austere 4-fold is degenerate, then it must have rank at most 2. Austere submanifolds with rank 2 Gauss map were classified by Dajczer and Florit [4].) These submanifolds, which all lie in a totally geodesic $\mathbb{R}^6$, turn out to be either holomorphic submanifolds, products of minimal surfaces, or else 2-ruled submanifolds. The latter have the property that the image of the map $\gamma : M \to G(2,6)$, taking point $p \in M$ into the subspace of $\mathbb{R}^6$ parallel to the ruling through $p$, is a holomorphic curve. Such curves are not arbitrary, however; we also show how these ruled submanifolds may be constructed by instead choosing a general holomorphic curve in $\mathbb{CP}^3$.

We plan to carry out a full classification of 2-ruled austere 4-folds in our next paper.

## 2. Moving frames

In this section, we discuss the first applications of moving frames to the geometry of austere 4-folds. In particular, we obtain upper bounds on the effective codimension of the submanifold, and show that adapted moving frames correspond to integrals of a certain Pfaffian exterior differential system on the appropriate frame bundle.

**2.1. Codimension.** Let $M^4 \subset \mathbb{R}^{r+4}$ be austere. For $p \in M$, let $N_p$ denote the orthogonal complement of $T_pM$ in $\mathbb{R}^{r+4}$. A first approximation to the effective codimension of $M$ is the dimension of its *first normal space* $N_p^1 M$, which is the image of the second fundamental form $\mathrm{II} : S^2 T_pM \to N_p$. Let $\delta(p)$ denote the dimension of the first normal space, which we will refer to as the *normal rank* of $M$ at $p$. This is a lower semicontinuous function on $M$, bounded above by the codimension of $M$. (For an austere submanifold of a given type, $\delta(p)$ is also bounded above by the dimension of the maximal austere subspace in which $|\,\mathrm{II}_p\,|$ lies.) Hence, $\delta(p)$ will be constant on an open set in $M$, so without loss of generality we will assume that $\delta(p)$ is constant.

PROPOSITION 3. *Suppose that $M \subset \mathbb{R}^{r+4}$ is type B or type C. Then the effective codimension of $M$ equals $\delta$.*

*Proof.* Let $e_1, \ldots, e_4, \nu_1, \ldots, \nu_r$ be a moving frame along $M$, such that at each point $p$, $e_1, \ldots, e_4$ span $T_pM$, $\nu_1, \ldots, \nu_\delta$ span $N_p^1 M$, and $\nu_{\delta+1}, \ldots, \nu_r$ are orthogonal to $T_pM \oplus N_p^1 M$.

Let $\mathrm{II}(e_i, e_j) = S^a_{ij}(p)\nu_a$. These symmetric matrices span the subspace $|\,\mathrm{II}_p\,|$, when expressed in terms of the basis $e_1, \ldots, e_4$. (We will use index ranges $1 \le i,j,k \le 4$, $1 \le a,b,c \le \delta$ and $\delta < \beta \le r$.) Let

$$\nabla_{e_i} \nu_a(p) = T^\beta_{ai} \nu_\beta(p) \mod T_pM \oplus N_p^1 M. \tag{4}$$

(We will use summation convention from now on.) Differentiating

$$\nabla_{e_j} e_i \equiv S^a_{ij} \nu_a \mod T_pM \tag{5}$$

along the $e_k$ direction, skew-symmetrizing in $j$ and $k$, and taking the component in the direction of $\nu_\beta$ gives

$$T^\beta_{ak} S^a_{ij} - T^\beta_{aj} S^a_{ik} = 0. \tag{6}$$

Let $V = T_pM$ and let $\mathcal{Q} = |\,\mathrm{II}_p\,| \subset V^* \otimes V^*$. Define the *prolongation of $\mathcal{Q}$* as

$$\mathcal{Q}^{(1)} := \mathcal{Q} \otimes V^* \cap V^* \otimes S^2 V^*. \tag{7}$$

(This is a special case of the definition of the prolongation of a tableau $\mathcal{Q} \subset W \otimes V^*$; see [7], Chapter 4.) Then the equation (6) implies that for each $\beta$ the tensor $U^\beta_{ijk} := T^\beta_{ak} S^a_{ij}$ lies in the space $\mathcal{Q}^{(1)}$.

However, by Lemma 4 below, the space $\mathcal{Q}^{(1)}$ has dimension zero. If all the matrices $S^a_{ij}$ are identically zero, then $M$ is totally geodesic and the proposition is true with $\delta = 0$. Otherwise, the second fundamental form is nonzero on an open set in $M$, and it follows that $T^\beta_{ak} = 0$. In that case, (4) and (5) show that the span $\{e_1, \ldots, e_4, \nu_1, \ldots, \nu_\delta\}$ is fixed as we move along $M$. Thus, $M$ lies in an affine linear subspace of dimension $\delta + 4$. □

LEMMA 4. *Let $\mathcal{Q} \subset \mathcal{Q}_B$ or $\mathcal{Q} \subset \mathcal{Q}_C$. Then $\mathcal{Q}^{(1)}$, as defined by* (7), *has dimension zero.*

*Proof.* It suffices to verify that the prolongations of $\mathcal{Q}_B$ and $\mathcal{Q}_C$ have dimension zero.

It is convenient for us to compute the prolongation as the space of integral elements for a linear Pfaffian system with independence condition. (See [7], Chapters 4–5, or [3], Chapter 4 for more examples.) In the case of $\mathcal{Q}_B$, the 2-forms of such a system would take the form

$$\begin{bmatrix} \pi_0 & 0 & \pi_1 & \pi_2 \\ 0 & \pi_0 & \pi_3 & \pi_4 \\ \pi_1 & \pi_3 & -\pi_0 & 0 \\ \pi_2 & \pi_4 & 0 & -\pi_0 \end{bmatrix} \wedge \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}, \tag{8}$$

where $\pi_0, \ldots, \pi_4, \omega^1, \ldots, \omega^4$ are linearly independent 1-forms and $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \neq 0$ is the independence condition for integral elements. Thus, an integral element satisfying this condition is described by setting $\pi_a = P_{ak}\omega^k$ for some coefficients $P_{ak}$ such that the above 2-forms vanish. The possible values for these coefficients give a parametrization of the space $\mathcal{Q}_B^{(1)}$, since the prolongation is kernel of the composition

$$\mathcal{Q}_B \otimes V^* \longrightarrow S^2V^* \otimes V^* \longrightarrow V^* \otimes \Lambda^2 V^*,$$

where the first map is inclusion and the second skew-symmetization.

The vanishing of the first 2-form in (8) implies that $\pi_0, \pi_1, \pi_2$ cannot contain any $\omega^2$ terms. Applying the same idea to each of the other 2-forms implies, in particular, that on any integral element $\pi_0$ cannot contain terms involving $\omega^1, \omega^3$ or $\omega^4$ either. Thus, $\pi_0 = 0$ on any integral element. Then, examining the first and third rows in (8) shows that, on any integral element, $\pi_1$ must lie in the intersection of spans $\{\omega^1, \omega^2\}$ and $\{\omega^3, \omega^4\}$, and thus must be zero. We similar find the $\pi_2, \pi_3, \pi_4$ must vanish on any integral element, and the prolongation space has dimension zero.

The proof that $\mathcal{Q}_C^{(1)} = 0$ is similar. □

The argument of Lemma 4 does not automatically apply to subspaces of $\mathcal{Q}_A$, because the prolongation of $\mathcal{Q}_A$ is nonzero. In fact, $\mathcal{Q}_A$ is easily seen to be the space of symmetric matrices that anticommute with the complex structure represented by

$$J = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{9}$$

Thus, because $-J = {}^tJ$, a quadratic form $\mathsf{S}$ on the tangent space represented by a matrix in $\mathcal{Q}_A$ is $J$-*linear*, that is, $\mathsf{S}(X, JY) = \mathsf{S}(JX, Y)$. The prolongation of $\mathcal{Q}_A$ is the space of $J$-linear cubic forms, which is spanned by the real and imaginary parts of complex-linear cubic forms in two complex variables, and thus has real dimension 8.

Likewise, the conclusions of Proposition 3 do not apply to all austere 4-folds of type A. For this, we may construct holomorphic submanifolds of real dimension 4 inside $\mathbb{C}^N$, for arbitrarily high $N$, which lie in no lower-dimensional subspace. One example is the Segre embedding of the product of $\mathbb{CP}^1$ with a rational normal curve in $\mathbb{CP}^n$, given in terms of homogeneous coordinates $[u, v]$ and $[z, w]$ by

$$([u, v], [z, w]) \mapsto [zu^n, wu^n, zu^{n-1}v, wu^{n-1}v, \ldots, zv^n, wv^n].$$

This embedding maps $\mathbb{CP}^1 \times \mathbb{CP}^1$ into $\mathbb{CP}^{2n+1}$, and we obtain an austere 4-fold in $\mathbb{C}^{2n+1}$ by intersecting with the domain of a standard chart in projective space.

**2.2. Moving frames and Pfaffian systems.** Let $\mathcal{F}$ be the subbundle of the general linear frame bundle of $\mathbb{R}^{4+r}$ whose fiber at a point $p$ consists of all bases $(e_1, \ldots, e_4, \nu_1, \ldots, \nu_r)$ for $T_p\mathbb{R}^{4+r}$ such that the $e_i$ are orthonormal and orthogonal to the $\nu_a$. (Here, we use index ranges $1 \le i, j, k \le 4$ and $1 \le a, b, c \le r$.) We'll refer to $\mathcal{F}$ as the *semi-orthonormal* frame bundle.

As in Section 2.1, along a submanifold $M^4 \subset \mathbb{R}^{4+r}$ we may adapt a moving frame $e_1, \ldots, e_4$, $\nu_1, \ldots, \nu_r$ such that at each $p \in M$ the frame vectors $e_1(p), \ldots, e_4(p)$ are an orthonormal basis for $T_pM$. (The reason we do not also choose the $\nu_a$ to be orthonormal is that we will adapt them so that the quadratic forms $\mathrm{II}_{\nu_a}$ are represented by a particular basis for an austere subspace.) Then our moving frame along $M$ is a section of $\mathcal{F}|_M$. We will characterize such sections in terms of the canonical and connection 1-forms on $\mathcal{F}$.

These 1-forms are defined in terms of the exterior derivatives of the base-point $p$ and the frame vectors, regarded as $\mathbb{R}^{4+\delta}$-valued functions on $\mathcal{F}$. We let

$$\begin{aligned} dp &= e_i\omega^i + \nu_a\theta^a, \\ de_i &= e_j\phi_i^j + \nu_a\eta_i^a, \\ d\nu_a &= e_j\xi_a^j + \nu_b\kappa_a^b, \end{aligned} \tag{10}$$

define the canonical forms $\omega^i, \theta^a$ and connection forms $\phi^i_j$, $\eta^a_i$, $\xi^i_a$ and $\kappa^a_b$. These 1-forms span the cotangent space of $\mathcal{F}$ at each point but are not linearly independent; differentiating the equations $e_i \cdot e_j = \delta_{ij}$ and $e_i \cdot \nu_a = 0$ yields the relations

$$\phi^i_j = -\phi^j_i, \qquad \xi^i_a = -\eta^b_i g_{ba}, \tag{11}$$

where $g_{ab} = \nu_a \cdot \nu_b$. The exterior derivatives of the canonical forms satisfy structure equations

$$d\begin{bmatrix}\omega\\ \theta\end{bmatrix} = -\begin{bmatrix}\phi & -{}^t\eta g\\ \eta & \kappa\end{bmatrix} \wedge \begin{bmatrix}\omega\\ \theta\end{bmatrix}, \tag{12}$$

where $\omega, \theta, \phi, \eta, \kappa$ denote vector and matrix-valued 1-forms with components $\omega^i$, $\theta^a$, $\phi^i_j$, $\eta^a_i$, and $\kappa^a_b$ respectively, and $g$ has entries $g_{ab}$. Differentiating these equations, and noting that $\mathbb{R}^{4+\delta}$ is flat, gives the derivatives of the matrices of connection forms as

$$\begin{aligned} d\phi &= -\phi \wedge \phi + {}^t\eta \wedge g\eta, \\ d\eta &= -\eta \wedge \phi - \kappa \wedge \eta, \\ d\kappa &= \eta \wedge {}^t\eta g - \kappa \wedge \kappa, \end{aligned} \tag{13}$$

along with

$$dg = g\kappa + {}^t\kappa g.$$

We note the following fundamental fact relating adapted frames and submanifolds of $\mathcal{F}$:

A submanifold $\Sigma^4 \subset \mathcal{F}$ is a section given by an adapted frame along some submanifold $M \subset \mathbb{R}^{4+\delta}$ if and only if $\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4|_\Sigma \neq 0$ and $\theta^a|_\Sigma = 0$.

(We adopt the convention that $|_\Sigma$ for a differential form denotes the pullback to $\Sigma$ of that form under the inclusion map.) The first of these conditions is a nondegeneracy assumption called the *independence condition*. The second condition implies, by differentiation, that

$$\eta^a_i|_\Sigma = S^a_{ij}\omega^j$$

for some functions $S^a_{ij}$. These functions give the components of the second fundamental form in this frame, that is,

$$\mathrm{II}(e_i, e_j) = S^a_{ij}\nu_a. \tag{14}$$

*The standard system.* We will now describe a class of exterior differential system (EDS), for later use, whose integral submanifolds are adapted frames along austere submanifolds. (Being an integral submanifold of an EDS $\mathcal{I}$ means that the pullback to the submanifold of any 1-forms in $\mathcal{I}$ is zero.) To avoid tedious repetition, we will only consider integral submanifolds (and integral elements) satisfying the above independence condition. Unless otherwise stated, we will limit our attention to austere submanifolds whose effective codimension $r$ equals the normal rank $\delta$.

First, suppose we wish to construct an austere submanifold $M$ such that at each point $p$, $|\,\mathrm{II}_p\,|$ is conjugate to a fixed austere subspace $\mathcal{Q}$ of dimension $\delta$ (i.e, $M$ is of type $\mathcal{Q}$). Let symmetric matrices $\hat{S}^1, \ldots, \hat{S}^\delta$ be a fixed basis for this subspace. Then any such submanifold can be locally equipped with an adapted frame such that

$$\mathrm{II}(e_i, e_j) = \hat{S}^a_{ij} \nu_a \tag{15}$$

holds. Conversely, if submanifold $\Sigma^4 \subset \mathcal{F}$ is such that

$$\theta^a|_\Sigma = 0, \qquad (\eta^a_i - \hat{S}^a_{ij}\omega^j)|_\Sigma = 0,$$

then it is the image of a section of $\mathcal{F}|_M$ for some austere manifold $M$ of type $\mathcal{Q}$. Thus, we may define on $\mathcal{F}$ a Pfaffian exterior differential system, the *standard system*,

$$\mathcal{I} = \{\theta^a, \eta^a_i - \hat{S}^a_{ij}\omega^j\}$$

whose integral submanifolds correspond to austere manifolds of this type.

We will also need to consider austere manifolds $M$ where $|\,\mathrm{II}\,|$ is conjugate to an austere subspace $\mathcal{Q}_\lambda$ of fixed dimension $\delta$ but which depends on parameters $\lambda^1, \ldots, \lambda^\ell$ which are allowed to vary along $M$. Suppose that a basis of this subspace is given by symmetric matrices $S^1(\lambda), \ldots, S^\delta(\lambda)$, and the parameters are allowed to range over an open set $L \subset \mathbb{R}^\ell$. Then we may define the *standard system with parameters*

$$\mathcal{I} = \{\theta^a, \eta^a_i - S^a_{ij}\omega^j\},$$

which is analogous to the above, but now defined on the product $\mathcal{F} \times L$. Given any austere manifold $M$ of this kind, we may construct an adapted frame along $M$ such that

$$\mathrm{II}(e_i, e_j) = S^a_{ij}(\lambda)\nu_a$$

for functions $\lambda^1, \ldots, \lambda^\ell$ on $M$. Then the image of the fibered product of the mappings $p \mapsto (p, e_i(p), \nu_a(p))$ and $p \mapsto (\lambda^1(p), \ldots, \lambda^\ell(p))$ will be an integral submanifold of $\mathcal{I}$. Conversely, any integral submanifold of $\mathcal{I}$ satisfying the independence condition gives (by projecting onto the first factor in $\mathcal{F} \times L$) a section of $\mathcal{F}|_M$ which is an adapted frame for an austere manifold $M$.

For later use, we compute the 1-forms of $\mathcal{I}$. We note that $d\theta^a \equiv 0$ modulo the 1-forms of $\mathcal{I}$, so that the only algebraic generator 2-forms are obtained from differentiating the 1-forms $\theta^a_i := \eta^a_i - S^a_{ij}\omega^j$. Using (12) and (13), we obtain

$$d\theta^a_i \equiv -(dS^a_{ij} - S^a_{kj}\phi^k_i - S^a_{ik}\phi^k_j + \kappa^a_b S^b_{ij}) \wedge \omega^j \tag{16}$$

modulo $\theta^a$ and $\theta^a_i$. The 2-forms for the standard system without parameters are obtained replacing $S^a$ in (16) with a constant $\hat{S}^a$.

## 3. Austere submanifolds of type A

In this section, we classify maximal austere submanifolds of type A. Of course, holomorphic submanifolds are of this type, but we will show that general type A austere 4-folds are much more plentiful than holomorphic submanifolds. Before specializing to submanifolds of maximal type A, we will first characterize those type A submanifolds for which the metric is Kähler.

**3.1. Real Kähler submanifolds.** As mentioned in the Introduction, Dajczer and Gromoll [5] observed that a real Kähler submanifold (i.e., a submanifold for which the metric inherited from ambient space is Kähler) is austere. On the other hand, the maximal austere space $\mathcal{Q}_A$ may be characterized as the set of symmetric matrices which anti-commute with a complex structure on $\mathbb{R}^4$ represented by the matrix $J$ given by (9). Thus, the subgroup $U(2)^{\mathbb{R}} \subset \mathrm{GL}(4,\mathbb{R})$ of matrices that commute with $J$ preserve $\mathcal{Q}_A$. In general, we can associate a well-defined almost complex structure[2] to type A austere 4-folds $M$. Thus, it is natural to ask under what circumstances the metric on $M$ is Kähler.

Below, we will give a partial converse to Dajczer and Gromoll's result. In order to state our result precisely, we will need some algebraic preliminaries.

We split the space $\mathfrak{so}(4)$ of skew-symmetric matrices as

$$\mathfrak{so}(4) = \mathfrak{u}(2)^{\mathbb{R}} \oplus \mathcal{P},$$

where $\mathfrak{u}(2)^{\mathbb{R}}$ is the subspace of matrices that commute with $J$ (which is isomorphic to the Lie algebra $\mathfrak{u}(2)$) and $\mathcal{P}$ is the subspace of matrices that anticommute with $J$, which is spanned by the matrices

$$T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad U = -JT = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Proposition 5. *Let $M$ be an austere 4-fold of type $A$, and for $p \in M$ let $|\,\mathrm{II}_p\,|$ be $O(4)$-conjugate to a subspace $\mathcal{Q}(p) \subset \mathcal{Q}_A$. Define the map*

$$\mathsf{K} : S \mapsto ([S,T] | [S,U])$$

*from the space of $4 \times 4$ matrices $S$ into the space of $4 \times 8$ matrices. Suppose that for every $p$ the image of $\mathcal{Q}(p)$ under $\mathsf{K}$ has the property that the common nullspace of all matrices in the image is zero. Then $M$ is Kähler with respect to the complex structure defined by $J$.*

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[2] In fact, for a generic two-dimensional subspace $\mathcal{Q} \subset \mathcal{Q}_A$ the complex structure which anticommutes with matrices in $\mathcal{Q}$ is uniquely defined up to a minus sign.

*Proof.* Let $\hat{S}^1, \dots, \hat{S}^6$ be a fixed basis for $\mathcal{Q}_A$. Let $r$ be the effective codimension of $M$, not assumed to be the same as the normal rank of $M$. Locally on $M$, we may construct an adapted frame $e_1, \dots, e_4, \nu_1, \dots, \nu_r$ such that

$$\mathrm{II}(e_i, e_j) = v^a_h \hat{S}^h_{ij} \nu_a,$$

where $v^a_h$ are some functions on $M$ and $1 \le h \le 6$. Then the adapted frame defines a local section $f : M \to \mathfrak{F}$ such that $f^*\omega^i$ span the cotangent space of $M$ and the image of $f$ is an integral of the 1-forms $\theta^a$ and

$$\theta^a_i := \eta^a_i - v^a_h \hat{S}^h_{ij} \omega^j.$$

By specializing the computation (16) to the case where $S^a_{ij} = v^a_h \hat{S}^h_{ij}$, we obtain $d\theta^a_i \equiv -\Omega^a_i$ modulo the forms $\theta^a_i$, where

$$\Omega^a_i := \big((dv^a_h + \kappa^a_b v^b_h)\hat{S}^h_{ij} - v^a_h [\hat{S}^h, \phi]_{ij}\big) \wedge \omega^j \tag{17}$$

and $[\hat{S}^h, \phi]$ denotes the commutator.

These 2-forms must vanish under pullback via $f$. Consider the 4-forms

$$\Xi^a_i := \Omega^a_i \wedge U_{k\ell}\omega^k \wedge \omega^\ell - J_{im}\Omega^a_m \wedge T_{k\ell}\omega^k \wedge \omega^\ell.$$

Using the fact that $U = -JT$, we can expand these as

$$\begin{aligned} \Xi^a_i = &-(dv^a_h + \kappa^a_b v^b_h) \wedge \big(\hat{S}^h_{ij}(JT)_{k\ell} + J_{im}\hat{S}^h_{mj} T_{k\ell}\big) \wedge \omega^j \wedge \omega^k \wedge \omega^\ell \\ &+ v^a_h \big([\hat{S}^h, \phi]_{ij}(JT)_{k\ell} + J_{im}[\hat{S}^h, \phi]_{mj} T_{k\ell}\big) \wedge \omega^j \wedge \omega^k \wedge \omega^\ell. \end{aligned}$$

Next, write $\phi = \widetilde{\phi} + \psi$, where $\widetilde{\phi}$ takes value in $\mathfrak{u}(2)^{\mathbb{R}}$ and $\psi$ takes value in $\mathcal{P}$. Using the fact that the matrices $\hat{S}^h$ and $[\hat{S}^h, \widetilde{\phi}]$ anticommute with $J$ while $[\hat{S}^h, \psi]$ commutes with $J$, we have

$$\begin{aligned} \Xi^a_i = \big((dv^a_h &+ \kappa^a_b v^b_h) \wedge \hat{S}^h_{ij}(-\omega \wedge {}^t\omega J + J\omega \wedge {}^t\omega)_{jk} \\ &+ v^a_h \big([\hat{S}^h, \widetilde{\phi}]_{ij} \wedge (-\omega \wedge {}^t\omega J + J\omega \wedge {}^t\omega)_{jk} \\ &+ [\hat{S}^h, \psi]_{ij} \wedge (\omega \wedge {}^t\omega J + J\omega \wedge {}^t\omega)_{jk}\big)\big) \wedge T_{k\ell}\omega^\ell. \end{aligned}$$

It is easy to verify that $(-\omega \wedge {}^t\omega J + J\omega \wedge {}^t\omega) \wedge T\omega = 0$. Computing the remaining terms gives

$$\Xi^a_i = -4v^a_h [\hat{S}^h, \psi]_{ij} U_{jk} \wedge \omega_{(k)},$$

where $\omega_{(k)}$ denotes the 3-form which is the wedge product of the $\omega^i$ such that $\omega^j \wedge \omega_{(k)} = \delta^j_k \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4$.

Write $\psi = \psi_1 T + \psi_2 U$, where $\psi_1 = \frac{1}{2}(\phi^2_1 - \phi^4_3)$ and $\psi_2 = \frac{1}{2}(\phi^3_2 - \phi^4_1)$. Suppose that $f^*\psi_1 = a_k f^*\omega^k$ and $f^*\psi_2 = b_k f^*\omega^k$. Then the vanishing of $\Xi^a_i$ and the fact that $f^*(\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4) \neq 0$, implies that the $a_k$ and $b_k$ must satisfy

$$v^a_h([\hat{S}^h, T]_{ij} U_{jk} a_k + [\hat{S}^h, U]_{ij} U_{jk} b_k) = 0.$$

Thus, if $\mathcal{Q}(p)$ satisfies the given conditions, then $a_k$ and $b_k$ must vanish. Therefore, the connection forms $\phi$ of $M$ take value in $\mathfrak{u}(2)^{\mathbb{R}}$, the connected component of the holonomy group of $M$ lies in $U(2)^{\mathbb{R}}$, and it follows that $M$ is Kähler. $\square$

Note that the vanishing of $\psi$ implies the vanishing of additional polynomials in the coefficients $v^a_h$. To express these, introduce the notation $\{\cdot\}_{\mathcal{P}}$ for the projection of an $\mathfrak{so}(4)$-valued function (or differential form) into the subspace $\mathcal{P}$. By (13),

$$d\psi \equiv \{{}^t\eta \wedge g\eta\}_{\mathcal{P}} \mod \psi.$$

Furthermore, the $(i,j)$ entry of the matrix within braces is congruent, modulo the $\theta^a_i$, to the 2-form $Y^i_{jk\ell}\omega^k \wedge \omega^\ell$, where

$$Y^i_{jk\ell} := g_{ab} v^a_h v^b_{h'} (\hat{S}^h_{ik}\hat{S}^{h'}_{j\ell} - \hat{S}^h_{i\ell}\hat{S}^{h'}_{jk}).$$

Then these additional conditions take the form

$$Y^2_{1k\ell} = Y^4_{3k\ell}, \qquad Y^3_{2k\ell} = Y^4_{1k\ell} \tag{18}$$

for all $k < \ell$.

Proposition 6. *The only 2-dimensional subspaces of $\mathcal{Q}_A$ which do not satisfy the hypothesis of Proposition 5 are conjugate, via the action of $U(2)^{\mathbb{R}}$, to the following*:

$$\mathcal{Q}_x = \{S_x, \widetilde{J}S_x\}, \quad \textit{where } S_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -x \end{bmatrix}, \widetilde{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

*and $x$ is a real parameter.*

*Proof.* Any nonzero matrix in $\mathcal{Q}_A$ can be diagonalized using $U(2)^{\mathbb{R}}$, and scaled to be equal to $S_x$ for some $x \in \mathbb{R}$. The kernel of $\mathsf{K}(S_x)$ is a 4-dimensional subspace of $\mathbb{R}^8$. Requiring that a general element $S \in \mathcal{Q}_A$, linearly independent from $S_x$, has the property that the restriction of $\mathsf{K}(S)$ to $\ker \mathsf{K}(S_x)$ is singular, implies that $S$ must be a multiple of $\widetilde{J}S_x$. $\square$

Corollary 7. *All type A submanifolds with $\delta \geq 2$ are Kähler.*

*Proof.* Because of Proposition 5, we need only check those submanifolds such that $|\,\mathrm{II}_p\,|$ is at every point conjugate to a space of the form $\mathcal{Q}_x$ defined in Proposition 6. Note that matrices in $\mathcal{Q}_x$ anticommute with $\widetilde{J}$, and $\widetilde{J} = PJP^{-1}$ for a permutation matrix $P$. Thus, we may repeat the argument of the proof of Proposition 5 with all matrices in $\mathfrak{so}(4)$ replaced by their conjugates under $P$. We conclude that the submanifold is Kähler with respect to the complex structure defined by $\widetilde{J}$. $\square$

**3.2. Maximal type A.** In the rest of this section, we discuss austere submanifolds of type A with $\delta=6$, that is, whose second fundamental forms span the entire space $\mathcal{Q}_A$. We fix the following basis for this space:

$$\hat{S}^1=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{bmatrix},\qquad \hat{S}^2=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{bmatrix},$$
$$\hat{S}^3=\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix},\qquad \hat{S}^4=\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix},$$
$$\hat{S}^5=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\qquad \hat{S}^6=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&1&0&0\end{bmatrix}.$$

Let $M$ be an austere submanifold of this type; for the sake of simplicity, we first assume that $\operatorname{cod} M=\delta=6$, that is, $M$ lies in $\mathbb{R}^{10}$. Let $\mathcal{F}$ be the bundle of semi-orthonormal frames on $\mathbb{R}^{10}$, as described Section 2.2. The manifold $\mathcal{F}$ has dimension $10+24+36+6$; referring to the structure equations (12), we note that the components of $\omega$, $\theta$, $\eta$, $\kappa$, and the lower triangle of $\phi$ give a coframe on $\mathcal{F}$.

By hypothesis, along $M$ there is a moving frame $(e_1,e_2,e_3,e_4,\nu_1,\ldots,\nu_6)$ such that

$$\mathrm{II}(e_i,e_j)=\hat{S}^a_{ij}\nu_a. \tag{19}$$

(Here, we take the convention that indices $i,j,k$ run between 1 and 4, while indices $a,b$ run between 1 and $\delta$.) The image $\Sigma\subset\mathcal{F}$ is an integral of the standard system $\mathcal{I}$ defined in Section 2.2. By (16), we see that the 2-forms of $\mathcal{I}$ are given by $\pi^a_{ij}\wedge\omega^j$ where

$$\pi^a_{ij}:=\kappa^a_b\hat{S}^b_{ij}-\hat{S}^a_{ik}\phi^k_j-\hat{S}^a_{jk}\phi^k_i.$$

Applying Propositions 5 and 6, we see that $M$ must be Kähler; in particular, the differential forms $\psi_1=\frac{1}{2}(\phi^2_1-\phi^4_3)$ and $\psi_2=\frac{1}{2}(\phi^3_2-\phi^4_1)$ must also vanish along $\Sigma$. Therefore, an adapted frame along an austere 4-fold of this type will be an integral of the augmented Pfaffian system

$$\mathcal{I}^+=\{\theta^a,\eta^a_i-\hat{S}^a_{ij}\omega^j,\psi_1,\psi_2\}.$$

With the addition of the 1-forms $\psi_1,\psi_2$ come the additional integrability conditions (18). In this case, these generate only two linearly independent conditions,

$$g_{13}-g_{22}-g_{46}+g_{55}=0,\qquad g_{16}-2g_{25}+g_{34}=0. \tag{20}$$

Since these conditions are constraints on how the normal vectors $\nu_a$ may be arranged, they hold only on a codimension-two submanifold of the frame bundle $\mathcal{F}$. Let $\mathcal{F}' \subset \mathcal{F}$ denote this submanifold. We now apply Cartan's test for involutivity to the pullback of the Pfaffian system $\mathcal{I}^+$ to $\mathcal{F}'$.

PROPOSITION 8. *On $\mathcal{F}'$, the system $\mathcal{I}^+$ is involutive with Cartan characters $s_1 = 24$, $s_2 = 10$.*

*Proof.* As in the proof of Proposition 5, let $\widetilde{\phi}$ be the projection of $\phi$ into $\mathfrak{u}(2)^{\mathbb{R}}$. Then

$$d\theta_i^a \equiv \widetilde{\pi}_{ij}^a \wedge \omega^j \mod \theta^a, \theta_i^a, \psi_1, \psi_2,$$

where we define

$$\widetilde{\pi}_{ij}^a := \kappa_b^a \hat{S}_{ij}^b - \hat{S}_{ik}^a \widetilde{\phi}_j^k - \hat{S}_{jk}^a \widetilde{\phi}_i^k.$$

Next, let $\widetilde{\pi}^a$ stand for the matrix-valued 1-form whose entries are $\widetilde{\pi}_{ij}^a$. On $\mathcal{F}$, the 36 1-forms $\kappa_b^a$ are linearly independent. Because, for each $a$, $\widetilde{\pi}^a$ takes value in the 6-dimensional space $\mathcal{Q}_A$, it follows that on $\mathcal{F}$ there are exactly 36 linearly independent forms among the $\widetilde{\pi}_{ij}^a$. Differentiating (20) shows that two of these forms pull back to $\mathcal{F}'$ to be linearly dependent on the others; for example, one can solve for $\pi_{22}^4$ and $\pi_{24}^4$ in terms of the other $\pi_{ij}^a$. It follows that, when pulled back to $\mathcal{F}'$, there are 24 linearly independent 1-forms among the $\pi_{1j}^a$ and 10 further independent 1-forms among the $\pi_{2j}^a$. This gives us the claimed values for the Cartan characters.

To apply Cartan's test, we need to calculate the fiber dimension of the space of 4-dimensional integral elements at points on $\mathcal{F}'$. Suppose that an integral element is defined by

$$\widetilde{\pi}_{ij}^a = p_{ijk}^a \omega^k,$$

where $p_{ijk}^a$ is symmetric in $i, j, k$, and for any fixed $a$ and $k$ is in the space $\mathcal{Q}_A$. For each $a$, the space of symmetric tensors satisfying these conditions is isomorphic to the prolongation $\mathcal{Q}_A^{(1)}$, which has dimension 8. As $a$ varies, we obtain a 48-dimensional space of solutions $p_{ijk}^a$. However, the corresponding integral 4-planes must be tangent to the submanifold $\mathcal{F}'$. This requirement imposes 4 additional linearly independent homogeneous conditions on the $p_{ijk}^a$, so we conclude that the fiber dimension of the space of integral elements tangent to $\mathcal{F}'$ is 44. Since this dimension coincides with $s_1 + 2s_2$, the system is involutive. □

We now state the following theorem.

THEOREM 9. *Austere 4-folds in $\mathbb{R}^{10}$ of maximal type A exist and depend locally on a choice of 10 functions of 2 variables. Each of them carries a complex structure with respect to which the metric inherited from ambient space is Kähler, but they are generically not complex submanifolds.*

*Proof.* The first assertion follows by applying the Cartan–Kähler theorem (cf. Theorem 7.3.3 in [7]), given the fact that the system $\mathcal{I}^+$ is involutive with characters computed in Proposition 8. The second assertion follows from Proposition 5. Evidence for the third assertion is provided by the fact that holomorphic submanifolds of real dimension four in $\mathbb{R}^{10} \simeq \mathbb{C}^5$ are locally the graphs of three holomorphic functions of two complex variables. The Cauchy–Riemann system for one function of two complex variables is involutive with Cartan character $s_2 = 2$, so such submanifolds $M$ depend locally on a choice of 6 functions of 2 real variables. However, we will provide a more concrete argument as to how the solutions of the above Pfaffian system fail, in general, to be holomorphic submanifolds.

Suppose that $M^4 \subset \mathbb{C}^5$ is a holomorphic submanifold with normal space of real dimension 6, and let $\mathsf{J}$ denote the ambient complex structure. Adapt a framing along $M$ so that (19) holds. Then equation (2) implies that $\mathrm{II}(e_i, \mathsf{J}e_j) = -\mathsf{J}\,\mathrm{II}(e_i, e_j)$. Applying (19) to both sides and observing that $\hat{S}^1 J = -\hat{S}^4$, $\hat{S}^2 J = -\hat{S}^5$, $\hat{S}^3 J = -\hat{S}^6$ shows that

$$\mathsf{J}\nu_1 = \nu_4, \qquad \mathsf{J}\nu_2 = \nu_5, \qquad \mathsf{J}\nu_3 = \nu_6. \tag{21}$$

By abuse of notation, we can assume that $\mathsf{J}$ is a constant matrix. Differentiating, for example, the equation $\mathsf{J}\nu_1 = \nu_4$, and using (10), yields

$$\mathsf{J}(e_j \xi^j_1 + \nu_a \kappa^a_1) = e_j \xi^j_4 + \nu_a \kappa^a_4.$$

In particular, such framings satisfy $\kappa^1_1 = \kappa^4_4$, $\kappa^2_1 = \kappa^5_4$ and $\kappa^3_1 = \kappa^6_4$. More generally, differentiating the equations (21) shows that the matrix $\kappa$ of 1-forms $\kappa^a_b$ must commute with the matrix

$$L = \begin{bmatrix} 0 & -I_{3\times 3} \\ I_{3\times 3} & 0 \end{bmatrix}.$$

Because $L^a_b \hat{S}^b = \hat{S}^a K$, we must have $\kappa^{a+3}_{b+3} = \kappa^a_b$ and $\kappa^{a+3}_b = -\kappa^a_{b+3}$ for $1 \le a, b \le 3$. It follows that the 36 1-forms $\widetilde{\pi}^a_{ij}$ must satisfy

$$L^a_b \widetilde{\pi}^b_{ij} = \widetilde{\pi}^a_{ik} K^k_j = \widetilde{\pi}^a_{jk} K^k_i.$$

(In this equation, we revert to $1 \le a, b \le 6$.) Because involutivity implies that integral manifolds may be constructed passing through any given initial integral element, we see that a generic solution will not satisfy these extra necessary conditions. □

As noted in Section 2.1, space $\mathcal{Q}^{(1)}_A$ has nonzero dimension, so austere 4-folds of type A with $\delta = 6$ may in fact have codimension $r > 6$. To see how many of these there are, suppose that along such a submanifold $M$ we adapt moving frames, as in the proof of Proposition 3, so that $\nu_1, \ldots, \nu_6$ span $N^1_p M$, and $\nu_7, \ldots, \nu_r$ are orthogonal to $T_p M \oplus N^1_p M$. (As before, let indices $a, b$ run from 1 to 6, but now let indices $\alpha, \beta$ run between 7 and $r$.)

Such moving frames, as sections of $\mathcal{F}$, give integral submanifolds of the following Pfaffian system:

$$\mathcal{I}^+ = \{\theta^a, \theta^\beta, \psi_1, \psi_2, \eta_i^a - \hat{S}_{ij}^a \omega^j, \eta_i^\beta\}.$$

Again, we restrict to the submanifold $\mathcal{F}'$ where the integrability conditions (20) hold. We compute

$$d\eta_i^\beta \equiv -\kappa_b^\beta \hat{S}_{ij}^b \wedge \omega^j$$

modulo the 1-forms in $\mathcal{I}^+$. For every fixed index $\beta$, the tableau component given by $\kappa_b^\beta \hat{S}_{ij}^b$ is isomorphic to $\mathcal{Q}_A$, and is involutive with characters $s_1 = 4, s_2 = 2$. Combining this with the results of Proposition 8 we conclude that the EDS $\mathcal{I}^+$ is involutive with characters $s_1 = 24 + 4(r-6) = 4r$ and $s_2 = 10 + 2(r-6) = 2r-2$.

We conclude that type A austere 4-folds in $\mathbb{R}^{4+r}$ with maximal first normal space (so that $r \geq 6$) depend on a choice of $2(r-1)$ functions of 2 variables. By contrast, when $r$ is even, holomorphic submanifolds of real dimension 4 depend on $r$ functions of 2 variables.

## 4. Maximal types B and C

**4.1. Submanifolds of maximal type B.** Let $M$ be an austere submanifold of type $B$ of normal rank $\delta$. By hypothesis, there is a moving frame $(e_1, e_2, e_3, e_4, \nu_1, \ldots, \nu_\delta)$ such that the $e_i$ are orthonormal and tangent to $M$, and in each normal direction $\nu_a$ the shape operator takes the form

$$\nu_a \cdot \mathrm{II} = \begin{bmatrix} m^a I & B^a \\ {}^t B^a & -m^a I \end{bmatrix}. \tag{22}$$

We consider the standard system with parameters

$$\mathcal{I} = \{\theta^a, \eta_i^a - S_{ij}^a \omega^j\}$$

on $\mathcal{F} \times \mathbb{R}^{5\delta}$, where $\mathcal{F}$ is the semi-orthonormal frame bundle of $M$ and $\mathrm{II}(e_i, e_j) = S_{ij}^a \nu_a$ for matrices $S^a$ of the form (22). (For each $a$, the parameters are the scalar $m^a$ and the entries of $B^a$.) The integral submanifolds of this EDS correspond to austere submanifolds of type B. As in (16), we compute the system 2-forms as

$$d(\eta_i^a - S_{ij}^a \omega^j) \equiv -(dS_{ij}^a - [S^a, \phi]_{ij} + \kappa_b^a S_{ij}^b) \wedge \omega^j$$

modulo the 1-forms of the ideal $\mathcal{I}$, where $[S^a, \phi]$ denotes the commutator. Hence, the tableau of the system is spanned by the 1-forms

$$\pi_{ij}^a := dS_{ij}^a - [S^a, \phi]_{ij} + \kappa_b^a S_{ij}^b. \tag{23}$$

The 5-dimensional space $\mathcal{Q}_B$ of type B second fundamental forms is spanned symmetric matrices that anticommute with the reflection

$$R=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix},$$

along with $R$ itself. The space $\mathfrak{so}(4)$ of skew-symmetric matrices splits as a direct sum of two subspaces $W_1$ and $W_2$, where $W_1=\{[\begin{smallmatrix}A_1&0\\0&A_2\end{smallmatrix}], A_1, A_2\in\mathfrak{so}(2)\}$ is the 2-dimensional subspace of matrices which commute with $R$ and $W_2=\{[\begin{smallmatrix}0&B\\-{}^tB&0\end{smallmatrix}], B\in\mathcal{M}_2(\mathbb{R})\}$ is the 4-dimensional subspace of matrices that anticommute with $R$. We write the $\mathfrak{so}(4)$-valued connection form $\phi$ as $\phi=\phi_0+\psi$, where $\phi_0$ takes value in $W_1$ and $\psi$ takes value in $W_2$.

If $S\in\mathcal{Q}_B$, then $[S,\phi_0]$ anticommutes with the reflection $R$. To see this, suppose $S$ belongs to the subspace of matrices in $\mathcal{Q}_B$ that anticommute with $R$. Then

$$[S,\phi_0]R=S\phi_0R-\phi_0SR=SR\phi_0+\phi_0RS=-RS\phi_0+R\phi_0S=-R[S,\phi_0].$$

On the other hand, if $S$ is a multiple of $R$ we can see easily that $[S,\phi_0]$ again anticommutes with $R$.

The following result gives an upper bound for the fiber dimension of the set of integral elements of the Pfaffian system $\mathcal{I}$. We point out that this result is independent of the normal rank of the submanifold.

PROPOSITION 10. *The fiber dimension of the set of integral* 4*-planes* (*satisfying the independence condition*) *of the Pfaffian system* $\mathcal{I}$ *is at most* 16.

*Proof.* As the 2-forms of the system are $\pi^a_{ij}\wedge\omega^j$, it follows that $\pi^a_{ij}=P^a_{ijk}\omega^k$ on any integral element, where

$$P^a_{ijk}=P^a_{ikj} \tag{24}$$

for every $a,i,j,k$. Moreover, an integral element at a point in $\mathcal{F}\times\mathbb{R}^{5\delta}$ is uniquely determined by these coefficients $P^a_{ijk}$. We decompose the space of symmetric $4\times4$ matrices as $\mathcal{Q}_B\oplus\mathcal{U}$ (where $\mathcal{U}$ is the orthogonal complement) and write

$$P^a_{ijk}=Q^a_{ijk}+R^a_{ijk},$$

where $Q^a_{ijk}, R^a_{ijk}$ take value in $\mathcal{Q}_B\otimes\mathbb{R}^4$ and $\mathcal{U}\otimes\mathbb{R}^4$ respectively, for every index $a$. Then (24) is a set of linear equations satisfied by the $Q^a_{ijk}$ and $R^a_{ijk}$. Since $\phi=\phi_0+\psi$ and $[S^a,\phi_0]\in\mathcal{Q}_B$, it follows that the projection of $\pi^a$ into the space $\mathcal{U}$ is the projection of $[S^a,\psi]$ onto $\mathcal{U}$. Therefore, $R^a_{ijk}$ is completely determined by the value of the $W_2$-valued 1-form $\psi$ on the integral element. Because $W_2$ is 4-dimensional, these $R^a_{ijk}$ depend on at most 16 parameters. Now rewrite (24) as a nonhomogeneous linear system for $Q^a_{ijk}$:

$$Q^a_{ijk}-Q^a_{jik}=-R^a_{ijk}+R^a_{jik}. \tag{25}$$

The dimension of the solution space of this system is the same for any set of values for the $R^a_{ijk}$. In particular, when $R^a_{ijk}$ is zero, (25) implies that $Q^a_{ijk}$ takes value in $\mathcal{Q}_B^{(1)}$, which by Lemma 4 is zero-dimensional. Therefore, the values of $Q^a_{ijk}$ satisfying (25) are uniquely determined by the parameters that give the $R^a_{ijk}$. $\square$

The next result shows that there are no type B austere 4-folds of maximal normal rank.

PROPOSITION 11. *A type B austere submanifold $M^4$ cannot have first normal space of dimension $\delta = 5$.*

*Proof.* If $\delta = 5$, then at each point the second fundamental form spans all of $\mathcal{Q}_B$. We can therefore choose smooth, linearly independent normal vector fields $\nu_1, \ldots, \nu_5$ so that $m^1, \ldots, m^4$ are identically zero, $B^1$ through $B^4$ are given by

$$B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B^3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

and $m^5 = 1$, $B^5 = 0$. The system 2-forms, which we denote by

$$\Omega^a_i := \pi^a_{ij} \wedge \omega^j$$

are simplified considerably. Moreover, addition 1-forms, not in the ideal, are forced to vanish on all integral 4-planes. Such 1-forms may be determined by examining the *tableau* of the Pfaffian system. For example, the system 2-forms $\Omega^5_i$ may be written in matrix-vector form as follows

$$\begin{bmatrix} \Omega^5_1 \\ \Omega^5_2 \\ \Omega^5_3 \\ \Omega^5_4 \end{bmatrix} = \begin{bmatrix} \kappa^5_5 & 0 & \kappa^5_1 + 2\phi^3_1 & \kappa^5_2 + 2\phi^4_1 \\ 0 & \kappa^5_5 & \kappa^5_3 + 2\phi^3_2 & \kappa^5_4 + 2\phi^4_2 \\ \kappa^5_1 + 2\phi^3_1 & \kappa^5_3 + 2\phi^3_2 & -\kappa^5_5 & 0 \\ \kappa^5_2 + 2\phi^4_1 & \kappa^5_4 + 2\phi^4_2 & 0 & -\kappa^5_5 \end{bmatrix} \wedge \begin{bmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \end{bmatrix}.$$

This piece of the tableau allows us to deduce that $\kappa^5_5$ must vanish on any integral 4-plane $E$: for, the vanishing of the 2-forms $\Omega^5_1$ and $\Omega^5_2$ implies that $\kappa^5_5$ restricts to $E$ to be a linear combination of $\omega^3$ and $\omega^4$, while the vanishing of $\Omega^5_3$ and $\Omega^5_4$ implies that $\kappa^5_5$ restricts to be a linear combination of $\omega^1$ and $\omega^2$. Thus, $\kappa^5_5 = 0$ on any such integral element. (See the proof of Proposition 13 below for a more subtle example of this kind of calculation.)

In all, the additional 1-forms that vanish on all integral elements are

$$\begin{aligned} \psi_1 &= \kappa^5_1 + 2\phi^3_1, & \psi_6 &= \kappa^1_2 - \kappa^4_3 + 2\phi^4_3, \\ \psi_2 &= \kappa^5_2 + 2\phi^4_1, & \psi_7 &= \kappa^1_3 - \kappa^4_2 + 2\phi^2_1, \\ \psi_3 &= \kappa^5_3 + 2\phi^3_2, & \psi_8 &= \kappa^2_1 - \kappa^3_4 - 2\phi^4_3, \end{aligned}$$

$$\begin{aligned} \psi_4 &= \kappa^5_4 + 2\phi^4_2, & \psi_9 &= \kappa^3_1 - \kappa^2_4 - 2\phi^2_1,\\ \psi_5 &= \kappa^5_5, & \psi_{10} &= \kappa^1_1 - \kappa^2_2 - \kappa^3_3 + \kappa^4_4,\\ & & \psi_{11} &= \kappa^1_4 + \kappa^2_3 + \kappa^3_2 + \kappa^4_1. \end{aligned}$$

Thus, any integral 4-fold of the EDS $\mathcal{I}$ will also be an integral of the 1-forms $\psi_1, \dots, \psi_{11}$. Let $\mathcal{J}$ be the differential ideal resulting from adding these 1-forms to $\mathcal{I}$. The exterior derivatives of the $\psi$'s, modulo the 1-forms of $\mathcal{J}$, are linear combinations of wedge products of the $\kappa$'s with each other, and with the $\phi$'s. Thus, $\mathcal{J}$ is a nonlinear Pfaffian system. In particular, if we substitute the values given by $\pi^a_{ij} = P^a_{ijk}\omega^k$ into the new 2-forms, and take coefficients with respect to the 2-forms $\omega^1 \wedge \omega^2$, $\omega^1 \wedge \omega^3$, $\omega^1 \wedge \omega^4$, $\omega^2 \wedge \omega^3$, $\omega^2 \wedge \omega^4$, $\omega^3 \wedge \omega^4$, we obtain 66 quadratic polynomials in the $P^a_{ijk}$ which must vanish in order for an integral element of $\mathcal{I}$ to be an integral element of $\mathcal{J}$. Eliminating the $P^a_{ijk}$ from these polynomials yields integrability conditions in terms of the $g_{ab}$ which include $g_{11} + g_{55} = g_{22} + g_{44} = 0$. Since this is impossible for components of a positive definite metric, we conclude that the set of integral 4-planes of $\mathcal{J}$ satisfying the independence condition is empty. □

**4.2. Submanifolds of maximal type C.** We begin by noting that the space $\mathcal{Q}_C$ of quadratic forms is invariant under conjugation by a discrete subgroup of $O(4)$ that simultaneously permutes $x_1, x_2, x_3$ and $\lambda_1, \lambda_2, \lambda_3$. These permutations will, of course, preserve the equation in (1) satisfied by the $\lambda$'s, but will not preserve the inequalities in (1).

We now discuss submanifolds of type C whose first normal space is of dimension $\delta = 3$. These submanifolds lie in $\mathbb{R}^7$ as seen in Proposition 3. As was the case with submanifolds of type B whose second fundamental form had maximal span, we can choose an orthonormal frame $e_1, e_2, e_3, e_4$ for the tangent space and a basis $\nu_1, \nu_2, \nu_3$ for the first normal space with respect to which the second fundamental form is represented by any basis for the space $\mathcal{Q}_C$ we choose. Accordingly, let $\mathcal{F}$ be the bundle of such frames on $\mathbb{R}^7$ and use the basis matrices

$$S^1 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda_1\\ 0 & 0 & \lambda_1 & 0 \end{bmatrix}, \qquad S^2 = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & \lambda_2\\ 1 & 0 & 0 & 0\\ 0 & \lambda_2 & 0 & 0 \end{bmatrix},$$
$$S^3 = \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & \lambda_3 & 0\\ 0 & \lambda_3 & 0 & 0\\ 1 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

where we assume that

$$\lambda_1\lambda_2\lambda_3 + \lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{28}$$

Choosing a moving frame so that the second fundamental form in the direction of $\nu_a$ is represented by matrix $S^a$ means that $S^a$ gives the components of the 1-forms $\eta_i^a$ in terms of the $\omega^j$. So, the moving frame will be an integral of the standard system with parameters:

$$\mathcal{I} = \{\theta^a, \theta_i^a\}, \quad \theta_i^a := \eta_i^a - S_{ij}^a \omega^j, \tag{29}$$

where $1 \leq i, j, k \leq 4$ and $1 \leq a, b \leq 3$.

This Pfaffian system is defined on $\mathcal{F} \times L$, where $L \subset \mathbb{R}^3$ is the smooth affine algebraic variety defined by (28), minus the origin. (We assume that not all the $\lambda$'s are zero, at least on an open set in the submanifold; otherwise, the submanifold must be a generalized helicoid.) Using the permutation symmetry of $L$, we may assume without loss of generality that $\lambda_1 \geq \lambda_2 \geq 0$, and thus we may solve (28) for $\lambda_3$ in terms of $\lambda_1$ and $\lambda_2$.

The 2-forms of this EDS are given by (16). Let $\pi_{ij}^a = \pi_{ji}^a$ be the 1-forms defined as in (23). These 30 1-forms are not all independent; in fact, they are linear combinations of the 17 independent 1-forms $d\lambda_1, d\lambda_2$, $\phi_j^i$ and $\kappa_b^a$. The span of the components $\pi_{ij}^a$ within the cotangent space of $\mathcal{F} \times L$ will be the same as the span of these 17 1-forms, provided that $\lambda_2$ is nonzero. (Otherwise, nontrivial Cauchy characteristics will be present.) Let $L_0$ denote the open subset of $L$ where $\lambda_1 \geq \lambda_2 > 0$. At each point of $\mathcal{F} \times L_0$, the set of integral 4-planes of $\mathcal{I}$ has dimension 8, while the Cartan characters of the system are $s_1 = 12$ (for the 12 independent 1-forms $\pi_{1j}^a$, for example) and $s_2 = 5$. Since $8 < s_1 + 2s_2$, the system fails to be involutive.

Without prolongation, we can obtain more information about the system by calculating its characteristics.

Proposition 12. *At points of $\mathcal{F} \times L_0$ where neither of $\lambda_1$ or $\lambda_2$ is equal to* 1, *the characteristic variety of $\mathcal{I}$ is empty. At points where $\lambda_1 = 1$ or $\lambda_2 = 1$, the characteristic variety consists of a pair of complex lines.*

Consequently (using Theorem V.3.12 in [3]), the set of integral 4-folds which lie in the open subset satisfying $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$ is at most finite-dimensional.

*Proof of Proposition* 12. Let $E$ be an integral 4-plane annihilated by the $\pi_{ij}^a$. (Because the Pfaffian system $\mathcal{I}$ is linear, the characteristic variety is the same for every integral element at a point.) Let $\xi = \xi_i \omega^i$ be a nonzero element of $E^*$, and let $\xi^\perp \subset E$ be the hyperplane annihilated by $\xi$. Then the polar equations of $\xi^\perp$ are generated by the 1-forms of $\mathcal{I}$ and the 1-forms $\pi_{ij}^a \xi_k - \pi_{ik}^a \xi_j$ for $j < k$. By definition, a point $[\xi] \in \mathbb{P}(E^* \otimes \mathbb{C})$ is in the characteristic variety of $E$ if these equations fail to have full rank, that is, the 72 1-forms $\pi_{ij}^a \xi_k - \pi_{ik}^a \xi_j$ have rank less than 17.

Expressing these 1-forms terms of the 1-forms $d\lambda_1, d\lambda_2$, $\phi_j^i$ and $\kappa_b^a$ yields a 72-by-17 matrix whose entries are linear functions of the $\xi_i$ with coefficients which are rational functions of $\lambda_1$ and $\lambda_2$. We find that, for any nonzero $\xi$, the matrix has full rank at points where neither $\lambda_1$ nor $\lambda_2$ is equal to one, for any nonzero $\xi$. On the other hand, the matrix drops rank to 16 when $\lambda_1 = 1$ and $\xi$ lies on one of the two lines described by

$$\xi_1\xi_4 + \xi_2\xi_3 = 0, \qquad \xi_3 = \pm i\xi_1.$$

Similarly, it drops rank to 16 when $\lambda_2 = 1$ and $\xi$ lies on one of the lines given by

$$\xi_1\xi_3 - \xi_2\xi_4 = 0, \qquad \xi_2 = \pm i\xi_1. \qquad \square$$

It turns out that in the case when $\lambda_1 = 1$ or $\lambda_2 = 1$, there are no integral submanifolds.

PROPOSITION 13. *A type C austere submanifold $M^4$ cannot have normal rank $\delta = 3$ and either $\lambda_1$ or $\lambda_2$ identically equal to* 1.

*Proof.* Suppose $\lambda_1 = 1$. Equation (28) forces $\lambda_2 = -1$ or $\lambda_3 = -1$. Without loss of generality, we take the case where $\lambda_3 = -1$. Denote the remaining parameter $\lambda_2$ by $\lambda$.

The Pfaffian system (29) is defined on $\mathcal{F} \times \mathbb{R}$ and its 2-forms are given by (16). The 1-forms (23) are linear combinations of the 16 independent 1-forms $d\lambda, \phi_i^j$ and $\kappa_a^b$. The Cartan characters of $\mathcal{I}$ are computed to be $s_1 = 12$ and $s_2 = 4$. At each point of $\mathcal{F} \times \mathbb{R}$, the set of integral 4-planes of $\mathcal{I}$ has fiber dimension 12. Since $12 < s_1 + 2s_2$, the system fails to be in involution. It turns out that there are four additional 1-forms that vanish on all integral 4-planes and should be added to the ideal (29). These are obtained by studying the tableau $\pi_{ij}^a$ of the Pfaffian system.

For example, if we consider the first four lines of the tableau (given by $a = 1$), the 2-forms obtained can be written in matrix form as

$$\begin{bmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\\ \Omega_4\end{bmatrix} := \begin{bmatrix} 2\phi_2^1 & -\kappa_1^1 & \phi_1^4 - \kappa_2^1 - \phi_2^3 & -\phi_2^4 - \kappa_3^1 + \phi_1^3 \\ -\kappa_1^1 & -2\phi_1^2 & \phi_2^4 + \kappa_3^1 - \phi_1^3 & \phi_2^3 - \phi_1^4 - \lambda\kappa_2^1 \\ \phi_1^4 - \kappa_2^1 - \phi_2^3 & \phi_2^4 + \kappa_3^1 - \phi_1^3 & 2\phi_3^4 & -\kappa_1^1 \\ -\phi_2^4 - \kappa_3^1 + \phi_1^3 & \phi_2^3 - \phi_1^4 - \lambda\kappa_2^1 & -\kappa_1^1 & -2\phi_3^4 \end{bmatrix}$$
$$\wedge \begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{bmatrix}.$$

We calculate that

$$\begin{aligned}\Omega_1 \wedge \omega^3 \wedge \omega^4 + \Omega_3 \wedge \omega^4 \wedge \omega^1 + \Omega_4 \wedge \omega^2 \wedge \omega^4 \\ = \frac{1}{2}(\phi_1^2 + \phi_3^4) \wedge \omega^1 \wedge \omega^3 \wedge \omega^4,\end{aligned} \tag{30}$$

$$
\begin{aligned}
&\Omega_1 \wedge \omega^2 \wedge \omega^3 + \Omega_2 \wedge \omega^4 \wedge \omega^2 + \Omega_3 \wedge \omega^1 \wedge \omega^2 \\
&\quad = \frac{1}{2}(\phi_1^2 + \phi_3^4) \wedge \omega^1 \wedge \omega^2 \wedge \omega^3, \\
&\Omega_2 \wedge \omega^3 \wedge \omega^4 + \Omega_3 \wedge \omega^3 \wedge \omega^1 + \Omega_4 \wedge \omega^2 \wedge \omega^3 \\
&\quad = \frac{1}{2}(\phi_1^2 + \phi_3^4) \wedge \omega^3 \wedge \omega^2 \wedge \omega^4, \\
&\Omega_1 \wedge \omega^1 \wedge \omega^3 + \Omega_2 \wedge \omega^4 \wedge \omega^1 + \Omega_4 \wedge \omega^1 \wedge \omega^2 \\
&\quad = \frac{1}{2}(\phi_1^2 + \phi_3^4) \wedge \omega^2 \wedge \omega^1 \wedge \omega^4.
\end{aligned}
$$

Because the 2-forms $\Omega_i$ vanish on any integral 4-plane $E$, the same is true for the 4-forms on the right. Then, since $\phi_1^2 + \phi_3^4$ must restrict to $E$ to be a linear combination of the $\omega^i$, the simultaneous vanishing of the 4-forms on the right in (30) implies that this linear combination must be zero.

Similarly, one can use the other pieces of the tableau to show that there are three more 1-forms that must vanish on the integral elements. In all, these additional forms are

$$
\begin{aligned}
\psi_1 &= \phi_1^2 + \phi_3^4, & \psi_3 &= \frac{4}{\lambda - 1}\phi_1^4 + \kappa_2^1, \\
\psi_2 &= \phi_1^4 + \phi_2^3, & \psi_4 &= \frac{4}{\lambda + 1}\phi_1^2 + \kappa_2^3,
\end{aligned}
$$

where we now assume that $\lambda \neq 1$ and $\lambda \neq -1$. (We will consider the case where $\lambda = \pm 1$ below.) Let $\mathcal{J}$ be the differential ideal obtained by adding the above four 1-forms to $\mathcal{I}$. This yields a nonlinear Pfaffian system, since the exterior derivatives of the new added forms will contain linear combinations of wedges of the $\pi_{ij}^a$. Computing $d\psi_1$ modulo, the 1-forms of $\mathcal{J}$ gives

$$d\psi_1 = -\lambda\omega^3 \wedge \omega^4 - \lambda\omega^1 \wedge \omega^2.$$

Thus, integral elements exist only on the submanifold where $\lambda = 0$.

We restrict $\mathcal{J}$ to the submanifold where $\lambda = 0$. The integral 4-planes will be defined by the equations

$$\pi_{ij}^a = s_{ijk}^a \omega^k, \tag{31}$$

where now only 15 of the 1-forms $\pi_{ij}^a$ are linearly independent, as linear combinations of $\phi_i^j$ and $\kappa_a^b$. Now we substitute the values in (31) into the new 2-forms $d\psi_i$, $i = 1, \ldots, 4$. For each of these, the coefficients with respect to $\omega^i \wedge \omega^j$ for $i < j$ should all be zero. From these conditions, we get 12 quadratic polynomials in the $s_{ij}^a$ which must vanish on any integral submanifold of $\mathcal{J}$. A Gröbner basis calculation shows that these polynomials have no common zero, so the set of 4-integral elements of $\mathcal{J}$ is empty.

If $\lambda = 1$ or $\lambda = -1$, the conclusion is the same. It turns out that in this case there are 7 more 1-forms that vanish on any integral element of $\mathcal{I}$ and which have to be added to the ideal. Among the 1-forms of the augmented ideal $\mathcal{J}$

is $\phi^3_2$; when we compute the derivative of this 1-form modulo the 1-forms of $\mathcal{J}$ we get

$$d\phi^3_2 = 2\omega^3 \wedge \omega^2,$$

which can never vanish on integral elements satisfying the independence condition. □

The following result shows that in maximal codimension $\delta = 3$ and when all parameters $\lambda_i$ are constant (i.e., the austere subspace does not vary from point to point), the $\lambda$'s are all forced to be equal to zero. This means that the second fundamental forms in various normal directions are rank one, and contain a common linear factor; in Bryant's terminology, $M$ is called *simple*. By Bryant's Theorem 3.1 in [2] it is congruent to a generalized helicoid.

PROPOSITION 14. *An austere 4-fold $M$ of type $C$, with first normal space of dimension $\delta = 3$ and such that the parameters $\lambda_1, \lambda_2, \lambda_3$ are constant, must be a generalized helicoid.*

*Proof.* First, assume that none of the parameters $\lambda_i$ are zero. Because of Proposition 13, we can also assume that none of them are equal to $\pm 1$. We take the standard system $\mathcal{I}$ on $\mathcal{F}$ with basis matrices $S^1, S^2, S^3$ given by (27), and calculate the system 2-forms $\Omega^a_i := \pi^a_{ij} \wedge \omega^j$, where

$$\pi^a_{ij} = -[S^a, \phi]_{ij} + \kappa^a_b S^b_{ij}.$$

The components of $\pi^a_{ij}$ are linear combinations of the 15 linearly independent forms $\phi^i_j$ and $\kappa^a_b$. We claim that all of these forms must vanish on any integral element of $\mathcal{I}$. For, substituting $\phi^i_j = s^i_{jk}\omega^k$ and $\kappa^a_b = t^a_{bk}\omega^k$ in the 2-forms, and equating the coefficients of the 6 2-forms $\omega^i \wedge \omega^j$ to zero yields a system of 72 homogeneous linear equations for the 60 variables $s^i_{jk}$ and $t^a_{bk}$. By a permutation of rows and columns, the matrix for this linear system is equivalent to one with four nonzero $15 \times 15$ blocks, each of which is nonsingular under our assumptions about the values of the $\lambda_i$. In particular, the connection forms $\phi^i_j$ must vanish identically on any integral submanifold of $\mathcal{I}$, implying that the corresponding submanifold $M^4 \subset \mathbb{R}^7$ is totally geodesic. This contradicts our assumption that $\delta = 3$.

Next, we assume that exactly one of the parameters is identically zero. Without loss of generality, we may assume that $\lambda_3 = 0$ and $\lambda_2 = -\lambda_1 \neq 0$. Then the fiber of the space of integral elements of $\mathcal{I}$ has dimension two, but then the following additional 1-forms vanish on all integral elements:

$$\begin{gathered} \phi^2_1 - \lambda_1\phi^4_3, \qquad \phi^3_1 + \lambda_1\phi^4_2, \qquad \kappa^1_2 + \kappa^2_1, \\ \kappa^2_2 - \kappa^1_1, \qquad \kappa^3_1 - \phi^4_2, \qquad \kappa^3_2 - \phi^4_3, \qquad \kappa^3_3. \end{gathered} \tag{32}$$

We adjoin these 1-forms to obtain a larger Pfaffian system $\mathcal{J}$. However, taking the exterior derivatives of the first two 1-forms in (32) implies that $g_{11} = g_{22} = 0$, which is impossible for components of the metric on the normal bundle.

Thus, we conclude that the only possible solutions with parameters $\lambda_i$ all constant are those for which all these parameters are zero. $\square$

## 5. Examples

In this section, we examine some interesting examples of austere submanifolds whose normal rank $\delta$ is not maximal. In particular, we describe some nongeneric austere 4-folds of type A with $\delta = 2$. More precisely, we normalize the 2-dimensional subspaces of $\mathcal{Q}_A$ which lie on nonprincipal orbits of the symmetry group, and classify the corresponding austere 4-folds.

As stated in Section 3.1, the symmetry group of $\mathcal{Q}_A$ is $U(2)^{\mathbb{R}} = \{M \in \mathrm{SO}(4) | JM = MJ\}$, with $J$ given by (9), and its action on $\mathcal{Q}_A$ is $M \cdot S = MS\,{}^tM$. This group is isomorphic to the usual group $U(2)$ of $2 \times 2$ unitary matrices, which acts in a similar way on the space $\mathcal{V}$ of $2 \times 2$ complex matrices. In fact, we can define an isomorphism $\rho : U(2)^{\mathbb{R}} \to U(2)$ that intertwines these actions: if we let

$$\rho : \begin{bmatrix} E & F \\ -F & E \end{bmatrix} \mapsto E + \mathrm{i}F, \qquad \overline{\rho} : \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \mapsto A - \mathrm{i}B,$$

then $\overline{\rho}(M \cdot S) = \rho(M) \cdot \overline{\rho}(S)$. In what follows, we will use this action to normalize real subspaces of $\mathcal{V}$.

Let $\mathcal{Q} \subset \mathcal{V}$ be a subspace of real dimension 2, and let $S, T$ span $\mathcal{Q}$. We first consider the following special cases:

1. *$S, T$ are linearly dependent over* $\mathbb{C}$. In this case, we can use $U(2)$ to simultaneously diagonalize $S$ and $T$. Using linear combinations with real coefficients, we can arrange that

$$S = \begin{bmatrix} 1 & 0 \\ 0 & x + \mathrm{i}y \end{bmatrix}, \qquad T = \mathrm{i}S, \quad x, y \in \mathbb{R}. \tag{33}$$

We distinguish two subcases:

(a) every matrix in $\mathcal{Q}$ has full rank, so that $x, y$ are not both zero; and
(b) the matrices $S$ and $T$ are singular (i.e., $x = y = 0$).

2. *$S, T$ are linearly independent over* $\mathbb{C}$. We first note that there must be a singular matrix in the complex span of $S$ and $T$, that is,

$$\det(T - \lambda S) = 0. \tag{34}$$

We distinguish several subcases:

(a) *$\mathcal{Q}$ contains a singular matrix* (i.e., $\lambda \in \mathbb{R}$). In this case, we can linearly combine $S$ and $T$ so that $T$ has rank 1. Using $U(2)$, we can arrange that $\ker T$ is spanned by ${}^t[1, 0]$; then using the diagonal subgroup $U(1) \times U(1)$ and real scale factors, we can assume that

$$S = \begin{bmatrix} 1 & x + \mathrm{i}y \\ x + \mathrm{i}y & \mathrm{i}u \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{35}$$

for real parameters $u, x, y$.

(b) *Quadrics in $\mathcal{Q}$ have a common nullspace.* Assuming that the previous cases do not apply, in this case we can arrange that $S$ and $T$ have the form

$$S = \begin{bmatrix} 0 & 1 \\ 1 & \mathrm{i}x \end{bmatrix}, \qquad T = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & y - x \end{bmatrix} \tag{36}$$

for positive real parameters $x, y$.

(c) *$S$ and $T$ commute.* Assuming the previous cases do not apply, in this case we can arrange that

$$S = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{i}y \end{bmatrix}, \qquad T = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & p - y \end{bmatrix}$$

for nonzero real parameters $p, y$.

Finally, we have the

(d) *Generic case.* When none of the above hold, we may arrange that

$$S = \begin{bmatrix} 1 & u \\ u & x + \mathrm{i}y \end{bmatrix}, \qquad T = \begin{bmatrix} \mathrm{i} & \mathrm{i}u \\ \mathrm{i}u & p - y + \mathrm{i}x \end{bmatrix}$$

for real parameters $p, u, x, y$ with $u, p$ nonzero. We now classify the austere 4-folds corresponding to the nongeneric cases above.

THEOREM 15. *Let $M \subset \mathbb{R}^{4+r}$ be an austere submanifold of type $A$, with $\delta = 2$, such that $|\,\mathrm{II}\,|$ is of fixed nongeneric type on an open dense subset of $M$, and the Gauss map of $M$ is nondegenerate. Then $M$ lies in a totally geodesic $\mathbb{R}^6$. Furthermore,*

(i) *if $|\,\mathrm{II}\,|$ is of type* 1(a) *or* 2(c), *then $M$ is holomorphic submanifold with respect to a complex structure on $\mathbb{R}^6$ given by a constant matrix $\widehat{J}$;*

(ii) *if $|\,\mathrm{II}\,|$ is of type* 2(a), *then $M = \Sigma_1 \times \Sigma_2$ for minimal surfaces $\Sigma_1, \Sigma_2$ in $\mathbb{R}^3$;*

(iii) *if $|\,\mathrm{II}\,|$ is of type* 2(b), *then $M$ is 2-ruled, and the image of the map $\gamma : M \to G(2,6)$ assigning to each point $p \in M$ the subspace of $\mathbb{R}^6$ parallel to the ruling through $p$ is a holomorphic curve.*

Note that the Grassmannian $G(2,6)$ is endowed with a complex structure that enables us to identify it with the standard quadric in $\mathbb{CP}^5$ (see [9], Chapter XI, Example 10.6). Note also that if the Gauss map of $M$ is degenerate, then it falls into case 1(b) and the Gauss map has rank 2. Austere manifolds with Gauss map rank 2, and $\delta \geq 2$, were classified by Dajczer and Florit [4]. For submanifolds of this type, the prolongation $|\,\mathrm{II}\,|^{(1)}$ is nonzero, so we cannot conclude that they lie in a totally geodesic $\mathbb{R}^6$; in fact, the examples of Dajczer and Florit exist in arbitrarily high effective codimension.

*Proof of Theorem* 15. For each case, let $S$ and $T$ be the normalized basis matrices for the subspace, let $S^{\mathbb{R}}, T^{\mathbb{R}}$ denote their inverse images under the map $\overline{\rho}$, and let $\mathcal{Q}$ be the span of $S^{\mathbb{R}}$ and $T^{\mathbb{R}}$. It is easy to check that $\mathcal{Q}^{(1)}=0$ in each case, so by the argument of Proposition 3, $M$ lies in a totally geodesic $\mathbb{R}^6$.

(i) Assume that $|\,\mathrm{II}\,|$ is of type 1(a); then $\mathcal{Q}$ is parametrized by $x,y$. Taking $S^{\mathbb{R}}$ and $T^{\mathbb{R}}$ [where $S$ and $T$ are given by (33)] as basis matrices, let $\mathcal{I}$ be the standard system with parameters, defined on $\mathcal{F}\times L$ where $L=\mathbb{R}^2$ minus the origin. By Proposition 6, any austere manifold of this type will be Kähler with respect to the complex structure given by (9). Thus, the connection 1-forms must satisfy $\phi^2_1=\phi^4_3$ and $\phi^3_2=\phi^4_1$ for any adapted frame that makes $|\,\mathrm{II}\,|$ conjugate to $\mathcal{Q}$. Therefore, such adapted frames give integrals of the augmented system

$$\mathcal{I}^+=\{\theta^1,\theta^2,\eta^1_i-S^{\mathbb{R}}_{ij}\omega^j,\eta^2_i-T^{\mathbb{R}}_{ij}\omega^j,\phi^2_1-\phi^4_3,\phi^3_2-\phi^4_1\}. \tag{37}$$

Taking exterior derivatives of the last two 1-forms modulo the algebraic ideal generated by forms in $\mathcal{I}^+$ shows that integral submanifolds exist only at points where

$$g_{11}=g_{22},\qquad g_{12}=0. \tag{38}$$

In other words, it is necessary that the frame vectors $\nu_1,\nu_2$ be orthogonal and have the same length. We pull back the system $\mathcal{I}^+$ to the submanifold $V\subset\mathcal{F}\times L$ where these conditions hold. (Pulled back to $V$, the connection forms satisfy the additional relations $\kappa^1_1=\kappa^2_2$ and $\kappa^2_1=-\kappa^1_2$.) On $V$, the system is involutive with Cartan characters $s_1=4$, $s_2=2$.

To see that the corresponding austere submanifolds are holomorphic, we need to endow $\mathbb{R}^6$ with the appropriate complex structure $\mathsf{J}$ which restricts to $J$ on $M$. Because $T^{\mathbb{R}}=S^{\mathbb{R}}J$, equation (2) implies that this ambient complex structure must satisfy $\mathsf{J}\nu_1=-\nu_2$. Thus, if we let $F$ be a matrix whose columns are the vectors $e_1,\dots,e_4,\nu_1,\nu_2$, then $\mathsf{J}$ must satisfy

$$\mathsf{J}F=FC,\qquad C:=\left[\begin{array}{c|c} J & 0\\ \hline 0 & \begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\end{array}\right]. \tag{39}$$

If $\mathsf{J}$ is to be the standard complex structure on $\mathbb{R}^6$, then it must be given by a constant matrix. By (39), this matrix must equal $FCF^{-1}$. Thus, we have only to show that, for any integral of $\mathcal{I}^+$, this matrix is a constant.

The structure equations (12) imply that $dF=F\Phi$, where $\Phi$ is the $6\times 6$ matrix of connection forms:

$$\Phi=\begin{bmatrix}\phi & -{}^t\eta g\\ \eta & \kappa\end{bmatrix}.$$

Using this, we compute that $d\mathsf{J}=F[\Phi,C]F^{-1}$. Then, it is easy to verify that, for any integral of $\mathcal{I}^+|_V$, the values of the connection forms imply that $[\Phi,C]=0$.

The argument in the case that $|\,\mathrm{II}\,|$ is conjugate to a space of type 2(c) is similar, save that in that case $M$ is Kähler with respect to the complex structure represented by $\widetilde{J}$.

(ii) We again set up the standard system with basis matrices $S^{\mathbb{R}}$ and $T^{\mathbb{R}}$ (where $S, T$ are given by (35), and parameters $u, x, y$ range over all of $L = \mathbb{R}^3$). The metric on $M$ is Kähler with respect to $J$, so we pass to the augmented system $\mathcal{I}^+$ as in (37). Again, differentiating the last two 1-forms in $\mathcal{I}^+$ yield integrability conditions, which in this case are

$$u = 2xy, \qquad g_{12} = (x^2 - y^2)g_{11}.$$

Let $V \subset \mathfrak{F} \times L$ be the submanifold on which these conditions hold. The pullback of $\mathcal{I}^+$ to $V$ fails to be involutive, In fact, integral elements exist only on the submanifold $V'$ defined by $u = x = y = 0$ and $g_{12} = 0$. The pullback of $\mathcal{I}^+$ to this submanifold is involutive after one prolongation, with character $s_1 = 4$.

On $V'$, we compute [using the structure equations (10)] that

$$d\Big(e_1 \bigwedge e_3\Big) \equiv \Big(\nu_1 \bigwedge e_3\Big)\omega^1 + \Big(\nu_1 \bigwedge e_1\Big)\omega^3 \mod \mathcal{I}^+,$$
$$\nu_1 \bigwedge d\nu_1 \equiv g_{11}\Big(\Big(\nu_1 \bigwedge e_3\Big)\omega^3 - \Big(\nu_1 \bigwedge e_1\Big)\omega^1\Big) \mod \mathcal{I}^+,$$

where $\bigwedge$ is the exterior product on $\mathbb{R}^6$. This shows that, for the framed austere manifold corresponding to any solution of this EDS, the 3-plane through the origin in $\mathbb{R}^6$ spanned by $e_1, e_3, \nu_1$ is fixed, and the orthogonal projection of $M$ onto this 3-plane is a rank 2 mapping. The same is true for the 3-plane spanned by $e_2, e_4, \nu_2$. Thus, $M$ is the product of surfaces in these two copies of $\mathbb{R}^3$. The austere condition implies that these must be minimal surfaces.

(iii) We set up the standard system with $S, T$ given by (36) for positive parameters $x, y$; let $L \subset \mathbb{R}^2$ be the first quadrant. The derivatives of the last two 1-forms in $\mathcal{I}^+$ yield integrability conditions which are the same as (38). The restriction of $\mathcal{I}^+$ to the submanifold $V \subset \mathfrak{F} \times L$ where these conditions hold is involutive, with character $s_1 = 8$.

To see that the corresponding austere manifolds are ruled, we compute the system 2-forms

$$d\big(\eta_1^1 - \eta_3^2 - (S_{1j}^{\mathbb{R}} - T_{3j}^{\mathbb{R}})\omega^j\big) \equiv y(\phi_3^4 \wedge \omega^2 - \phi_1^4 \wedge \omega^4) \mod \mathcal{I}_1^+,$$
$$d\big(\eta_3^1 + \eta_1^2 - (S_{3j}^{\mathbb{R}} + T_{1j}^{\mathbb{R}})\omega^j\big) \equiv y(\phi_1^4 \wedge \omega^2 + \phi_3^4 \wedge \omega^4) \mod \mathcal{I}_1^+,$$

where $\mathcal{I}_1^+$ denotes the algebraic ideal generated by the 1-forms of $\mathcal{I}^+$. It follows that on any solution there are functions $u_1, u_2$ such that

$$\phi_1^2 = \phi_3^4 = u_1\omega^2 + u_2\omega^4, \qquad \phi_2^3 = \phi_1^4 = u_1\omega^4 - u_2\omega^2. \tag{40}$$

Thus, we have

$$de_1 = e_3\phi_1^3 + (u_1e_2 + u_2e_4 + \nu_1)\omega^2 + (u_1e_4 - u_2e_2 + \nu_2)\omega^4,$$
$$de_3 = e_1\phi_3^1 + (u_1e_4 - u_2e_2 + \nu_2)\omega^2 - (u_1e_2 + u_2e_4 + \nu_1)\omega^4.$$

These equations show that, as we move along directions tangent to the $e_1$–$e_3$ plane in $T_pM$ (i.e., directions annihilated by $\omega^2, \omega^4$) the span of $e_1, e_3$ is fixed.

Thus, the map $\gamma : M \to G(2,6)$ has rank two. To see that the image is a holomorphic curve, we must examine the complex structure on $G(2,6)$.

As in ([9], *loc. cit.*), we think of $G(2,6)$ as $\mathrm{SO}(6)/\mathrm{SO}(2) \times \mathrm{SO}(4)$. Differential forms on $G(2,6)$—in particular, $(1,0)$-forms for the complex structure—may be lifted up to the group $\mathrm{SO}(6)$. In this case, we use a lifting of the map $\gamma$ to a map $\Gamma : M \to \mathrm{SO}(6)$ defined by

$$\Gamma : p \mapsto (e_1(p), e_3(p), e_2(p), e_4(p), e_5(p), e_6(p)), \quad e_5 = \frac{1}{r}\nu_1, e_6 = \frac{1}{r}\nu_2, \tag{41}$$

where $r = \sqrt{g_{11}}$. (Note the change in order, chosen so that the vectors tangent to the ruling at $p$ are the first two columns of $\Gamma(p)$.) Let $\Psi = G^{-1}\,dG$ be the Maurer–Cartan form on $\mathrm{SO}(6)$, with components $\psi^i_j = -\psi^j_i$. Then the forms $\psi^m_1, \psi^m_2$ for $3 \le m \le 6$ are semibasic for the quotient map $q :\ \mathrm{SO}(6) \to G(2,6)$, which sends $G$ to the 2-plane spanned by its first two columns. Moreover, the complex span of the 1-forms $\psi^m_1 - \mathrm{i}\psi^m_2$ is well defined on the quotient, and spans the space of $(1,0)$-forms on $G(2,6)$. Note also that by comparing the Maurer–Cartan equation $dG = G\Psi$ with the defining properties (10) of the connection forms shows that

$$\Gamma^*\psi^i_j = \phi^{\sigma(i)}_{\sigma(j)}, \qquad \Gamma^*\psi^5_j = r\eta^1_{\sigma(j)}, \qquad \Gamma^*\psi^6_j = r\eta^2_{\sigma(j)}, \tag{42}$$

for $1 \le i,j \le 4$, where $\sigma$ is the permutation that exchanges indices 2 and 3.

To show holomorphicity of $\gamma$, we have to show that the pullback under $\gamma = q \circ \Gamma$ of the $(1,0)$-forms on $G(2,6)$ are $(1,0)$-forms on $M$, that is, in the span of $\omega^1 + \mathrm{i}\omega^3$ and $\omega^2 + \mathrm{i}\omega^4$. (Note that it is equivalent to show this for the pullbacks under $\Gamma$ for the forms $\psi^m_1 - \mathrm{i}\psi^m_2$.) Using (40) and (42), we compute

$$\begin{aligned}
\Gamma^*(\psi^3_1 - \mathrm{i}\psi^3_2) &= \phi^2_1 - \mathrm{i}\phi^2_3 = (u_1 - \mathrm{i}u_2)(\omega^2 + \mathrm{i}\omega^4),\\
\Gamma^*(\psi^4_1 - \mathrm{i}\psi^4_2) &= \phi^4_1 - \mathrm{i}\phi^4_3 = -(u_2 + \mathrm{i}u_1)(\omega^2 + \mathrm{i}\omega^4),\\
\Gamma^*(\psi^5_1 - \mathrm{i}\psi^5_2) &= r(\eta^1_1 - \mathrm{i}\eta^1_3) = r(\omega^2 + \mathrm{i}\omega^4),\\
\Gamma^*(\psi^6_1 - \mathrm{i}\psi^6_2) &= r(\eta^2_1 - \mathrm{i}\eta^2_3) = \mathrm{i}r(\omega^2 + \mathrm{i}\omega^4).
\end{aligned} \tag{43}$$

Notice that the holomorphic curve in $G(2,6)$ is not generic; indeed, if $M$ were determined by specifying an arbitrary holomorphic curve in the Grassmannian as the image $\gamma(M)$, then one would expect the Cartan character $s_1$ of $\mathcal{I}^+$ to be 6. Instead, as we will see below, $M$ is (in part) determined by a general holomorphic curve in a different Hermitian symmetric space. □

For the rest of this subsection, we will focus on 2-ruled austere submanifolds in $\mathbb{R}^6$ with $\delta = 2$, the last type discussed in Theorem 15. As in the proof of that theorem, an adapted frame along such a submanifold $M$ defines the map $\Gamma : M \to \mathrm{SO}(6)$ given by (41). Now let $\pi :\ \mathrm{SO}(6) \to \mathrm{SO}(6)/U(3)$ be the quotient map, where $U(3)$ is the intersection of SO(6) with

$$\mathrm{GL}(3,\mathbb{C})^{\mathbb{R}} = \{M \in \mathrm{GL}(6,\mathbb{R}) | M\widehat{J} = \widehat{J}M\}, \quad \widehat{J} := \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

On SO(6), define the complex-valued 1-forms

$$\begin{aligned} \beta^1 &= \psi^4_1 - \mathrm{i}\psi^4_2 + \mathrm{i}(\psi^3_1 - \mathrm{i}\psi^3_2), \\ \beta^2 &= \psi^6_1 - \mathrm{i}\psi^6_2 + \mathrm{i}(\psi^5_1 - \mathrm{i}\psi^5_2), \\ \beta^3 &= \psi^6_3 - \mathrm{i}\psi^6_4 + \mathrm{i}(\psi^5_3 - \mathrm{i}\psi^5_4). \end{aligned}$$

The quotient $\mathrm{SO}(6)/U(3)$ has real dimension 6, and the space of semibasic forms for the projection $\pi$ is spanned by the real and imaginary parts of the $\beta^\ell$ for $1 \le \ell \le 3$. (These forms annihilate the left-invariant vector fields in the subalgebra $\mathfrak{u}(3)$, which span tangent spaces of the fibres of $\pi$.) The forms $\beta^i$ satisfy

$$d \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} = \Upsilon \wedge \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix}, \tag{44}$$

where

$$\Upsilon := \frac{1}{2} \begin{bmatrix} 2\mathrm{i}(\psi^2_1 + \psi^4_3) & \psi^5_3 - \mathrm{i}\psi^5_4 + \psi^6_4 + \mathrm{i}\psi^6_3 & -(\psi^5_1 - \mathrm{i}\psi^5_2 + \mathrm{i}\psi^6_1 + \psi^6_2) \\ -\psi^5_3 - \mathrm{i}\psi^5_4 - \psi^6_4 + \mathrm{i}\psi^6_3 & 2\mathrm{i}(\psi^2_1 + \psi^6_5) & \psi^3_1 - \mathrm{i}\psi^3_2 + \mathrm{i}\psi^4_1 + \psi^4_2 \\ \psi^5_1 + \mathrm{i}\psi^5_2 - \mathrm{i}\psi^6_1 + \psi^6_2 & -(\psi^3_1 + \mathrm{i}\psi^3_2 - \mathrm{i}\psi^4_1 + \psi^4_2) & 2\mathrm{i}(\psi^4_3 + \psi^6_5) \end{bmatrix},$$

indicating that the complex span of the $\beta^\ell$ is a pullback of a well-defined Pfaffian system on $\mathrm{SO}(6)/U(3)$, and these are the $(1,0)$-forms of an invariant (integrable) complex structure on the quotient.

It is evident from (43) that $\Gamma^*\beta^1 = \Gamma^*\beta^2 = 0$. Moreover,

$$\Gamma^*\beta^3 = r\big(\phi^6_2 + \phi^5_4 + \mathrm{i}(\phi^5_2 - \phi^6_4)\big) \equiv yr(\omega^2 + \mathrm{i}\omega^4) \mod \mathcal{I}^+, \tag{45}$$

indicating that the map $\pi \circ \Gamma : M \to \mathrm{SO}(6)/U(3)$ has rank 2, and is holomorphic.

In general, the space $\mathrm{SO}(2n)/U(n)$ may be identified with the set of orthogonal complex structures on $\mathbb{R}^{2n}$; in this case, with $n = 3$, it may also be identified with $\mathbb{CP}^3$, in the following way.[3] Let $V=\mathbb{C}^4$ with the standard Hermitian metric, and let $W=\mathbb{R}^6$ with the Euclidean metric. Then $\Lambda^2 V=\mathbb{C}^6=W\otimes\mathbb{C}$, and each complex structure $\mathsf{J}$ on $W$ corresponds (by associating to $\mathsf{J}$ its $+\mathrm{i}$ eigenspace) to a totally isotropic subspace $E\subset W\otimes C$. Each such subspace $E$ is of the form $E_u=\{u\bigwedge v|v\in V\}$ for some vector $u\in\mathbb{C}^4$. The map $\mathsf{J}\mapsto E_u\mapsto u$ is well-defined up to complex multiple, and identifies $\mathrm{SO}(6)/U(3)$ with $\mathbb{CP}^3$. Moreover, the standard Kähler form on $\mathbb{CP}^3$ pulls back to $\beta^1\wedge\overline{\beta^1}+\beta^2\wedge\overline{\beta^2}+\beta^3\wedge\overline{\beta^3}$ on $\mathrm{SO}(6)$, and $\mathrm{SO}(6)$ may be identified with the unitary frame bundle of $\mathbb{CP}^3$, with connection forms given by the components of $\Upsilon$.

The following result shows that the association of $M$ with a holomorphic curve in $\mathbb{CP}^3$ is surjective but not 1-to-1.

THEOREM 16. *Let $\mathcal{C}$ be a holomorphic curve in $\mathbb{CP}^3$. Given a nonplanar point $p\in\mathcal{C}$, there is an open neighborhood $U\subset\mathcal{C}$ containing $p$ and a 2-ruled austere manifold $M\subset\mathbb{R}^6$ such that $\pi\circ\Gamma(M)=U$. Such manifolds $M$ depend on a choice of 4 functions of 1 variable.*

*Proof.* The proof of Theorem 15 part (iii) shows that along $M$ there is an *orthonormal* frame $(e_1,\dots,e_4,\nu_1,\nu_2)$ such that $\nu_1\cdot\mathrm{II}(e_i,e_j)=S^1_{ij}$ and $\nu_2\cdot\mathrm{II}(e_i,e_i)=S^2_{ij}$ for matrices

$$S^1=r\begin{bmatrix}0&1&0&0\\1&0&0&x\\0&0&0&-1\\0&x&-1&0\end{bmatrix},\qquad S^2=r\begin{bmatrix}0&0&0&1\\0&y-x&1&0\\0&1&0&0\\1&0&0&x-y\end{bmatrix}$$

and some positive functions $r,x,y$ along $M$. To construct $M$, we will set up a Pfaffian system, similar to the augmented standard system $\mathcal{I}^+$, satisfied by the orthonormal frame.

Let $\mathcal{F}_o$ be the orthonormal frame bundle of $\mathbb{R}^6$; in terms of the bundle $\mathcal{F}$ of semi-orthonormal frames defined in Section 2.2, $\mathcal{F}_o$ is the subbundle of $\mathcal{F}$ on which

$$g_{11}=g_{22}=1,\qquad g_{12}=0 \tag{46}$$

hold. The structure equations of $\mathcal{F}_o$ are the same as those given by equations (10) through (12), but with the specialization (46) taken into account, $\kappa$ is skew-symmetric and $\xi=-{}^t\eta$.

We adjoin $r,x,y$ as new variables, taking value in the positive octant $L\subset\mathbb{R}^3$, and define our Pfaffian system $\mathcal{J}$ on $\mathcal{F}_o\times L$ to be generated by

$$\theta^1,\qquad\theta^2,\qquad\eta^a_i-S^a_{ij}\omega^j,\qquad\phi^4_3-\phi^2_1,\qquad\phi^4_1-\phi^3_2.$$

As in (41), we define a map $\Gamma:\mathcal{F}_o\times L\to\mathrm{SO}(6)$, whose value is the matrix with columns $(e_1,e_2,e_3,e_4,\nu_1,\nu_2)$. We will now show how, given an arbitrary

[3] We learned this identification in a paper of Abbena, Garbiero and Salamon [1].

holomorphic curve $\mathcal{C} \subset \mathbb{CP}^3$, we construct an integral manifold of $\mathcal{J}$ whose image, under $\pi \circ \Gamma$, is an open neighborhood of $p \in \mathcal{C}$.

Let $N = \pi^{-1}(\mathcal{C})$, a codimension-4 submanifold of SO(6). We begin by constructing an integral of the 1-forms $\beta^1, \beta^2$ within $N$. On $N$, the complex span of the 1-forms $\beta^\ell$ is one-dimensional at each point. If $z$ is a local holomorphic coordinate on $\mathcal{C}$ near $p$, then there will be complex functions $f^\ell$ such that $\beta^\ell = f^\ell \pi^* dz$ on $N$. Substituting this in (44) gives

$$df^\ell \equiv \Upsilon^\ell_m f^m \mod dz. \tag{47}$$

Let $F$ be a fiber of $\pi : N \to \mathbb{CP}^3$. Since such fibers are left cosets of $U(3) \subset$ SO(6), $U(3)$ acts simply transitively on them by right multiplication. This action is generated infinitesimally by the left-invariant vector fields on SO(6) that are tangent to subalgebra $\mathfrak{u}(3)$ at the identity. Thus, such vector fields give a frame field tangent to $F$. Because $\Upsilon$ is $\mathfrak{u}(3)$-valued, (47) shows that as the action of $U(3)$ moves points along $F$, the corresponding action on the vector with components $f^\ell$ is isomorphic to the standard action of $U(3)$ on $\mathbb{C}^3$. Thus, in each fiber there is a subset where $f^1 = f^2 = 0$, and the union of these subsets is a smooth submanifold $N' \subset N$ of codimension 4 within $N$. (Note that the construction of $N'$ does not depend on the choice of $p$ or the local coordinate on $\mathcal{C}$.)

Because $\beta^1 = \beta^2 = 0$, then $\Upsilon^1_3 = -\psi^5_1 + \mathrm{i}\psi^5_2$ and $\Upsilon^2_3 = \psi^3_1 - \mathrm{i}\psi^3_2$. Thus, the restriction of (44) to $N'$ implies that there are complex-valued functions $f, k$ on $N'$ such that

$$\psi^3_1 - \mathrm{i}\psi^3_2 = f\beta^3, \qquad \psi^5_1 - \mathrm{i}\psi^5_2 = k\beta^3. \tag{48}$$

Besides $\beta^1 = \beta^2 = 0$, these are the only linearly dependencies among the left-invariant 1-forms of SO(6) when restricted to $N'$; thus, the 1-forms $\psi^2_1, \psi^4_3, \psi^5_3, \psi^5_4, \psi^6_3, \psi^6_4, \psi^6_5$ (which include the real and imaginary parts of $\beta^3$) form a coframe on $N'$.

Computing

$$\Gamma^*(\psi^5_1 - \mathrm{i}\psi^5_2) = \eta^1_1 - \mathrm{i}\eta^1_3 \equiv r(\omega^2 + \mathrm{i}\omega^4) \mod \mathcal{J}$$

and comparing with (45) shows that on the image under $\Gamma$ of a solution of $\mathcal{J}$, we must have $k$ equal to $1/y$. Thus, in order to construct a candidate for such an image, we need to restrict to the subset $N'' \subset N'$ where $k$ is real and positive. Differentiating (48) gives

$$dk = \mathrm{i}k(\psi^2_1 - \psi^4_3 - 2\psi^6_5) - f(\psi^6_4 + \mathrm{i}\psi^6_3) + w\beta^3 \tag{49}$$

for some complex function $w$ on $N'$. This equation shows that the subgroup of $U(3)$ stabilizing $N'$ can be used to make $k$ real and positive, provided that $f$ and $k$ are not both identically zero along a fiber. Since $f, k$ give the components of the second fundamental form of $\mathcal{C}$ as a holomorphic submanifold of $\mathbb{CP}^3$, then $f = k = 0$ along the fiber above $p$ means that $p$ is a planar point of $\mathcal{C}$. Thus, we will restrict to nonplanar points of $\mathcal{C}$. Then, in each fiber of $N'$

there is a subset where $k>0$, and the union $N''$ of these subsets is a smooth codimension-one submanifold of $N'$.

If we let $f=g+\mathrm{i}h$ and $w=u+\mathrm{i}v$ for real functions $g,h,u,v$, then taking the real and imaginary parts of (49) gives

$$dk = -v\psi^5_3 + u\psi^5_4 + (h+u)\psi^6_3 + (g-v)\psi^6_4 \tag{50}$$

and $k(\psi^2_1 - \psi^4_3 - 2\psi^6_5) + u\psi^5_3 + v\psi^5_4 - (g-v)\psi^6_3 - (h+u)\psi^6_4 = 0$ on $N''$. Because of the last equation, we may use the restrictions of the 1-forms $\psi^4_3, \psi^5_1, \psi^5_2, \psi^5_3, \psi^5_4, \psi^6_5$ as a coframe on $N''$.

Let $\Sigma$ be the smooth hypersurface in $\Gamma^{-1}(N'')$ defined by $y=1/k$. Because the fibers of $\Gamma$ have dimension 9, $\Sigma$ has dimension 14, with coframe given by $\omega^1,\dots,\omega^4$, $\theta^1,\theta^2$, $\phi^4_2$, $\eta^1_1$, $\eta^1_2$, $\eta^1_3$, $\eta^1_4$, $\kappa^2_1$, $dx$ and $dr$. From (50) we deduce that

$$dy = y^2(g\eta^1_2 + h\eta^1_4) - y^3\big((u+h)\eta^1_1 + (v-g)\eta^1_3\big) \tag{51}$$

on $\Sigma$. The 1-forms in $\mathcal{J}$ pull back to $\Sigma$ to give a rank 6 system generated by $\theta^1,\theta^2$ and

$$\begin{aligned}
\alpha^1 &= \eta^1_1 - r\omega^2,\\
\alpha^2 &= \eta^1_2 + r\omega^4,\\
\alpha^3 &= \eta^1_3 - r(\omega^1 + x\omega^4),\\
\alpha^4 &= \eta^1_4 + r(\omega^3 - x\omega^2).
\end{aligned}$$

On $\Sigma$, there are 1-forms $\pi_1,\pi_2,\pi_3,\pi_4$ which are linearly independent combinations of $\phi^4_2,\kappa^2_1,dx,dr$ modulo $\omega^1,\dots,\omega^4$, such that

$$d\begin{bmatrix}\alpha^1\\ \alpha^2\\ \alpha^3\\ \alpha^4\end{bmatrix} \equiv \begin{bmatrix} 0 & \pi_1 & 0 & \pi_2\\ 0 & \pi_2 & 0 & -\pi_1\\ \pi_1 & (y-2x)\pi_4 & \pi_2 & \pi_3\\ \pi_2 & \pi_3 & -\pi_1 & -(y-2x)\pi_4 \end{bmatrix} \wedge \begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\\ \omega^4\end{bmatrix} \quad \mod \alpha^1,\dots,\alpha^4.$$

This indicates that, at points where $x \neq y/2$, the system $\mathcal{J}|_\Sigma$ is involutive with terminal Cartan character $s_1 = 4$. Local existence of solutions then follows by applying the Cartan–Kähler Theorem at such points. □

Note that the construction of $M$ is global up to the point where Cartan–Kähler is applied; that is, the construction of $\Sigma$ does not depend on the choice of $p$, and $\Sigma$ covers all of the nonplanar points of $\mathcal{C}$. It is possible that this last step might be refined so as to enable one to deduce global information about $M$ from properties of $\mathcal{C}$.

## References


[1] E. Abbena, S. Garbiero and S. Salamon, *Almost Hermitian geometry in six-dimensional nilmanifolds*, Sci. École Norm. Sup. (4) **30** (2001), 147–170. MR 1882028

[2] R. L. Bryant, *Some remarks on the geometry of austere manifolds*, Bull. Braz. Math. Soc. (N.S.) **21** (1991), 133–157. MR 1139562

[3] R. L. Bryant, S.-S. Chern, R. B. Gardner, H. L. Goldschmidt and P. A. Griffiths, *Exterior differential systems*, MSRI Publications, vol. 18, Springer-Verlag, New York, 1989. MR 1083148

[4] M. Dajczer and L. Florit, *A class of austere submanifolds*, Illinois J. Math. **45** (2001), 735–755. MR 1879232

[5] M. Dajczer and D. Gromoll, *Gauss parametrizations and rigidity aspects of submanifolds*, J. Differential Geom. **22** (1985), 1–12. MR 0826420

[6] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157. MR 0666108

[7] T. A. Ivey and J. M. Landsberg, *Cartan for beginners: Differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, 2003. MR 2003610

[8] S. Karigiannis and M. Min-Oo, *Calibrated subbundles in noncompact manifolds of special holonomy*, Ann. Global Anal. Geom. **28** (2005), 371–394. MR 2199999

[9] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Wiley-Interscience, New York, 1969.

[10] M. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Math. **80** (1993), 151–163. MR 1233478

[11] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is $T$-duality*, Nuclear Phys. B **479** (1996), 243–259. MR 1429831

Marianty Ionel, Department of Mathematics, University of Toledo, 2801 W. Bancroft St., Toledo, OH 43606, USA

*E-mail address*: mionel@math.utoledo.edu

Thomas Ivey, Department of Mathematics, College of Charleston, 66 George St., Charleston, SC 29424, USA

*E-mail address*: iveyt@cofc.edu