International Journal of Geometric Methods in Modern Physics
Vol. 13, No. 2 (2016) 1650012 (19 pages)

DOI: 10.1142/S0219887816500122

Flat Lorentz surfaces in anti-de Sitter 3-space and Gravitational Instantons

Jun-Ichi Inoguchi

Institute of Mathematics
University of Tsukuba
1-1-1 Tennodai, Tsukuba 305-8571, Japan
inoguchi@math.tsukuba.ac.jp

Marianty Ionel

Departamento de Matemática Aplicada
Instituto de Matemática, Sala 112 D
Universidade Federal do Rio de Janeiro
Av. Athos da Silveira Ramos 149
Edifício do centro de Tecnologia
Bloco C Caixa Postal 68530, Ilha do
Fundão, CEP 21941-909, Rio de Janeiro, Brazil
marianty@im.ufrj.br

Sungwook Lee

Department of Mathematics
University of Southern Mississippi
Hattiesburg, MS 39406-5045, U.S.A
sunglee@usm.edu

Received 3 April 2015
Accepted 26 October 2015
Published 15 December 2015

In this paper, we study flat Lorentz surfaces in anti-de Sitter 3-space $\mathbb{H}^3_1(-1)$ in terms of the second conformal structure. Those flat Lorentz surfaces can be represented in terms of a Lorentz holomorphic and a Lorentz anti-holomorphic data similarly to Weierstraß representation formula. An analogue of hyperbolic Gauß map is considered for timelike surfaces in $\mathbb{H}^3_1(-1)$ and the relationship between the conformality (or the holomorphicity) of hyperbolic Gauß map and the flatness of a Lorentz surface is discussed. It is shown that flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ are associated with a hyperbolic Monge–Ampère equation. It is also known that Monge–Ampére equation may be regarded as a 2-dimensional reduction of the Einstein's field equation. Using this connection, we construct a class of anti-self-dual gravitational instantons from flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$.

Keywords: Anti-de Sitter space; flat surface; Lorentz surface; gravitational instanton.

Mathematics Subject Classification 2010: 32Q20, 53C42, 53C50, 53C80

0. Introduction

Let $\overline{M}$ be a 3-dimensional Riemannian or semi-Riemannian space form of constant sectional curvature, say κ, and $M \subset \overline{M}$ a non-degenerate surface. Then the Gauß equation is given by

$$K = \kappa + \epsilon \frac{\det II}{\det I}, \tag{1}$$

where K is the Gaußian curvature of M, I and II denote the first and the second fundamental forms of M respectively, and $\epsilon = \langle N, N \rangle$ is the sign of the unit normal vector field [14]. If M is flat, then $\det II = -\kappa\epsilon \det I$. If in addition $-\kappa\epsilon > 0$ then the signature of the second fundamental form II coincides with that of the first fundamental form. The second fundamental form determines a conformal structure on M. Such a conformal structure is called the *second conformal structure*. One can easily see that if M is Riemannian, $\overline{M} = \mathbb{H}^3(-c^2)$, hyperbolic 3-space of sectional curvature $-c^2$, is the only 3-dimensional space form in which flat surfaces can have the positive definite second conformal structure. If M is semi-Riemannian, only flat spacelike surfaces in $\mathbb{S}^3_1(c^2)$, de Sitter 3-space of sectional curvature c^2, and flat timelike surfaces in $\mathbb{H}^3_1(-c^2)$, anti-de Sitter 3-space of sectional curvature $-c^2$, can have the second conformal structure. Flat surfaces in $\mathbb{H}^3(-1)$ and flat spacelike surfaces in $\mathbb{S}^3_1(1)$ cases are studied by Gálvez, Martínez, and Milán in [8, 9], respectively.

In this paper, we study the only remaining case, flat timelike surfaces in $\mathbb{H}^3_1(-1)$. It turns out that flat timelike surfaces in $\mathbb{H}^3_1(-1)$ can be obtained by a representation formula in terms of a Lorentz holomorphic and a Lorentz anti-holomorphic data analogously to the cases of timelike surfaces of constant mean curvature 1 in $\mathbb{H}^3_1(-1)$ [11] and timelike minimal surfaces in Minkowski 3-space $\mathbb{E}^3_1$ [12], although the holomorphicity is due to the second conformal structure. This is discussed in Sec. 3. An analogue of hyperbolic Gauß map [7, 3] may be considered for timelike surfaces in $\mathbb{H}^3_1(-1)$ as seen in [11]. It is shown (Sec. 4) that the hyperbolic Gauß map of a timelike surface in $\mathbb{H}^3_1(-1)$ is conformal with respect to the second conformal structure if and only if the timelike surface is flat or totally umbilic. The relationship between the holomorphicity of (projected) hyperbolic Gauß map and the flatness of a Lorentz surface is also discussed in Sec. 4. Flat timelike surfaces are associated with a hyperbolic Monge–Amère equation (Sec. 2). It is well known [13] that Monge–Ampére equation may be regarded as a 2-dimensional reduction of the Euclidean Einstein's field equation. Using this connection, we construct (Sec. 5) a class of anti-self-dual gravitational instantons from flat timelike surfaces in $\mathbb{H}^3_1(-1)$.

1. Lorentz Surfaces in Anti-de Sitter 3-Space

Let $\mathbb{E}^4_2$ be the semi-Euclidean 4-space with coordinates (x^0, x^1, x^2, x^3) and the semi-Riemannian metric $\langle \cdot, \cdot \rangle$ of signature $(-, -, +, +)$ given by the quadratic form $-(dx^0)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2$. The *anti-de Sitter* 3*-space* $\mathbb{H}^3_1(-1)$ is a Lorentzian

3-manifold of constant sectional curvature -1 that can be realized as the hyperquadric in $\mathbb{E}_2^4$ [14]:

$$\mathbb{H}_1^3(-1) := \{(x^0, x^1, x^2, x^3) \in \mathbb{E}_2^4 \colon -(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1\}.$$

Let $\mathbb{D}$ be a 2-dimensional orientable domain and $\varphi : \mathbb{D} \to \mathbb{H}_1^3(-1)$ an immersion. The immersion φ is said to be *timelike* if the induced metric $I = \langle d\varphi, d\varphi\rangle$ on $\mathbb{D}$ is Lorentzian. The induced Lorentzian metric I determines a Lorentz conformal structure $\mathcal{C}_I$ on $\mathbb{D}$. More specifically, if (x', y') is a Lorentz isothermal coordinate system with respect to the conformal structure $\mathcal{C}_I$, then the first fundamental form is given by $I = e^\rho\{-(dx')^2 + (dy')^2\}$ where ρ is a real-valued smooth function defined on $\mathbb{D}$. Hence a timelike immersion φ being conformal is equivalent to the conditions:

$$\langle \varphi_{x'}, \varphi_{x'}\rangle = -e^\rho, \quad \langle \varphi_{y'}, \varphi_{y'}\rangle = e^\rho, \quad \langle \varphi_{x'}, \varphi_{y'}\rangle = 0. \tag{2}$$

These conditions are said to be *conformality conditions* and a conformal timelike surface is said to be a *Lorentz surface* hereafter. Let $u' := x' + y'$ and $v' := -x' + y'$. Then (u', v') defines a *null coordinate system* with respect to the conformal structure $\mathcal{C}_I$. The first fundamental form I is written in terms of (u', v') as

$$I = e^\rho du' dv'.$$

The differential operators $\frac{\partial}{\partial u'}$ and $\frac{\partial}{\partial v'}$ are given by

$$\frac{\partial}{\partial u'} = \frac{1}{2}\left(\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right), \quad \frac{\partial}{\partial v'} = \frac{1}{2}\left(-\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}\right).$$

With these differential operators, one can speak of Lorentz holomorphicity and Lorentz anti-holomorphicity. A map $f : \mathbb{D} \to \mathbb{E}_1^2$ is said to be *Lorentz holomorphic* (*Lorentz anti-holomorphic*) if $\frac{\partial f}{\partial v'} = 0$ ($\frac{\partial f}{\partial u'} = 0$, respectively).

The conformality conditions (2) are equivalent to

$$\langle \varphi_{u'}, \varphi_{u'}\rangle = \langle \varphi_{v'}, \varphi_{v'}\rangle = 0, \quad \langle \varphi_{u'}, \varphi_{v'}\rangle = \frac{1}{2}e^\rho.$$

Let N be a unit normal vector field along φ. Then

$$\langle N, N\rangle = 1, \quad \langle \varphi, N\rangle = \langle \varphi_{u'}, N\rangle = \langle \varphi_{v'}, N\rangle = 0.$$

The mean curvature H is computed to be $H = 2e^{-\rho}\langle \varphi_{u'v'}, N\rangle$. Let $Q := \langle \varphi_{u'u'}, N\rangle$ and $R := \langle \varphi_{v'v'}, N\rangle$. The quadratic differential

$$\mathcal{Q} := Q du'^2 + R dv'^2$$

is then called *Hopf differential*. The Hopf differential is defined globally on the Lorentz surface $(\mathbb{D}, \mathcal{C}_I)$. The second fundamental form II of φ is given by

$$II = \mathcal{Q} + HI.$$

A point $p \in \mathbb{D}$ is said to be an *umbilic point* if II is proportional to I at p or equivalently p is a common zero of Q and R, i.e. $\mathcal{Q}(p) = 0$.

If K is the Gaußian curvature, then the Gauß equation which describes a relationship between K, H, Q, R takes the form:

$$H^2 - K - 1 = 4e^{-2\rho} QR. \tag{3}$$

The semi-Euclidean 4-space $\mathbb{E}^4_2$ is identified with the linear space $\mathrm{M}(2, \mathbb{R})$ of all 2×2 real matrices via the correspondence

$$\mathbf{u} = (x^0, x^1, x^2, x^3) \leftrightarrow \begin{pmatrix} x^0 + x^3 & x^1 + x^2 \\ -x^1 + x^2 & x^0 - x^3 \end{pmatrix}. \tag{4}$$

The inner product of $\mathbb{E}^4_2$ corresponds to the inner product of $\mathrm{M}(2, \mathbb{R})$

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{2}\{\mathrm{tr}(\mathbf{uv}) - \mathrm{tr}(\mathbf{u})\,\mathrm{tr}(\mathbf{v})\}, \quad \mathbf{u}, \mathbf{v} \in \mathrm{M}(2, \mathbb{R}). \tag{5}$$

In particular, $\langle \mathbf{u}, \mathbf{u} \rangle = -\det \mathbf{u}$ so the correspondence is an isometry. The standard basis $\{e_0, e_1, e_2, e_3\}$ for $\mathbb{E}^4_2$ is then identified with the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}'\}$ satisfies the properties:

$$\begin{aligned} &\mathbf{i}^2 = -\mathbf{1}, && \mathbf{j}'^2 = \mathbf{k}'^2 = \mathbf{1}, \\ &\mathbf{ij}' = -\mathbf{j}'\mathbf{i} = \mathbf{k}', && \mathbf{j}'\mathbf{k}' = -\mathbf{k}'\mathbf{j}' = -\mathbf{i}, \quad \mathbf{k}'\mathbf{i} = -\mathbf{ik}' = \mathbf{j}'. \end{aligned}$$

A 2×2 matrix of the form $x^0\mathbf{1} + x^1\mathbf{i} + x^2\mathbf{j}' + x^3\mathbf{k}'$ is called a *split-quaternion* or a *paraquaternion*. The set $\mathbb{H}'$ of all split-quaternions is an algebra over real numbers and by (4) $\mathbb{H}'$ is identified with $\mathbb{E}^4_2$. The Lie group of timelike unit vectors corresponds to a special linear group

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}(2, \mathbb{R}) : ad - bc = 1 \right\}.$$

The metric of $\mathrm{SL}(2, \mathbb{R})$ induced by the inner product (5) is a bi-invariant Lorentz metric of constant curvature -1. Hence, $\mathrm{SL}(2, \mathbb{R})$ is identified with $\mathbb{H}^3_1(-1)$.

The Lie group $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ acts isometrically on $\mathbb{E}^4_2$ via the group action:

$$(g_1, g_2) \cdot \mathbf{u} = g_1 \mathbf{u} g_2^t \tag{6}$$

for $g_1, g_2 \in \mathrm{SL}(2, \mathbb{R})$ and $\mathbf{u} \in \mathbb{E}^4_2$. This action is transitive on $\mathbb{H}^3_1(-1)$. The isotropy group of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ at $\mathbf{1}$ is $\mathcal{K} = \{(g, (g^{-1})^t) : g \in \mathrm{SL}(2, \mathbb{R})\}$ and $\mathbb{H}^3_1(-1)$ is represented as the Lorentzian symmetric space $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/\mathcal{K}$. The natural projection $\pi : \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \to \mathbb{H}^3_1(-1)$ is given by $\pi(g_1, g_2) = g_1 g_2^t$. The action (6) induces a double covering $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SO}^+(2, 2)$, where $\mathrm{SO}^+(2, 2)$ denotes the identity component of the pseudo-orthogonal group $\mathrm{O}(2, 2)$.

The frame field $\{e_\alpha : \alpha = 0,1,2,3\}$ can be then parametrized as follows: for each $g = (g_1, g_2) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$,

$$e_0(g) = g_1 \mathbf{1} g_2^t, \quad e_1(g) = g_1 \mathbf{i} g_2^t, \quad e_2(g) = g_1 \mathbf{j}' g_2^t, \quad e_3(g) = g_1 \mathbf{k}' g_2^t.$$

Let $\mathbb{D}$ be a 2-dimensional simply connected orientable domain and $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ a Lorentz surface with unit normal vector field N. Then we can define an orthonormal frame field $\mathcal{F} : \mathbb{D} \to \mathrm{SO}^+(2)$ along φ by

$$\mathcal{F} = (\varphi, e^{-\rho/2}(\varphi_{u'} - \varphi_{v'}), e^{-\rho/2}(\varphi_{u'} + \varphi_{v'}), N). \tag{7}$$

By means of a double covering induced by the group action (6), one can find a lift $F = (F_1, F_2)$ (called a coordinate frame) of $\mathcal{F}$ to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ such that

$$F_1(\mathbf{1}, \mathbf{i}, \mathbf{j}', \mathbf{k}') F_2^t = \mathcal{F}. \tag{8}$$

Each component framing F_1 and F_2 satisfy the following system of first-order linear equations, so-called *Lax system*:

$$(F_i)_{u'} = F_i U_i, \quad (F_i)_{v'} = F_i V_i, \tag{9}$$

where $i = 1,2$ and

$$U_1 = \begin{pmatrix} \rho_{u'}/4 & \frac{1}{2}e^{\rho/2}(H+1) \\ -e^{-\rho/2}Q & -\rho_{u'}/4 \end{pmatrix}, \quad V_1 = \begin{pmatrix} -\rho_{v'}/4 & e^{-\rho/2}R \\ -\frac{1}{2}e^{\rho/2}(H-1) & \rho_{v'}/4 \end{pmatrix},$$

$$U_2 = \begin{pmatrix} -\rho_{u'}/4 & e^{-\rho/2}Q \\ -\frac{1}{2}e^{\rho/2}(H-1) & \rho_{u'}/4 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \rho_{v'}/4 & \frac{1}{2}e^{\rho/2}(H+1) \\ -e^{-\rho/2}R & -\rho_{v'}/4 \end{pmatrix}.$$

The compatibility condition $F_{u'v'} = F_{v'u'}$ gives the Maurer–Cartan equations

$$(U_i)_{v'} - (V_i)_{u'} - [U_i, V_i] = 0, \quad i = 1,2. \tag{10}$$

Each of these equations is equivalent to the Gauß–Mainardi–Codazzi equations

$$\rho_{u'v'} + \tfrac{1}{2}e^{\rho}(H^2 - 1) - 2QRe^{-\rho} = 0, \tag{11}$$

$$H_{u'} = 2e^{-\rho}Q_{v'}, \quad H_{v'} = 2e^{-\rho}R_{u'}. \tag{12}$$

2. Fundamental Equations

In this section we derive some fundamental equations that we need in order to study flat Lorentz surfaces in the following sections.

Proposition 1. *Let $\mathbb{D}$ be a 2-dimensional simply connected domain with isothermal coordinates (x', y') and $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ a flat Lorentz surface with the first* *fundamental form* $I = e^{\rho}(-dx'^2 + dy'^2)$. *Then there exist coordinates* (x, y) *in* $\mathbb{D}$ *so that* I *can be written as*

$$I = -dx^2 + dy^2. \tag{13}$$

Proof. From Eqs. (3) and (11), we obtain

$$\rho_{u'v'} = -\frac{1}{2}e^{\rho}K. \tag{14}$$

Since $K = 0$, (14) is simply the homogeneous wave equation

$$\rho_{u'v'} = 0.$$

The general solution is $\rho(u', v') = X(u') + Y(v')$ where $X, Y : \mathbb{D} \to \mathbb{R}$.

Let $u := \int e^X du'$ and $v := \int e^Y dv'$. Define x and y by

$$x := \frac{u-v}{2}, \quad y := \frac{u+v}{2}.$$

Then

$$I = dudv = -dx^2 + dy^2. \qquad \square$$

Let $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ be a simply connected flat Lorentz surface with globally defined isothermal coordinate system[a] (x, y) and the first fundamental form (13). Then the Gauß–Weingarten equations are given by

$$\varphi_{xx} = -\varphi + \ell N, \tag{15}$$

$$\varphi_{xy} = \mathfrak{m}N, \tag{16}$$

$$\varphi_{yy} = \varphi + \mathfrak{n}N, \tag{17}$$

$$N_x = \ell\varphi_x - \mathfrak{m}\varphi_y, \tag{18}$$

$$N_y = \mathfrak{m}\varphi_x - \mathfrak{n}\varphi_y. \tag{19}$$

Here N is a unit normal vector field along φ. The coefficient functions ℓ, $\mathfrak{m}$ and $\mathfrak{n}$ are defined by

$$\ell = \langle\varphi_{xx}, N\rangle, \quad \mathfrak{m} = \langle\varphi_{xy}, N\rangle, \quad \mathfrak{n} = \langle\varphi_{yy}, N\rangle.$$

The Gauß–Mainardi–Codazzi equations, which are the integrability conditions for Gauß–Weingarten equations, are equivalent to

$$\mathfrak{m}^2 - \ell\mathfrak{n} = 1, \tag{20}$$

$$\mathfrak{m}_x = \ell_y, \quad \mathfrak{n}_x = \mathfrak{m}_y. \tag{21}$$

Equations (21) guarantee the existence of a potential ϕ such that

$$\ell = \phi_{xx}, \quad \mathfrak{m} = \phi_{xy} = \phi_{yx}, \quad \mathfrak{n} = \phi_{yy}.$$

[a]In Lorentzian case, the Riemann Mapping Theorem or Köbe Uniformization Theorem does not hold. So the global existence of isothermal coordinates is not guaranteed even in a simply connected Lorentzian 2-manifold. See, for example, [16] for details.

The Gauß equation (11) then becomes the *hyperbolic Monge–Ampère equation*

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = -1. \tag{22}$$

The second fundamental form is given by

$$II = \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2. \tag{23}$$

Note that the second fundamental form (23) determines a conformal structure, the so-called *second conformal structure*, on $\mathbb{D}$. To see this let

$$x' = x - \phi_x, \quad y' = y + \phi_y. \tag{24}$$

Then by a straightforward calculation one obtains

$$\begin{aligned} dx &= \frac{1+\phi_{yy}}{2-\phi_{xx}+\phi_{yy}}dx' + \frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}dy', \\ dy &= -\frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}dx' + \frac{1-\phi_{xx}}{2-\phi_{xx}+\phi_{yy}}dy' \end{aligned} \tag{25}$$

and

$$II = \phi_{xx}dx^2 + 2\phi_{xy}dxdy + \phi_{yy}dy^2 = \frac{dudv}{2-\phi_{xx}+\phi_{yy}}, \tag{26}$$

where $u := x' + y'$ and $v := -x' + y'$. Hence we see that (u, v) defines a null coordinate system with respect to the conformal structure $\mathcal{C}_{II}$ on $\mathbb{D}$ determined by the second fundamental form II. The differential operators $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial y'}$ are computed, in terms of the coordinates (x, y), to be:

$$\begin{aligned} \frac{\partial}{\partial x'} &= \frac{1+\phi_{yy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial x} - \frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial y'} &= \frac{\phi_{xy}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial x} + \frac{1-\phi_{xx}}{2-\phi_{xx}+\phi_{yy}}\frac{\partial}{\partial y}. \end{aligned} \tag{27}$$

It follows from $\frac{\partial(x',y')}{\partial(x,y)} = 2 - \phi_{xx} + \phi_{yy}$ that:

Proposition 2. *The coordinates* (x', y') *in* (24) *exist globally on* $\mathbb{D}$ *if and only if* $2 - \phi_{xx} + \phi_{yy} \neq 0$.

On the other hand, we have

$$2 - \phi_{xx} + \phi_{yy} = 2(H + 1), \tag{28}$$

where H is the mean curvature of φ. So, it is required for flat Lorentz surfaces to satisfy the condition $H > -1$.

3. A Representation Formula for Flat Lorentz Surfaces in $\mathbb{H}^3_1(-1)$

In this section, it is shown that a flat Lorentz surface may be represented by a Lorentz holomorphic and a Lorentz anti-holomorphic data. We discuss this by means of the Lax system (9).

Suppose that $\mathbb{D}$ is a simply connected, oriented, 2-dimensional domain with globally defined null coordinate system (u', v'). Let $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with induced metric $I = e^{\rho} du' dv'$. Then by Proposition 1 we may assume that $\rho = 0$; hence the Gauß–Marnardi–Codazzi equations (11), (12) can be written as

$$H^2 - 1 = 4QR, \tag{29}$$

$$H_{u'} = 2Q_{v'}, \quad H_{v'} = 2R_{u'}. \tag{30}$$

From the Lax system (9), the 1-forms $F_1^{-1} dF_1$ and $F_2^{-1} dF_2$ are given by

$$\begin{aligned} F_1^{-1} dF_1 &= \begin{pmatrix} 0 & \dfrac{1}{2}(H+1)du' + R dv' \\ -Q du' - \dfrac{1}{2}(H-1)dv' & 0 \end{pmatrix}, \\ F_2^{-1} dF_2 &= \begin{pmatrix} 0 & Q du' + \dfrac{1}{2}(H+1)dv' \\ -\frac{1}{2}(H-1)du' - R dv' & 0 \end{pmatrix}. \end{aligned} \tag{31}$$

The Mainardi–Codazzi equations (30) imply that the nonzero entries of $F_1^{-1} dF_1$ and $F_2^{-1} dF_2$ are exact, i.e. there exist functions $u, \xi, v, \zeta \colon \mathbb{D}(u', v') \to \mathbb{E}^2_1$ such that

$$F_1^{-1} dF_1 = \begin{pmatrix} 0 & du \\ d\xi & 0 \end{pmatrix} \quad \text{and} \quad F_2^{-1} dF_2 = \begin{pmatrix} 0 & dv \\ d\zeta & 0 \end{pmatrix}. \tag{32}$$

Proposition 3. *The functions (u, v) constitute globally defined null coordinates of $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ if and only if the mean curvature $H \neq -1$.*

Proof. It follows from the Jacobian $\frac{\partial(u,v)}{\partial(u',v')} = \frac{1}{2}(H+1)$. □

Remark 1. In order to consider flat Lorentz surfaces with respect to the new null coordinate system (u, v), it is required that $H > -1$.

Remark 2. Flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$ with $H = 1$ are interesting because they resemble horospheres in hyperbolic 3-space: they are flat and totally umbilic, or equivalently they have constant (hyperbolic) Gauß map.[b]

The following proposition tells that $F_1^{-1} dF_1$ is a Lorentz holomorphic 1-form and $F_2^{-1} dF_2$ is a Lorentz anti-holomorphic 1-form.

Proposition 4. $\frac{\partial \xi}{\partial v} = 0$ *and* $\frac{\partial \zeta}{\partial u} = 0$ *i.e. ξ is Lorentz holomorphic and ζ is Lorentz anti-holomorphic.*

[b]The notion of hyperbolic Gauß map will be introduced in Sec. 4.

Proof. It follows from (31) and (32) by straightforward calculations. $\square$

Let $f = \frac{\partial \xi}{\partial u}$ and $g = \frac{\partial \zeta}{\partial v}$. Then $d\xi = f du$ and $d\zeta = g dv$ since ξ and ζ are, respectively, Lorentz holomorphic and Lorentz anti-holomorphic. The 1-forms $F_1^{-1}dF_1$ and $F_2^{-1}dF_2$ are then written as

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \quad \text{and} \quad F_2^{-1}dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv.$$

The induced metric I is computed to be

$$I = \langle d\varphi, d\varphi \rangle = f du^2 + g dv^2 + (1 + fg) du dv.$$

The functions f and g are given by

$$f = -\frac{2Q}{H+1}, \quad g = -\frac{2R}{H+1}. \tag{33}$$

The ∂' and ∂'' forms[c] du' and dv' which are given by

$$du' = du + g dv, \quad dv' = f du + dv.$$

Finally the second fundamental form II is computed to be

$$II = (1 - fg) du dv.$$

Therefore, we have the following theorem holds.

Theorem 5. *Let $\mathbb{D}$ be a simply connected, oriented, 2-dimensional domain with globally defined null coordinate system (u', v'). Let $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with induced metric $I = du'dv'$, mean curvature $H > -1$, and Hopf differential $Q du'^2 + R dv'^2$. Then there exists a globally defined null coordinate system (u, v) in $\mathbb{D}$ such that*

$$du = \frac{1}{2}(H+1)du' + R dv' \quad \text{and} \quad dv = Q du' + \frac{1}{2}(H+1)dv'.$$

Define $f, g : \mathbb{D} \to \mathbb{E}^2_1$ by (33). Then f is a Lorentz holomorphic function and g is a Lorentz anti-holomorphic function with respect to the null coordinate system (u, v), and that $fg < 1$. The flat Lorentz surface $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ may be described by

$$\varphi = F_1 F_2^t,$$

where $F_1, F_2 : \mathbb{D} \to \mathrm{SL}(2, \mathbb{R})$ are immersions that satisfy the system of decoupled ODEs:

$$F_1^{-1}dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \quad \text{and} \quad F_2^{-1}dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv.$$

[c] They are analogues of $(1, 0)$ and $(0, 1)$ forms in complex analysis.

The first and the second fundamental forms are, respectively, given in terms of f and g by

$$I = f du^2 + (1 + fg) du dv + g dv^2, \tag{34}$$

$$II = (1 - fg) du dv. \tag{35}$$

The converse of Theorem 5 also holds, namely:

Theorem 6. *Let $\mathbb{D}$ be an oriented 2-dimensional domain with globally defined null coordinate system (u', v'). Suppose that $F_1, F_2 : \mathbb{D} \to \mathrm{SL}(2, \mathbb{R})$ are immersions that satisfy the Lax system (9). If there exist a null coordinate system (u, v) globally defined in $\mathbb{D}$ and functions $f, g : \mathbb{D} \to \mathbb{E}^2_1$ such that*

$$F_1^{-1} dF_1 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} du \quad \text{and} \quad F_2^{-1} dF_2 = \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix} dv,$$

where f is Lorentz holomorphic, g is Lorentz anti-holomorphic with respect to (u, v), and that $fg < 1$, then

$$\varphi := F_1 F_2^t : \mathbb{D} \to \mathbb{H}^3_1(-1)$$

is a flat Lorentz surface whose first and second fundamental forms are given by (34) and (35) respectively. Furthermore $H > -1$ and

$$f = -\frac{2Q}{e^\rho (H + 1)}, \quad g = -\frac{2R}{e^\rho (H + 1)},$$

where ρ is a constant and $Q du'^2 + R dv'^2$ is the Hopf differential of φ.

Proof. Suppose that $F_1, F_2 : \mathbb{D} \to \mathrm{SL}(2, \mathbb{R})$ satisfy the Lax system (9). Then

$$\begin{aligned} F_1^{-1} dF_1 &= \begin{pmatrix} \frac{\rho_{u'}}{4} du' - \frac{\rho_{v'}}{4} dv' & \frac{1}{2} e^{\rho/2} (H + 1) du' + e^{-\rho/2} R dv' \\ -e^{-\rho/2} Q du' - \frac{1}{2} e^{\rho/2} (H - 1) dv' & -\frac{\rho_{u'}}{4} du' + \frac{\rho_{v'}}{4} dv' \end{pmatrix} \\ &= \begin{pmatrix} 0 & du \\ d\xi & 0 \end{pmatrix} \end{aligned} \tag{36}$$

and

$$\begin{aligned} F_2^{-1} dF_2 &= \begin{pmatrix} -\frac{\rho_{u'}}{4} du' + \frac{\rho_{v'}}{4} dv' & e^{-\rho/2} Q du' + \frac{1}{2} e^{\rho/2} (H + 1) dv' \\ -\frac{1}{2} e^{\rho/2} (H - 1) du' - e^{-\rho/2} R dv' & \frac{\rho_{u'}}{4} du' - \frac{\rho_{v'}}{4} dv' \end{pmatrix} \\ &= \begin{pmatrix} 0 & dv \\ d\zeta & 0 \end{pmatrix}, \end{aligned} \tag{37}$$

where $d\xi = fdu$ and $d\zeta = gdv$. Clearly $\rho = c$, a constant. Since ξ is Lorentz holomorphic with respect to (u, v),

$$\frac{\partial \xi}{\partial u'} = \frac{1}{2}e^{\frac{c}{2}}(H+1)\frac{\partial \xi}{\partial u}, \quad \frac{\partial \xi}{\partial v'} = e^{-\frac{c}{2}}R\frac{\partial \xi}{\partial u}. \tag{38}$$

From (36) we also find

$$\frac{\partial \xi}{\partial u'} = -e^{-\frac{c}{2}}Q, \quad \frac{\partial \xi}{\partial v'} = -\frac{1}{2}e^{\frac{c}{2}}(H-1). \tag{39}$$

Combining (38) and (39) we obtain the equation

$$(H^2 - 1 - 4e^{-2c}QR)\frac{\partial \xi}{\partial u} = K\frac{\partial \xi}{\partial u} = 0. \tag{40}$$

If $\frac{\partial \xi}{\partial u} = 0$ then $d\xi = 0$ so $Q = 0$ and $H = 1$. This implies that $K = 0$ by the Gauß equation (3). Moreover this is the case that $\varphi = F_1F_2^t$ is an analogue of horospheres. If $\frac{\partial \xi}{\partial u} \neq 0$ then again $K = 0$ by (40). Hence in either case $\varphi = F_1F_2^t$ is a flat Lorentz surface in $\mathbb{H}_1^3(-1)$.

The following identities can be calculated from (36) and (37):

$$f = -\frac{2Q}{e^c(H+1)}, \quad g = -\frac{2R}{e^c(H+1)},$$

$$du' = e^{-2/c}du + e^{-c/2}gdv, \quad dv' = e^{-c/2}fdu + e^{-c/2}dv.$$

Using these identities, the first and the second fundamental forms can be written in terms of f and g as:

$$I = fdu^2 + (1+fg)dudv + gdv^2, \quad II = (1-fg)dudv.$$

Since $1 - fg = \frac{2}{H+1} > 0$, $H > -1$. $\square$

4. Flat Lorentz Surfaces in $\mathbb{H}_1^3(-1)$ and the Hyperbolic Gauss Map

Let $\varphi : \mathbb{D} \to \mathbb{H}_1^3(-1)$ be a Lorentz surface. At each point $p \in \mathbb{D}$, the oriented normal geodesic in $\mathbb{H}_1^3(-1)$ emanating from $\varphi(p)$, which is tangent to the normal vector $N(p)$ meets the null cone $\mathbb{N}^3 = \{\mathbf{u} \in \mathbb{E}_2^4 : \langle \mathbf{u}, \mathbf{u} \rangle = 0\}$ at exactly two points $[\varphi + N](p)$ and $[\varphi - N](p)$ in $\mathbb{N}^3$. Here, $[\varphi \pm N]$ denotes the null lines spanned by the null vectors $\varphi \pm N$. The orientation of φ allows us to name $[\varphi + N](p)$ the initial point and $[\varphi - N](p)$ the terminal point. The maps $G_+, G_- : \mathbb{D} \to \mathbb{N}^3$ defined by $G_+(p) = [\varphi + N](p)$ and $G_-(p) = [\varphi - N](p)$, respectively are called the *hyperbolic Gauss map*[d] of φ. In this paper, we are particularly interested in the hyperbolic Gauß map $G_- = [\varphi - N]$.

Let $\mathbb{N}_+^3$ and $\mathbb{N}_-^3$ respectively denote the future and the past null cones with respect to the coordinate time[e] x_0. The multiplicative group $\mathbb{R}^+$ acts on $\mathbb{N}_+^3$ and

[d]The notion of hyperbolic Gauß map was first introduced by Epstein [7] and was also used by Bryant in the study of constant mean curvature one surfaces in hyperbolic 3-space [3].
[e]Without loss of generality, we may assume that $x_0 \neq 0$.

also on $\mathbb{N}^3_-$ by scalar multiplication. Let us denote by $\mathbb{N}^3_+/\mathbb{R}^+$ and $\mathbb{N}^3_-/\mathbb{R}^+$ the orbit spaces of $\mathbb{N}^3_+$ and $\mathbb{N}^3_-$, respectively. Then $\mathbb{N}^3_+/\mathbb{R}^+$ ($\mathbb{N}^3_-/\mathbb{R}^+$) is diffeomorphic to $\mathbb{S}^2_1$, where $\mathbb{S}^2_1$ denotes the *de Sitter 2-space* [14]:

$$\mathbb{S}^2_1 = \{(\xi^0, \xi^1, \xi^2) \in \mathbb{E}^3_1 : -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 = 1\}.$$

Let $\mathcal{N} = (0, 0, 1)$ and $\mathcal{S} = (0, 0, -1) \in \mathbb{S}^2_1$ be the *north pole* and the *south pole* of $\mathbb{S}^2_1$. Let $\wp_+ : \mathbb{S}^2_1 \setminus \{\xi^3 = -1\} \to \mathbb{E}^2_1 \setminus \mathbb{H}^1_0$ be the stereographic projection from the south pole $\mathcal{S}$, where $\mathbb{H}^1_0 = \{(\xi^1, \xi^2) \in \mathbb{E}^2_1 : -(\xi^1)^2 + (\xi^2)^2 = -1\}$ is the hyperbola in $\mathbb{E}^2_1$. Then

$$\wp_+(\xi^1, \xi^2, \xi^3) = \left(\frac{\xi^1 + \xi^2}{1 + \xi^3}, \frac{-\xi^1 + \xi^2}{1 + \xi^3}\right) \in \mathbb{E}^2_1(u, v), \tag{41}$$

where (u, v) is a null coordinate system in $\mathbb{E}^2_1$. If $\wp_- : \mathbb{S}^2_1 \setminus \{\xi^3 = 1\} \to \mathbb{E}^2_1 \setminus \mathbb{H}^1_0$ denotes the stereographic projection from the north pole $\mathcal{N}$, then

$$\wp_-(\xi^1, \xi^2, \xi^3) = \left(\frac{\xi^1 + \xi^2}{1 - \xi^3}, \frac{-\xi^1 + \xi^2}{1 - \xi^3}\right) \in \mathbb{E}^2_1(u, v). \tag{42}$$

Hence, the projected hyperbolic Gauß map is mapped into $\mathbb{E}^2_1(u, v)$.

Let $F_1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{13} & F_{14} \end{pmatrix}$ and $F_2 = \begin{pmatrix} F_{21} & F_{22} \\ F_{23} & F_{24} \end{pmatrix}$, where $F = (F_1, F_2)$ is a coordinate frame. Then by (8)

$$\varphi - N = F_1(\mathbf{1} - \mathbf{k}')F_2^t = 2\begin{pmatrix} F_{12} \\ F_{14} \end{pmatrix}\begin{pmatrix} F_{22} \\ F_{24} \end{pmatrix}^t.$$

That is, the hyperbolic Gauß map $[\varphi - N]$ is written as

$$[\varphi - N] = \left[\begin{pmatrix} F_{12} \\ F_{14} \end{pmatrix}\begin{pmatrix} F_{22} \\ F_{24} \end{pmatrix}^t\right].$$

By the identification (4), the projected hyperbolic Gauß map is given by

$$[\varphi - N] \overset{\wp_-}{=} \left(\frac{F_{12}}{F_{14}}, \frac{F_{22}}{F_{24}}\right) \in \mathbb{E}^2_1(u, v) \tag{43}$$

and

$$[\varphi - N] \overset{\wp_+}{=} \left(\frac{F_{24}}{F_{22}}, \frac{F_{14}}{F_{12}}\right) \in \mathbb{E}^2_1(u, v). \tag{44}$$

By (18), (19) and (27), one obtains

$$(\varphi - N)_{x'} = \varphi_x, \quad (\varphi - N)_{y'} = \varphi_y. \tag{45}$$

It then follows that

$$\begin{aligned} &\langle(\varphi - N)_u, (\varphi - N)_u\rangle = \langle(\varphi - N)_v, (\varphi - N)_v\rangle = 0, \\ &\langle(\varphi - N)_u, (\varphi - N)_v\rangle = \frac{1}{2}. \end{aligned} \tag{46}$$

Let $d\rho^2$ denote the induced metric on $\mathbb{N}^3$. Then the pullback of $d\rho^2$ by $\varphi - N$ is

$$d\rho^2_{\varphi-N} = \langle d(\varphi - N), d(\varphi - N)\rangle = dudv$$

by (46). This means that the hyperbolic Gauß map $[\varphi - N]$ is conformal with respect to the second conformal structure on $\mathbb{D}$. In fact, more can be said about conformal hyperbolic Gauß maps.

Theorem 7. *Let* $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ *be a Lorentz surface with unit normal vector field* N *and mean curvature* $H \geq 1$. *The hyperbolic Gauß map* $[\varphi - N] : \mathbb{D} \to \mathbb{S}^2_1$ *is conformal with respect to the second fundamental form if and only if* φ *is flat or totally umbilic. Here,* $\mathbb{S}^2_1 = \mathbb{N}^3_+/\mathbb{R}^+$ *or* $\mathbb{N}^3_-/\mathbb{R}^+$.

Proof. Let $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ be a Lorentz surface with induced metric $I = e^\rho du'dv'$ where (u', v') is a globally defined null coordinate system in $\mathbb{D}$. Let $\mathfrak{s} := (\varphi, \varphi_{u'}, \varphi_{v'}, N)$. The $\mathfrak{s}$ defines a moving frame on φ and satisfies the Gauß–Weingarten equations:

$$\mathfrak{s}_{u'} = \mathfrak{s}\mathcal{U}, \quad \mathfrak{s}_{v'} = \mathfrak{s}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & \frac{1}{2}e^\rho & 0 \\ 1 & \rho_{u'} & 0 & -H \\ 0 & 0 & 0 & -2Qe^{-\rho} \\ 0 & Q & \frac{1}{2}e^\rho H & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & \frac{1}{2}e^\rho & 0 & 0 \\ 0 & 0 & 0 & -2Re^{-\rho} \\ 1 & 0 & \rho_{v'} & -H \\ 0 & \frac{1}{2}e^\rho H & R & 0 \end{pmatrix}.$$

Using the Gauß–Weingarten equations, one can calculate

$$\begin{aligned} d\rho^2 &= \langle d(\varphi - N), d(\varphi - N)\rangle = -KI + 2(H+1)II \\ &= \left[2(H+1) - \frac{K}{H}\right] II + \frac{K\mathcal{Q}}{H}. \end{aligned}$$

Therefore $[\varphi - N]$ is conformal with respect to the second fundamental form II if and only if φ is flat or totally umbilic. □

Remark 3. If $H = -1$ then $d\rho^2 = \langle d(\varphi - N), d(\varphi - N)\rangle = KII - K\mathcal{Q}$. If $K = 0$ in addition then $d\rho^2$ is degenerate and $d(\varphi - N) = 0$, i.e. the hyperbolic Gauß map $[\varphi - N]$ is constant. This is the case when φ is an analogue of horosphere.

In light of theorem 7, one may wonder if there is any connection between the flatness of a Lorentz surfaces and the holomorphicity of the hyperbolic Gauß map.

Theorem 8. *Let* $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ *be a Lorentz surface with mean curvature* $H \geq 1$. *Then* φ *is flat or totally umbilic if and only if the first and the second coordinates of the projected hyperbolic Gauß map* (43) *are Lorentz anti-holomorphic and Lorentz holomorphic, respectively, with respect to null coordinates* (u, v) *determined by the second fundamental form.*

Proof. If φ is totally umbilic, then the first and the second coordinates of the projected Gauß map are Lorentz anti-holomorphic and Lorentz-holomorphic with respect to null coordinates determined by Lorentz isothermal coordinates as shown in [11, Sec. 13]. Since $I\!I = HI$, the same is true for null coordinates (u, v) determined by the second fundamental form.

Let $\varphi : \mathbb{D} \to \mathbb{H}^3_1(-1)$ be a Lorentz surface with the first fundamental form $ds_\varphi = e^\rho du'dv'$. Suppose that (u, v) is another null coordinate system globally defined in $\mathbb{D}$. From the Lax system (9) we obtain

$$(F_{12})_v F_{14} - F_{12}(F_{14})_v = \frac{1}{2}e^{\rho/2}(H+1)\frac{\partial u'}{\partial v} + e^{-\rho/2}R\frac{\partial v'}{\partial v}, \tag{47}$$

$$(F_{22})_u F_{24} - F_{22}(F_{24})_u = e^{-\rho/2}Q\frac{\partial u'}{\partial u} + \frac{1}{2}e^{\rho/2}(H+1)\frac{\partial v'}{\partial u}. \tag{48}$$

If φ is flat and (u, v) is a null coordinate system determined by the second fundamental form, then without loss of generality we may take

$$\frac{\partial u'}{\partial v} = -\frac{2Re^{-\rho/2}}{e^\rho(H+1)}, \quad \frac{\partial v'}{\partial v} = e^{-\rho/2},$$
$$\frac{\partial u'}{\partial u} = e^{-\rho/2}, \quad \frac{\partial v'}{\partial u} = -\frac{2Qe^{-\rho/2}}{e^\rho(H+1)}.$$

It then follows that $(\frac{F_{12}}{F_{14}})_v = 0$ and $(\frac{F_{22}}{F_{24}})_u = 0$.

In order to show the "if" part of the statement, suppose that the first fundamental form is given by $I = e^\rho du'dv'$ while the second fundamental form is given by $I\!I = e^\lambda dudv$. If the first and the second coordinates of the projected hyperbolic Gauß map are, respectively, Lorentz anti-holomorphic and Lorentz holomorphic, then by (47) and (48)

$$\frac{1}{2}e^{-\rho/2}(H+1)\frac{\partial u'}{\partial u} + e^{-\rho/2}R\frac{\partial v'}{\partial u} = 0,$$
$$e^{-\rho/2}Q\frac{\partial u'}{\partial v} + \frac{1}{2}e^{-\rho/2}(H+1)\frac{\partial v'}{\partial v} = 0.$$

Since $\frac{\partial(u',v')}{\partial(u,v)} \neq 0$,

$$\frac{\partial v}{\partial u'} = \frac{1}{2}e^{-\rho/2}(H+1), \quad \frac{\partial v}{\partial v'} = e^{-\rho/2}R,$$
$$\frac{\partial u}{\partial u'} = e^{-\rho/2}Q, \quad \frac{\partial u}{\partial v'} = \frac{1}{2}e^{-\rho/2}(H+1).$$

Consequently,

$$I\!I = \frac{1}{2}e^\lambda Q(H+1)du'^2 + \frac{1}{4}e^\lambda e^\rho[2H(H+1) - K]du'dv' + \frac{1}{2}e^\lambda R(H+1)dv'^2$$

and hence we have

$$[e^\lambda(H+1) - 2]Q = 0, \quad [e^\lambda(H+1) - 2]R = 0, \quad 2[H+1-2e^{-\lambda}]H = K.$$

If $H = 2e^{-\lambda} - 1$ then $K = 0$. Otherwise $Q = R = 0$. This completes the proof. □

Remark 4. Since we require that $H \geq 1$, λ must satisfy $\lambda \leq 0$.

Remark 5. If φ is both flat and totally umbilic then by the Gauß equation (3) $H = 1$ i.e. $\lambda = 0$.

5. Flat Lorentz Surfaces in $\mathbb{H}^3_1(-1)$ and Gravitational Instantons

Any self-dual or anti-self-dual curvature 2-form gives a vanishing Ricci tensor, so any metric yielding a self-dual or anti-self-dual connection satisfies the Euclidean Einstein's field equations. There are also self-dual or anti-self-dual solutions of the Einstein's equations that have additional property that the metric approaches a flat metric at infinity. Solutions satisfying such property are called *asymptotically locally Euclidean* (ALE) metrics. This hints that a certain compactness of the base manifold is expected to ensure the existence of ALE metrics. Since Yang–Mills gauge potential approaches a pure gauge at infinity, ALE metrics also closely resemble the Yang–Mills instantons. For this reason, ALE metrics are also called *gravitational instantons*. (See for instance [4–6] for more details and some examples of gravitational instantons.)

The Euclidean Einstein's field equations for anti-self-dual gravitational fields reduce to a complex elliptic Monge–Ampére equation ([17, 18]). In [13], Nutku considered the following equation

$$(\partial\bar{\partial}u)^2 = C * \mathbf{1} \tag{49}$$

on a complex 2-manifold M, which is rather a simpler form of the complex Monge–Ampére equation given in [17] and [18]. Here u is the Kähler potential, $\partial, \bar{\partial}$ are the Dolbeault operators, $*\mathbf{1}$ is the normalized volume element of M, and C is a constant.[f]

In [13], the author studied a naive 2-dimensional reduction of (49) that the Kähler potential u depends only on two real coordinates t and x, where $\zeta_1 = t + iz$ and $\zeta_2 = x + iy$ are two complex local coordinates of M. So the resulting Kähler potential is translation-invariant in the z, y-directions when M is viewed as a 4-dimensional differentiable manifold. Since any complex manifold admits a Hermitian metric, let us denote by g a Hermitian metric of M. As is well-known a Hermitian metric on a complex manifold locally takes the form

$$g = g_{\mu\bar{\nu}} d\zeta^\mu \otimes d\bar{\zeta}^\nu + g_{\bar{\mu}\nu} d\bar{\zeta}^\mu \otimes d\zeta^\nu, \tag{50}$$

where $\mu, \nu = 1, 2$. Suppose that g is a Kähler metric, then locally $g_{\mu\bar{\nu}}$ is given by

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} u, \tag{51}$$

[f] The constant C is different from one in [13]. The way the equation is written in [13] results an extra negative sign.

where u is a function called the *Kähler potential* of the Kähler metric g. The 2-dimensional reduction $u = u(t, x)$ yields the Kähler metric[g]

$$g = u_{tt}(dt^2 + dz^2) + 2u_{tx}(dtdx + dydz) + u_{xx}(dx^2 + dy^2) \tag{52}$$

and the Kähler 2-form $\Omega = i\partial\bar{\partial}u$ of the metric g is given by

$$\Omega = u_{tt}dt \wedge dz + u_{tx}(dt \wedge dy + dx \wedge dz) + u_{xx}dx \wedge dy. \tag{53}$$

From this reduction, Eq. (49) yields the 2-dimensional Monge–Ampére equation

$$u_{tt}u_{xx} - u_{tx}^2 = \kappa, \tag{54}$$

where κ is a constant. The constant κ is determined by choosing an appropriate value of C, so that the Monge–Ampére equation (54) is elliptic if $\kappa > 0$ and hyperbolic if $\kappa < 0$. The Monge–Ampére equation (54) may be regarded as a 2-dimensional reduction of the Euclidean Einstein's field equations that govern anti-self-dual solutions.

Suppose that $\phi(t, x)$ is a solution to the equation

$$\phi_{tt}(1 + (\phi_x)^2) - 2\phi_t\phi_x\phi_{tx} + \phi_{xx}(\kappa + (\phi_t)^2) = 0. \tag{55}$$

Then the following equations hold:

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}\right) + \frac{\partial}{\partial x}\left(\frac{\kappa + \phi_t^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}\right) &= 0, \\ -\frac{\partial}{\partial t}\left(\frac{1 + \phi_x^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}\right) + \frac{\partial}{\partial x}\left(\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}\right) &= 0. \end{aligned} \tag{56}$$

This implies that there exists a solution $u(t, x)$ to the Monge–Ampére equation (54) such that

$$\begin{aligned} u_{tt} &= \frac{\kappa + (\phi_t)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}, \quad u_{tx} = \frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}, \\ u_{xx} &= \frac{1 + (\phi_x)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}. \end{aligned} \tag{57}$$

The transformation in (57) is a slight variation of what is discussed in Jörgens's 1954 paper [10]. Using this transformation we can write a metric for a class of

[g]With a scaling.

anti-self-dual gravitational instantons:

$$ds^2 = \frac{\kappa + (\phi_t)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}(dt^2 + dz^2) + 2\frac{\phi_t\phi_x}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}(dtdx + dydz)$$
$$+ \frac{1 + (\phi_x)^2}{\sqrt{\frac{\kappa(1+(\phi_x)^2)+(\phi_t)^2}{\kappa}}}(dx^2 + dy^2). \tag{58}$$

Equation (55) for $\kappa = 1$ is the well-known *minimal surface equation* for non-parametric minimal surfaces in Euclidean 3-space $\mathbb{E}^3$. (See for example [1, 15].) For $\kappa = -1$, the resulting equation is known to physicists as the *Born–Infeld equation*. Geometrically it is in fact the equation of timelike minimal surfaces in $\mathbb{E}^3_1$: The area functional of timelike surfaces $\phi(t, x)$ is given by $A = \int \sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}dt \wedge dx$ and the Euler–Lagrange equation for this action functional is then equivalent to the Born–Infeld equation. For $\kappa = -1$, (58) is written as

$$ds^2 = \frac{-1 + (\phi_t)^2}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}(dt^2 + dz^2)$$
$$+ 2\frac{\phi_t\phi_x}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}(dtdx + dydz) + \frac{1 + (\phi_x)^2}{\sqrt{-(\phi_t)^2 + (\phi_x)^2 + 1}}$$
$$\times (dx^2 + dy^2). \tag{59}$$

Hence we explicitly construct a class of anti-self-dual gravitational instantons described by the metric in (59) from non-parametric timelike minimal surfaces in $\mathbb{E}^3_1$. It is well-known that timelike minimal surfaces are bosonic string worldsheets. (For example, see [2]. A short account of the relationship between timelike minimal surfaces in Minkowski space and bosonic string worldsheets can be also found in the Appendix of [12].) This may hint that there is a correspondence between bosonic strings and a class of self-dual gravitational instantons. It may be worthwhile to investigate any physical meaning of such a correspondence. We leave it for future discussion.

Interestingly, we can also explicitly construct a class of anti-self-dual gravitational instantons described by the Kähler metric in (52) from flat Lorentz surfaces in $\mathbb{H}^3_1(-1)$. To see this, let $\varphi : \mathbb{D}(t, x) \to \mathbb{H}^3_1(-1)$ be a flat Lorentz surface with a pair of Lorentz holomorphic and Lorentz anti-holomorphic data (f, g) given by (33) as in Theorem 5. Then there exists a function $u(t, x)$ such that

$$u_{tt} = \ell,\ u_{tx} = \mathfrak{m},\ u_{xx} = \mathfrak{n},$$

and

$$u_{tt}u_{xx} - u_{tx}^2 = -1$$

as discussed in Sec. 2. The geometric quantities Q, R, H can be written in terms of $\ell, \mathfrak{m}, \mathfrak{n}$ as

$$Q = \frac{1}{4}(\ell + \mathfrak{n} + 2\mathfrak{m}), \quad R = \frac{1}{4}(\ell + \mathfrak{n} - 2\mathfrak{m}), \quad H = \frac{1}{2}(-\ell + \mathfrak{n}). \tag{60}$$

It follows from (33) and (60) that

$$u_{tt} = -\frac{(1+f)(1-g)}{1-fg}, \quad u_{tx} = -\frac{f-g}{1-fg}, \quad u_{xx} = \frac{(1-f)(1-g)}{1-fg}.$$

Therefore the Kähler metric (52) is written as

$$\begin{aligned} ds^2 = &-\frac{(1+f)(1-g)}{1-fg}(dt^2+dz^2) - 2\frac{f-g}{1-fg}(dtdx+dydz) \\ &+ \frac{(1-f)(1-g)}{1-fg}(dx^2+dy^2). \end{aligned} \tag{61}$$

Acknowledgment

The first author was partially supported by Kakenhi 24540062, 15K04834.

References

[1] J. Lucal, M. Barbosa and A. Gervasio Colares, *Minimal Surfaces* in $\mathbb{R}^3$, Lecture Notes in Mathematics, Vol. 1195 (Springer-Verlag, Berlin, 1986).
[2] L. Brink and M. Henneaux, *Principles of String Theory* (Plenum Press, New York, 1988).
[3] R. L. Bryant, Surfaces of mean curvature one in hyperbolic space, *Astérisque Soc. Math. France* **154–155** (1987) 321–347.
[4] T. Eguchi and A. J. Hanson, Asymptotically flat self-dual solutions to Euclidean gravity, *Phys. Lett. B* **74**(3) (1978) 249–251.
[5] T. Eguchi and A. J. Hanson, Self-dual solutions to Euclidean gravity, *Ann. Phys.* **120**(1) (1979) 82–106.
[6] T. Eguchi, P. B. Gilkey and A. J. Hanson, Gravitation, gauge theories and differential geometry, *Phys. Rep.* **66**(6) (1980) 213–393.
[7] C. Epstein, The hyperbolic Gauss map and quasiconformal reflections, *J. Reine Angew. Math.* **372** (1986) 96–135.
[8] J. A. Gálvez, A. Martínez and F. Milán, Flat surfaces in the hyperbolic 3-space, *Math. Ann.* **316**(3) (2000) 419–435.
[9] J. A. Gálvez, A. Martínez and F. Milán, Flat surfaces in $\mathbb{L}^4$, *Ann. Global Anal. Geom.* **20**(3) (2001) 243–251.
[10] K. Jörgens, Über die Lösungen der Differentialgleichung $rt - s^2 = 1$, *Math. Ann.* **127** (1954) 130–134.
[11] S. Lee, Timelike surfaces of constant mean curvature ± 1 in anti-de Sitter 3-Space $\mathbb{H}^3_1(-1)$, *Ann. Global Anal. Geom.* **29**(4) (2006) 355–401.
[12] S. Lee, Weierstrass type representation for timelike minimal surfaces in Minkowski 3-space, in *Proc. Midwest Geometry Conf.*, University of Oklahoma, May 5–7, 2006, Commun. Math. Anal., *Conf. 01* (2008), pp. 11–19 (Mathematical Research Publishers, Washington, DC).
[13] Y. Nutku, Gravitational instantons and minimal surfaces, *Phys. Rev. Lett.* **77**(23) (1996) 4702–4703.
[14] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Pure and Applied Mathematics, Vol. 130 (Academic Press, 1983).
[15] R. Osserman, *A Survey of Minimal Surfaces* (Dover Publications, 1986).

[16] T. Weinstein, *An Introduction to Lorentz Surfaces*, de Gruter Exposition in Mathematics, Vol. 22 (Walter de Gruyter, Berlin, 1996).
[17] S.-T. Yau, Calabi's conjecture and some new results in algebraic geometry, *Proc. Natl. Acad. Sci. USA* **74**(5) (1977) 1798–1799.
[18] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampére equation, I, *Comm. Pure Appl. Math.* **31** (1978) 339–411.