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COHOMOGENEITY ONE SPECIAL LAGRANGIAN 3-FOLDS IN THE DEFORMED AND THE RESOLVED CONIFOLDS

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ABSTRACT. In this paper, we describe the cohomogeneity one special Lagrangian 3-folds in both the deformed and the resolved conifolds. Our results give an explicit construction of the families of $SO(3)$ and T^2-invariant special Lagrangian submanifolds in these conifolds and describe their asymptotic behavior.

1. Introduction

Beginning with the seminal paper by Harvey and Lawson [HL] on calibrated geometry, there has been extensive research in the mathematics literature on special Lagrangian and other calibrated submanifolds. Recently, a lot of progress has been done in constructing special Lagrangian submanifolds using various techniques. To give some examples, Joyce used the method of ruled submanifolds and integrable systems in [J2], [J3] to construct explicit examples of special Lagrangian submanifolds in $\mathbb{C}^n$, Haskins exhibited examples of special Lagrangian cones in $\mathbb{C}^3$ [Ha], etc. Although the main emphasis has been on examples in $\mathbb{C}^n$, there has also been progress in studying the special Lagrangian submanifolds in nonflat Calabi–Yau manifolds. Schoen and Wolfson used variational methods to construct special Lagrangians in Calabi–Yau manifolds in [SW] and Goldstein constructed special Lagrangian torus fibrations on the Borcea–Voisin 3-fold and found a mirror to this fibration in [G2]. Anciaux found new $SO(n)$-invariant examples in T^*S^n [An], equipped with the Ricci-flat Stenzel metric. See also [IKM] and [KM], where a different method is used to construct calibrated submanifolds.

Special Lagrangian geometry in Calabi–Yau manifolds has become an important subject due to a phenomenon in physics known as *mirror symmetry.* In 1996, Strominger, Yau, and Zaslow [SYZ] conjectured that a compact Calabi–Yau 3-fold and its mirror should be foliated by special Lagrangian

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3-tori with possibly singular fibres and the fibrations are dual to each other. This conjecture proposes a way to construct the mirror of a compact Calabi–Yau manifold, by an appropriate compactification of the dual of the special Lagrangian fibration. One of the earliest examples of a pair of mirror Calabi–Yau metrics was found by Candelas and de la Ossa [CO] in 1990. The two manifolds arise from perturbing a singular cone on $S^2 \times S^3$ and are respectively known as the deformed and the resolved conifold. In fact, there is a 1-parameter family of Calabi–Yau metrics that passes through the singular metric and transforms the deformed conifold into the resolved conifold. The deformed conifold is a (trivial) $\mathbb{R}^3$-bundle over S^3 and the resolved conifold is a $\mathbb{C}^2$-bundle over S^2. In passing through the singularity the special Lagrangian, S^3 in the deformed conifold is pinched to a point and reappears in the resolved conifold as a holomorphic $\mathbb{C}P^1$. This is known as the *conifold transition* (see also [CO] and [STY]).

In this paper, we will find new examples of cohomogeneity one SL[1] submanifolds in both the deformed and the resolved conifolds. The main result is that we exhibit an explicit foliation of both these Calabi–Yau manifolds by special Lagrangian 3-folds, where the generic leaf is $T^2 \times \mathbb{R}$ and the T^2 is an orbit under the maximal torus of the isometry group $SO(4)$. We show that asymptotically these special Lagrangian submanifolds approach a special Lagrangian cone in the conifold. The conifold is topologically a cone on $S^2 \times S^3$, with the singular Calabi–Yau metric [CO]. We will also recover some of Anciaux's results about $SO(n)$-invariant special Lagrangian submanifolds in T^*S^n using different techniques. Our method uses the moment map of the $SO(4)$ action and is similar to Harvey–Lawson method of finding cohomogenity one special Lagrangian folds [HL] for the flat case. Also, in [G1] Goldstein proved the general result that a Calabi–Yau manifold with a cohomogeneity one torus action and with $H^1(M,\mathbb{R}) = 0$ is fibred by special Lagrangian tori. See also [Ch] for examples in the canonical bundle of $\mathbb{C}P^{n-1}$ of the same flavor.

The Calabi–Yau metric on T^*S^3, the deformed conifold, was first explicitly described by Candelas and de la Ossa [CO] in 1990, before Stenzel (independently) discovered it for any T^*S^n, and more generally, for any cotangent bundles of rank one symmetric spaces. The case T^*S^2 is already very interesting and the Ricci-flat hyper-Kähler metric on this manifold was first discovered by Eguchi and Hanson [EH] in 1978. This 4-dimensional metric is the basic model of a number of explicit special holonomy metrics discovered by physicists.

We now give a brief description of the contents of this paper. In Section 2, we review the basic facts about Calabi–Yau manifolds and special Lagrangian submanifolds. In Section 3, we describe the Calabi–Yau structure on the deformed conifold and the Stenzel metric on T^*S^n. We define the associated

[1] SL stands for Special Lagrangian.

moment map and discuss some of its basic properties in Section 4. In Section 5, we prove that the only homogenous example of special Lagrangian 3-fold in T^*S^3 is the zero section and we compute our main examples of cohomogeneity one special Lagrangian folds in T^*S^3, including a description of the asymptotics. Section 6 is devoted to the study of cohomogeneity one special Lagrangian 3-folds in the resolved conifold.

2. Special Lagrangian Geometry

Special Lagrangian submanifolds are a special class of minimal submanifolds in Calabi–Yau spaces and were introduced by Harvey and Lawson in their seminal paper [HL] using the notion of a calibration. We begin by briefly reviewing the basic definitions and setting up notations. For details, see [HL] and [J1].

Definition 2.1. A Calabi–Yau n-fold (M, J, ω, Ω) is a Kähler n-dimensional manifold (M, J, ω) with a Ricci-flat Kähler metric g and a nonzero holomorphic section Ω which trivializes the canonical bundle K_M.

Since the metric g is Ricci-flat, Ω is a parallel tensor with respect to the Levi–Civita connection ∇^g. By rescaling Ω, we can take it to be the holomorphic $(n, 0)$-form that satisfies:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}, \tag{2.1}$$

where ω is the Kähler form of g. The form Ω is called the *holomorphic volume form* of the Calabi–Yau manifold M.

Let φ be a closed p-form on the manifold M. We say φ is a calibrating form on M if

$$\varphi|_V \leq vol_V$$

for any oriented p-plane $V \subset T_xM, \forall x \in M$. A submanifold N of M is called *calibrated by* φ if $\varphi|_{T_xN} = vol_{T_xN}, \forall x \in N$.

Remark. The constant factor in (2.1) is chosen so that $\mathrm{Re}\,\Omega$ becomes a calibration on M.

Definition 2.2. Let (M, J, ω, Ω) be a Calabi–Yau n-fold and $L \subset M$ a real oriented n-dimensional submanifold of M. Then L is called a *special Lagrangian submanifold* of M if it is calibrated by $\mathrm{Re}\,\Omega$.

The following gives an alternative description of the special Lagrangian submanifolds [HL].

Proposition 2.3. *Let (M, J, ω, Ω) be an n-dimensional Calabi–Yau manifold and $L \subset M$ a real n-dimensional submanifold of M. Then L is called a special Lagrangian submanifold if $\omega|_L \equiv 0$ and $\mathrm{Im}\,\Omega|_L \equiv 0$.*

Remark. The condition $\omega|_L \equiv 0$ says that L is a Lagrangian submanifold. Therefore, special Lagrangian submanifolds are Lagrangian with the extra condition $\operatorname{Im}\Omega|_L \equiv 0$.

The simplest example of a Calabi–Yau manifold is $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$, endowed with the flat metric g, the Kähler form ω_0 and the holomorphic volume form Ω_0, where:

$$g(z,w) = \operatorname{Re}\sum_{i=1}^{n} z_i \bar{w}_i, \tag{2.2}$$

$$\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n), \tag{2.3}$$

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_n, \tag{2.4}$$

$\mathbb{R}^n$ is a trivial example of a special Lagrangian submanifold in $\mathbb{C}^n$.

One important property of the special Lagrangian submanifolds is that they are absolutely area minimizing in their homology class, so they are in particular *minimal* submanifolds [HL]. Also, the graph of a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is special Lagrangian in $\mathbb{R}^{2n}$ if and only if $F = \nabla f$, for some function $f : \mathbb{R}^n \to \mathbb{R}$ that satisfies the differential equation:

$$\operatorname{Im} det_{\mathbb{C}}(I + i \operatorname{Hess} f) = 0 \quad \text{on } \mathbb{C}^n.$$

This is a nonlinear elliptic P.D.E. and it is difficult to solve in general. One idea, initiated by Harvey and Lawson, is to look for solutions invariant under certain group actions of $\mathbb{C}^n$. In [HL], Harvey and Lawson produced many examples of symmetric special Lagrangian submanifolds in $\mathbb{C}^n$. More precisely, they described the $SO(n)$ and T^n-invariant special Lagrangian in $\mathbb{C}^n$, where T^n is the maximal torus of $SU(n)$.

3. The deformed conifold

In 1995, Stenzel [St] showed that the cotangent bundle of the sphere can be endowed with a Ricci-flat metric. As mentioned in the Introduction, the lower dimensional cases were discovered in the physics literature. The case $n = 2$ is the Eguchi–Hanson metric [EH] and the case $n = 3$ (the deformed conifold) is due to Candelas and de la Ossa [CO].

Following Szöke [Sz], we will describe the cotangent bundle of the sphere as a complex affine quadric and define the Kähler potential of the Stenzel metric as in [St].

3.1. The Stenzel metric on T^*S^n. Let $T^*S^n = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, |x| = 1, x \cdot \xi = 0\}$ be the cotangent bundle of the n-sphere (which we have identified with the tangent bundle). The group $SO(n+1,\mathbb{R})$ acts with cohomogeneity one on T^*S^n, the generic orbit being $SO(n+1)/SO(n-1)$ (where

$|\xi|$ is constant). According to Szöke [Sz], one can identify T^*S^n with the affine quadric

$$Q^n = \left\{ z = (z_0, z_1, \dots, z_n) \in \mathbb{C}^{n+1} \Big| \sum_{i=0}^{n} z_i^2 = 1 \right\}$$

using the diffeomorphism $h : T^*S^n \to Q^n$ given by:

$$(x, \xi) \to z = x\cosh(|\xi|) + i\frac{\sinh(|\xi|)}{|\xi|}\xi. \tag{3.1}$$

This diffeomorphism is equivariant with respect to the action of $SO(n+1, \mathbb{R})$ on T^*S^n and the natural action of $SO(n+1, \mathbb{C})$ on Q^n. The complex structure on the cotangent bundle of the n-sphere is obtained by pulling back the complex structure of the affine quadric under the map h. On the complex quadric, there exists a Ricci-flat metric whose corresponding symplectic form is the Stenzel form given by:

$$\omega_{St} = i\,\partial\bar{\partial}u(r^2) = i\sum_{j=0}^{n}\sum_{k=0}^{n} \frac{\partial^2}{\partial z_j\,\partial\bar{z}_k} u(r^2)\, dz_j \wedge d\bar{z}_k,$$

where $r^2 = |z|^2 = \sum_{j=0}^{n} z_j\bar{z}_j = \cosh(2|\xi|)$ and $u(r^2)$ is a smooth real function satisfying the following differential equation:

$$\frac{d}{d\tau}(u'(\tau))^n = cn(\sinh\tau)^{n-1}, \tag{3.2}$$

where $\tau = \cosh^{-1}(r^2)$ and c is a positive constant (see [St]). In dimension $n = 2$, there is an explicit formula for the potential function: $u(r^2) = \sqrt{1 + r^2}$. In dimension $n = 3$, the derivative of the potential function is given by the relation:

$$u'(\tau)^3 = \frac{3c}{2}\left(\frac{\sinh(2\tau)}{2} - \tau\right), \tag{3.3}$$

which using the initial condition $u'(0) = 0$ integrates to:

$$u(r^2) = \int_0^{\cosh^{-1}(r^2)} \left[\frac{3c}{2}\left(\frac{\sinh(2\sigma) - \sigma}{2}\right)\right]^{\frac{1}{3}} d\sigma. \tag{3.4}$$

The form $\omega_{St} = i\,\partial\bar{\partial}u$ is exact on T^*S^n and $\omega_{St} = d\alpha_{St}$, where $\alpha_{St} = -\operatorname{Im}(\bar{\partial}u)$. α_{St} is related to the Liouville form $\alpha_0(v) = \frac{1}{2}\langle v, Jz\rangle$ on $\mathbb{C}^{n+1}$ by $\alpha_{St} = u'(|z|^2)\alpha_0$. Therefore, it follows that the 1-form α_{St} has the expression:

$$\alpha_{St}(v) = \frac{1}{2}u'(|z|^2)\omega_0(z, v) = \frac{1}{2}u'(|z|^2)\langle v, Jz\rangle, \quad v \in T_zQ, z \in Q, \tag{3.5}$$

where $\langle\cdot,\cdot\rangle$ and ω_0 are respectively, the flat Euclidean metric and the Kähler form of $\mathbb{C}^{n+1}$. It is well known that on the complex space $\mathbb{C}^{n+1}$, $\omega_0(v, w) = \langle Jv, w\rangle$, where J is the complex structure on $\mathbb{C}^{n+1}$.

Differentiating expression (3.5), we calculate:

$$\begin{aligned}\omega_{St}(v,w) &= d\alpha_{St}(v,w)\\ &= v(\alpha_{St}(w)) - w(\alpha_{St}(v)) - \alpha_{St}([v,w])\\ &= v\left(\frac{1}{2}u'(|z|^2)\langle w, Jz\rangle\right) - w\left(\frac{1}{2}u'(|z|^2)\langle v, Jz\rangle\right) - \alpha_{St}([v,w])\\ &= \frac{1}{2}\{u''(|z|^2)v(|z|^2)\langle w, Jz\rangle + u'(|z|^2)v(\langle w, Jz\rangle)\\ &\quad - u''(|z|^2)w(|z|^2)\langle v, Jz\rangle - u'(|z|^2)w(\langle v, Jz\rangle)\\ &\quad - u'(|z|^2)\langle [v,w], Jz\rangle\}\\ &= u'(|z|^2)\omega_0(v,w) + u''(|z|^2)\big(\langle v,z\rangle\omega_0(z,w) - \langle w,z\rangle\omega_0(z,v)\big).\end{aligned}$$

In the above calculation, we used that $v(|z|^2) = 2\langle v, z\rangle$, $\nabla_v w - \nabla_w v = [v,w]$ and the fact that $\langle v, Jw\rangle = -\langle w, Jv\rangle$.

Therefore, the Kähler form of the Stenzel metric at a point z on the quadric Q is given by:

$$\begin{aligned}\omega_{St}(v,w) &= u'(|z|^2)\omega_0(v,w)\\ &\qquad + u''(|z|^2)\big(\langle w,z\rangle\omega_0(v,z) - \langle v,z\rangle\omega_0(w,z)\big),\\ &v,w \in T_zQ.\end{aligned} \tag{3.6}$$

Formulas (3.5) and (3.6) will prove to be useful when we compute the moment maps for group actions on the quadric.

On the quadric Q define the holomorphic $(n,0)$-form Ω_{St} by the relation:

$$\Omega_{St}(v_1, v_2, \ldots, v_n) = \Omega_0(z, v_1, \ldots, v_n), \quad v_1, \ldots, v_n \in T_zQ, z \in Q, \tag{3.7}$$

where $\Omega_0 = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n$ is the holomorphic volume form of $\mathbb{C}^{n+1}$.

The quadric Q^n becomes a Calabi–Yau manifold since equation (2.1) holds for ω_{St} and the corresponding holomorphic n-form Ω_{St}, up to a multiplicative constant (see also [An]).

3.2. The conifold in Dimension 3. Let Q_0 be the quadric in $\mathbb{C}^4$ defined by the equation:

$$\sum_{i=0}^{3} z_i^2 = 0.$$

This quadric, called the conifold, is singular at the origin and represents a cone on $T_1(S^3) \cong S^2 \times S^3$. The complex structure of the conifold is given by the embedding $h_0 : T_1(S^3) = \{(x,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4 | |x| = |\xi| = 1, x \cdot \xi = 0\} \to Q_0$, $h_0(x,\xi) = x + i\xi = z$. Deforming the conifold equation to $\sum_{i=0}^{3} z_i^2 = \epsilon^2$, with ϵ a positive constant, yields the complex quadric Q_ϵ in Dimension 3, where ϵ is the radius of the zero section S^3. This is equivalent to replacing the tip of the conifold by an S^3 (see [CO]) and as $\epsilon \to 0$, this sphere collapses into the singular point of Q_0. In the physical literature, the complex quadric Q_ϵ is also known as a *deformed conifold*.

Candelas and de la Ossa [CO] showed that the conifold Q_0 admits a Ricci-flat metric g_{cone} with Kähler potential $u_{cone}(r^2) = \frac{3}{2} r^{\frac{4}{3}}$. The holomorphic $(n,0)$-form Ω_{cone} defined by:

$$\frac{1}{2} d(z_0^2 + z_1^2 + \cdots + z_3^2) \wedge \Omega_{cone} = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_3 \tag{3.8}$$

makes the conifold into a singular Calabi–Yau manifold.

We note that the above relation can be used to compute Ω_{cone} as:

$$\Omega_{cone}(v_1, v_2, v_3) = \frac{1}{|z|^2}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(\bar{z}, v_1, v_2, v_3), \tag{3.9}$$

where $v_1, v_2, v_3 \in T_z Q_0$, $z \in Q_0$ and $\bar{z} = \bar{z_0}\frac{\partial}{\partial z_0} + \bar{z_1}\frac{\partial}{\partial z_1} + \bar{z_2}\frac{\partial}{\partial z_2} + \bar{z_3}\frac{\partial}{\partial z_3}$.

4. Moment maps and special Lagrangians with symmetry

The group $SO(n+1,\mathbb{R})$ acts with cohomogeneity one on the cotangent bundle of the sphere

$$T^*S^n = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, |x| = 1, x \cdot \xi = 0\}. \tag{4.1}$$

The action is transitive on the sets $|\xi| = \rho = $ constant and is given by:

$$g.(x,\xi) = (gx, g\xi), \quad g \in SO(n+1), (x,\xi) \in T^*S^n. \tag{4.2}$$

In what follows, we use similar techniques as in [HL], [J1], [G1] to find explicit examples of G-invariant special Lagrangian 3-folds in T^*S^3, where G is an appropriate subgroup of the isometry group $SO(4)$.

Let $Q \cong T^*S^n$ be endowed with the Calabi–Yau structure described in Section 3.1. As we have seen, the group of automorphisms of T^*S^n preserving the Calabi–Yau structure is $SO(n+1,\mathbb{R}) \subset SO(n+1,\mathbb{C})$. Let G be a Lie subgroup of $SO(n+1,\mathbb{R})$, with Lie algebra $\mathfrak{g}$. Let $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$. Then the induced vector field on Q is given by: $z \mapsto X_A(z) = Az$ with flow $z \mapsto e^{tA}z$ for $z \in Q$. Using well-known techniques (see [MS]), we compute the moment map $\mu : T^*S^n \to \mathfrak{g}^*$ of the G-action.

Proposition 4.1. *Let $G \subset SO(n+1)$ be a connected Lie group. The moment map of the G-action on the complex quadric Q is given by:*

$$\mu_A : Q^n \to \mathbb{R}, \qquad \mu_A(z) = \alpha_{St}(Az) = \frac{1}{2} u'(|z|^2)\langle Az, Jz\rangle, \quad \forall A \in \mathfrak{g}, \tag{4.3}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean metric on $\mathbb{C}^{n+1}$ and the function u satisfies equation (3.2).

Proof. The forms ω_{St} and α_{St} are invariant under the flow of X_A, that is $\mathcal{L}_{X_A}\omega_{St} = \mathcal{L}_{X_A}\alpha_{St} = 0$, for any $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$.

From Cartan's formula, $\mathcal{L}_{X_A}\alpha_{St} = d(X_A \lrcorner \alpha_{St}) + X_A \lrcorner d\alpha_{St}$ and using $d\alpha_{St} = \omega_{St}$, we see that

$$-X_A \lrcorner \omega_{St} = d(\alpha_{St}(X_A)).$$

This shows that the action of G on T^*S^n is Hamiltonian, with moment map $\mu(z)$ given by $A \mapsto \alpha_{St}(Az)$ which can be rewritten as:

$$\langle \mu(z), A\rangle = \mu_A(z) = \alpha_{St}(X_A(z)),$$

where $z \in Q \subset \mathbb{C}^{n+1}$, $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$ and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Using formula (3.5) for α_{St}, the conclusion follows. □

We define the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ to be the subspace of $\mathfrak{g}$ fixed by the coadjoint action of G. Since the moment map is equivariant, a level set of the moment map $\mu^{-1}(c)$ for $c \in \mathfrak{g}^*$ is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$. In order to find examples of SL submanifolds in Q, we will use the following special case of a more general result due to Goldstein [G1] and Gross [Gr].

Proposition 4.2. *Let $G \subset SO(n+1)$ be a connected Lie subgroup with Lie algebra $\mathfrak{g}$ and moment map $\mu : T^*S^n \to \mathfrak{g}^*$ and $\mathcal{O}$ an orbit of G in T^*S^n. Then the orbit is isotropic, i.e., $\omega_{St}|_{\mathcal{O}} \equiv 0$ if and only if $\mathcal{O} \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

Remark. The result ensures that all the isotropic G-orbits are contained in the level sets of the moment map. A simple calculation shows that a 3-dimensional G-invariant extension L^3 of an isotropic orbit is a Lagrangian submanifold if and only if L lies in the level set of the moment map. Since SL submanifolds are in particular Lagrangian, we have the following corollary.

Corollary 4.3. *If L is a connected special Lagrangian submanifold in T^*S^n with symmetry group $G \subseteq SO(n+1)$, then $L \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$, where $\mu : T^*S^n \to \mathfrak{g}^*$ is the moment map of the action of G.*

5. Special Lagrangian submanifolds of T^*S^3

5.1. Homogenous special Lagrangian 3-folds. We will start by looking at homogenous SL 3-folds, i.e., those invariant under subgroups of $SO(4)$ that act on T^*S^3 with at least one orbit of Dimension 3. Let U be the group of unit quaternions and Φ the $2:1$ homomorphism $\Phi : U \times U \to SO(4)$ given by $\Phi(u_1, u_2)(x) = u_1 x \bar{u}_2$. It is easy to see that $\mathfrak{so}(4) \cong \mathfrak{so}(3)_1 \oplus \mathfrak{so}(3)_2$, where $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_2$ are two different copies of $\mathfrak{so}(3)$ whose intersection is the zero vector. By looking at the subgroups of $SO(4)$ of Dimension ≥ 3 (see also [Io]), one can see that the only connected subgroups of $SO(4)$ that act on Q with generic orbits of Dimension 3 are:

1. The full group $SO(4)$. The generic orbit of the action is an $S^2 \times S^3$, but the zero section is an orbit of Dimension 3.
2. The two nonconjugate $SU(2)$ subgroups in $SO(4)$, with Lie algebras $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_2$, respectively.

3. The subgroup $U(2)$, with Lie algebra $\mathfrak{so}(3)_1 \oplus \mathfrak{so}(2)_1$ (or $\mathfrak{so}(2)_1 \oplus \mathfrak{so}(3)_2$), whose infinitesimal generators are given by:

$$\left\{ \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}. \tag{5.1}$$

   This group acts with generic orbit of Dimension 4.
4. The subgroup $SO(3)$ acting trivially on $(1,0,0,0)$ with generic orbit of Dimension 3.

Proposition 5.1. *Every homogeneous special Lagrangian 3-fold in T^*S^3 is conjugate under the action of $SO(4)$ to the zero section $S^3 \subset T^*S^3$.*

Proof. Let L be a homogeneous special Lagrangian submanifold and $G \subset SO(4)$ its symmetry group. Then G is one of the subgroups of $SO(4)$ described above. We will sketch the proof for $G = SO(4)$. Similar arguments work for the rest of the subgroups.

From Equation (4.3), the moment map of the $SO(4)$-action is given by $\mu : Q \to \mathfrak{so}(4)^*$ with: $\mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_0\bar{z_1}), \mathrm{Im}(z_1\bar{z_2}), \mathrm{Im}(z_2\bar{z_3}), \mathrm{Im}(z_3\bar{z_0}), \mathrm{Im}(z_1\bar{z_3}), \mathrm{Im}(z_2\bar{z_0}))$.

Since $Z(\mathfrak{so}(4)^*) = \{0\}$, it follows from Corollary 4.3 that any $SO(4)$-invariant special Lagrangian 3-fold in Q^3 lies in the level set $\mu^{-1}(0)$. Applying an appropriate rotation by an element of $SO(4)$, one can assume that $x = (x_0, x_1, x_2, x_3) = (\cot t, \sin t, 0, 0), t \in [0, \pi)$. Now, since the special Lagrangian has to be in the zero level set of the moment map above, we have

$$\mathrm{Im}(z_0\bar{z_1}) = \mathrm{Im}(z_1\bar{z_2}) = \mathrm{Im}(z_2\bar{z_3}) = \mathrm{Im}(z_3\bar{z_0}) = \mathrm{Im}(z_1\bar{z_3}) = \mathrm{Im}(z_2\bar{z_0}).$$

Using the diffeomorphism (3.1) between the complex quadric Q^3 and T^*S^3, we see that ξ has to be of the form $\xi = \rho(-\sin t, \cos t, 0, 0)$, where $\rho = |\xi|$ and it has to also satisfy $\mathrm{Im}(z_0\bar{z_1}) = 0$, i.e., $\rho = 0$. Therefore, L is the zero section of the cotangent bundle. $\square$

The next most symmetric case is when the symmetry group of the special Lagrangian submanifold acts with cohomogeneity one. In this case, the differential equation of a special Lagrangian simplifies and we can find examples by solving an O.D.E. The idea is to find subgroups G of $SO(4)$ which has orbits of Dimension 2 in T^*S^3. In order to have orbits of Dimension 2, one must have that $\dim G \geq 2$. Then, the strategy is to choose an isotropic 2-dimensional orbit of the group action and then to find an extra direction in the level set of the moment map such that the resulting submanifold is special Lagrangian.

The subgroups of $SO(4)$ that act on Q with orbits of Dimension 2 are the maximal torus T^2 and the subgroup $SO(3)$ which leaves a direction invariant.

5.2. T^2-invariant special Lagrangian in T^*S^3. Let G be the maximal torus T^2 of $SO(4)$, described as:

$$\left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}, \theta_1, \theta_2 \in [0, 2\pi) \right\}.$$

The following result describes the family of T^2-invariant special Lagrangian 3-folds of T^*S^3.

THEOREM 5.2. *The special Lagrangian submanifolds in $T^*S^3 = Q = \{z \in \mathbb{C}^4 | \sum_{i=0}^{3} z_i^2 = 1\}$, which are invariant under the action of the maximal torus T^2 of $SO(4)$ are given by:*

$$\begin{aligned} u'(|z|^2)\,\mathrm{Im}(z_0\bar{z_1}) &= c_1, \\ u'(|z|^2)\,\mathrm{Im}(z_2\bar{z_3}) &= c_2, \\ \mathrm{Im}(z_0^2 + z_1^2) &= c_3, \end{aligned} \tag{5.2}$$

where u is given by (3.4) *and c_1, c_2, and c_3 are any real constants.*

Proof. Using the two infinitesimal generators of the T^2-action:

$$B_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and the expression (4.3), the moment map for the T^2-action on Q^3 is given by:

$$\mu : Q^3 \to (\mathfrak{t}^2)^*, \qquad \mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_0\bar{z_1}), \mathrm{Im}(z_2\bar{z_3})).$$

Since $(\mathfrak{t}^2)^* = \mathbb{R}^2$, it follows from Corollary 4.3 that any T^2-invariant special Lagrangian 3-fold L in Q^3 lies in a level set $\mu^{-1}(c)$, where $c = (c_1, c_2) \in \mathbb{R}^2$. We choose an orbit $\mathcal{O}$ lying in the level set of the moment map, hence isotropic. Any extension that lies in the level set of the moment map is Lagrangian. This can also be checked using directly formula (3.6). We want to find the direction in which the extension will be special Lagrangian. This is done by imposing the special Lagrangian condition $\mathrm{Im}\,\Omega_{St}|_L = 0$ at a given point z. We compute Ω_{St} on the three tangent vectors $Y_1 = B_1 z, Y_2 = B_2 z$, $Y_3 = \dot{z}$ and get:

$$\begin{aligned} \Omega_{St}(Y_1, Y_2, Y_3) &= (dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(z, Y_1, Y_2, Y_3) \\ &= \begin{vmatrix} z_0 & -z_1 & 0 & \dot{z_0} \\ z_1 & z_0 & 0 & \dot{z_1} \\ z_2 & 0 & -z_3 & \dot{z_2} \\ z_3 & 0 & z_2 & \dot{z_3} \end{vmatrix} \\ &= (z_0^2 + z_1^2)(z_2\dot{z_2} + z_3\dot{z_3}) - (z_2^2 + z_3^2)(z_0\dot{z_0} + z_1\dot{z_1}). \end{aligned}$$

Now using $z_0^2+z_1^2+z_2^2+z_3^2=1, \mathrm{Im}(z_0^2+z_1^2)=-\mathrm{Im}(z_2^2+z_3^2)$ and $\mathrm{Im}(z_2\dot{z}_2+z_3\dot{z}_3)=-\mathrm{Im}(z_0\dot{z}_0+z_1\dot{z}_1)$, we finally obtain $\mathrm{Im}\,\Omega_{St}(Y_1,Y_2,Y_3)=\mathrm{Im}(z_0\dot{z}_0+z_1\dot{z}_1)$, hence the conclusion follows. □

Remark. Since equations (5.2) are linearly independent, the above family foliates T^*S^3. The generic orbit is two disconnected copies of $T^2\times\mathbb{R}$ and the zero section is obtained by setting $c_1=c_2=c_3=0$.

Asymptotic behavior. We now study the asymptotic behavior of this family of SL 3-folds, i.e., the limiting behavior of the family as $\rho=|\xi|\to\infty$. For this, we will rewrite equations (5.2) in terms of x and ξ.

Note that one can also view the T^2-invariant SL 3-folds constructed above as being obtained by rotating a curve in T^*S^3 by the torus action. To see this, let $\gamma(t)=(x(t),\xi(t))\in T^*S^3$ be a curve in the complex quadric. By applying an appropriate rotation with an element of T^2, we can assume that

$$x(t)=\begin{pmatrix}\cos t\\0\\\sin t\\0\end{pmatrix},\quad t\in[0,\pi)$$

(since $|x|=1$). Denote the length of the vector

$$\xi=\begin{pmatrix}\xi_0\\\xi_1\\\xi_2\\\xi_3\end{pmatrix}\quad \text{by } \rho=|\xi|\geq 0.$$

Let $\rho_0=\xi_0^2+\xi_2^2$ and $\rho_1=\xi_1^2+\xi_3^2$. Since $\rho_0^2+\rho_1^2=\rho^2$, we let:

$$\begin{aligned}\rho_0&=\rho\cos\varphi,\\ \rho_1&=\rho\sin\varphi.\end{aligned}$$

Since $x\cdot\xi=0$, we can parameterize the vector as

$$\xi(t)=\begin{pmatrix}-\rho_0\sin t\\\rho_1\cos\psi\\\rho_0\cos t\\\rho_1\sin\psi\end{pmatrix}=\rho\begin{pmatrix}-\cos\varphi\sin t\\\sin\varphi\cos\psi\\\cos\varphi\cos t\\\sin\varphi\sin\psi\end{pmatrix}.$$

Using the diffeomorphism h given by relation (3.1), one gets that any point on the quadric is conjugate under the T^2-action to a point of the form:

$$z=\begin{pmatrix}\cos t\cosh\rho-i\sinh\rho\cos\varphi\sin t\\ i\sinh\rho\sin\varphi\cos\psi\\ \sin t\cosh\rho+i\sinh\rho\cos\varphi\cos t\\ i\sinh\rho\sin\varphi\sin\psi\end{pmatrix}\in Q.$$

In fact the whole quadric Q^6 can be parametrized as:

$$\begin{pmatrix} \cos\theta_1\cos t\cosh\rho - i\sinh\rho(\cos\theta_1\cos\varphi\sin t + \sin\theta_1\sin\varphi\cos\psi) \\ \sin\theta_1\cos t\cosh\rho + i\sinh\rho(\cos\theta_1\sin\varphi\cos\psi - \sin\theta_1\cos\varphi\sin t) \\ \cos\theta_2\sin t\cosh\rho + i\sinh\rho(\cos\theta_2\cos\varphi\cos t - \sin\theta_2\sin\varphi\sin\psi) \\ \sin\theta_2\sin t\cosh\rho + i\sinh\rho(\sin\theta_2\cos\varphi\cos t + \cos\theta_2\sin\varphi\sin\psi) \end{pmatrix},$$

where $t, \theta_1, \theta_2, \varphi, \psi \in S^1$ and $\rho \geq 0$.

Equations (5.2) become:

$$\begin{aligned} u'(\cosh(2\rho))\sinh(2\rho)\cos t\sin\varphi\cos\psi &= c_1, \\ u'(\cosh(2\rho))\sinh(2\rho)\sin t\sin\varphi\sin\psi &= c_2, \\ \sinh(2\rho)\sin(2t)\cos\varphi &= c_3. \end{aligned} \tag{5.3}$$

These equations describe a curve in the parameter space (ρ, t, φ, ψ) which under the T^2-action on Q gives the family of special Lagrangian 3-folds L_c.

As we have seen previously, T^*S^3 approaches the conifold Q_0 asymptotically as $|\xi| \to \infty$.

Notice that as $\rho \to \infty$, $u'(\cosh(2\rho)) \to \infty$ from relation (3.3), so in the limit, equations (5.3) become one of the following cases:

$$\begin{array}{lll} \text{(a)} & \sin\varphi = 0, & \sin t = 0, \\ \text{(b)} & \sin\varphi = 0, & \cos t = 0, \\ \text{(c)} & \cos t = 0, & \sin\psi = 0, \\ \text{(d)} & \cos\psi = 0, & \sin t = 0. \end{array} \tag{5.4}$$

We will study each case separately.

(a) $\sin\varphi = 0, \sin t = 0$. $z = (\cosh\rho, 0, i\sinh\rho, 0)$ and its unit vector is:

$$\frac{z}{|z|} = \frac{1}{\sqrt{\cosh(2\rho)}}(\cosh\rho, 0, i\sinh\rho, 0) \to \frac{1}{\sqrt{2}}(1, 0, i, 0) \in Q_0 \quad \text{as } \rho \to \infty.$$

Applying the T^2-action in the limit, one gets a surface Σ_1 diffeomorphic to T^2:

$$\Sigma_1 = \left\{ \frac{1}{\sqrt{2}}(\cos\theta_1, \sin\theta_1, i\cos\theta_2, i\sin\theta_2), \theta_1, \theta_2 \in [0, 2\pi) \right\} \subset Q_0.$$

We will show that the cone on $\Sigma_1, C(\Sigma_1) = \{sz | z \in \Sigma_1, s \in \mathbb{R}\}$, is special Lagrangian in the conifold Q_0, endowed with the Ricci-flat metric found by Candelas and de la Ossa [CO].

We first show that the cone $C(\Sigma_1)$ is Lagrangian, i.e., $\omega_{cone}|_{C(\Sigma_1)} = 0$. The moment map of the T^2-action on the cone is:

$$\mu_0 : Q_0 \to \mathbb{R}^2, \qquad \mu_0(z) = u'_{cone}(|z|^2)(\operatorname{Im}(z_0\bar{z_1}), \operatorname{Im}(z_2\bar{z_3}))$$

where $u_{cone}(r^2)$ is the potential function for the conifold given in Section 3.2. Since the cone on Σ_1 is seen to lie in $\mu_0^{-1}(0, 0)$, it follows from Proposition 4.2 that it is Lagrangian.

Next, we show that the cone is special Lagrangian, i.e., $\operatorname{Im}\Omega_{cone}|_{C(\Sigma_1)} \equiv 0$. For this, we compute Ω_{cone} on three tangent vectors Y_1, Y_2, Y_3 to the cone $C(\Sigma_1)$. One of them is the position vector and the other two vectors are the derivatives with respect to the parameters θ_1 and θ_2. The unit vector normal to the cone is given by

$$w = \bar{z} = \frac{1}{\sqrt{2}}(\cos\theta_1, \sin\theta_1, -i\cos\theta_2, -i\sin\theta_2)$$

and using formula (3.9), we get:

$$\begin{aligned}\operatorname{Im}\Omega_{cone}(Y_1,Y_2,Y_3) &= \operatorname{Im}(dz_0\wedge dz_1\wedge dz_2\wedge dz_3)(\bar{z},Y_1,Y_2,Y_3)\\ &= \operatorname{Im}\frac{1}{s^2}\begin{vmatrix} \frac{\cos\theta_1}{\sqrt{2}} & \frac{\cos\theta_1}{\sqrt{2}} & -s\frac{\sin\theta_1}{\sqrt{2}} & 0\\ \frac{\sin\theta_1}{\sqrt{2}} & \frac{\sin\theta_1}{\sqrt{2}} & s\frac{\cos\theta_1}{\sqrt{2}} & 0\\ -\frac{i\cos\theta_2}{\sqrt{2}} & \frac{i\cos\theta_2}{\sqrt{2}} & 0 & -s\frac{i\sin\theta_2}{\sqrt{2}}\\ -\frac{i\sin\theta_2}{\sqrt{2}} & \frac{i\sin\theta_2}{\sqrt{2}} & 0 & s\frac{i\cos\theta_2}{\sqrt{2}} \end{vmatrix} = 0.\end{aligned}$$

Therefore, the cone on Σ_1 is special Lagrangian.

(b) $\sin\varphi = 0, \cos t = 0$. In this case, in the limit as $\rho\to\infty$, the unit vector $\frac{z}{|z|}$ goes to: $\frac{1}{\sqrt{2}}(-i,0,1,0)\in Q_0$. Applying the T^2-action in the limit, one gets a surface Σ_2 in Q_0, diffeomorphic to T^2:

$$\Sigma_2 = \left\{\frac{1}{\sqrt{2}}(-i\cos\theta_1, -i\sin\theta_1, \cos\theta_2, \sin\theta_2), \theta_1,\theta_2\in[0,2\pi)\right\}\subset Q_0.$$

The unit vector $\frac{z}{|z|}$ in this case is the unit vector from case (a) rotated by J and so the limiting cone $C(\Sigma_2)$ is the special Lagrangian cone in (a) rotated by J.

(c) $\cos t = 0, \sin\psi = 0$. Similar computations as above yield the special Lagrangian cone in case (b).

(d) $\cos\psi = 0, \sin t = 0$. This case yields the special Lagrangian cone in (a).

Thus, we have showed the following result.

Proposition 5.3. *For generic $c\in\mathbb{R}^3$, the T^2-invariant special Lagrangian L_c in the complex quadric has two components, each of them asymptotic to a special Lagrangian submanifold with four ends, each end being a cone on a flat torus in the conifold, diffeomorphic to $T^2\times(0,\infty)$.*

Remark 1. When $\rho = 0$, equations (5.3) are identically satisfied and the solution is the zero section S^3 of the cotangent bundle, which was known to be special Lagrangian.

Remark 2. If we set $c_1 = c_2 = 0$ in equation (5.3), we obtain the equation

$$\sin(2t)\sinh(2\rho) = c$$

in the (t,ρ)-plane, $t\in[0,\pi)$. The phase portrait in this special case is shown in Figure 1. Note that as $t\to\frac{\pi}{2}$, $|\rho|\to\infty$. The SL are obtained by rotating

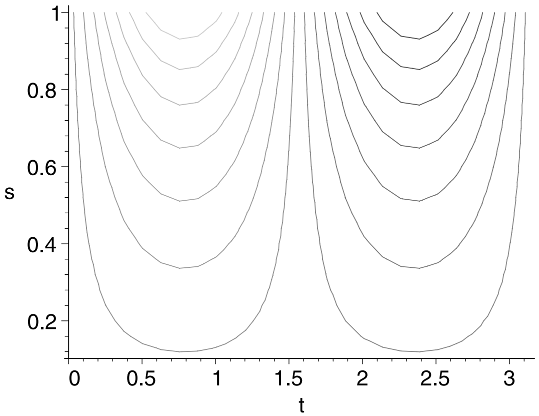

FIGURE 1. Phase portrait for the equation $\sin(2t)\sinh(2s) = c$.

these curves by the T^2-action.

5.3. The $SO(3)$-case. Let G be the subgroup $SO(3,\mathbb{R})$ of $SO(4,\mathbb{C})$, where $SO(3,\mathbb{R})$ sits in $SO(4,\mathbb{R})$ as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, A \in SO(3) \right\}.$$

THEOREM 5.4. *Let $T^*S^3 = \{(x,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4, |x| = 1, x \cdot \xi = 0\}$. Then:*

$$L_c = \left\{ (gx(t), g\xi(t)) \middle| (x(t),\xi(t)) \in T^*S^3, g \in SO(3), \right.$$

$$\left. x(t) = \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}, 2|\xi(t)| - \cos(2t)\sinh(2|\xi(t)|) = c \right\}$$

are the only $SO(3)$-invariant special Lagrangian 3-folds of T^*S^3.

Proof. Let $z = (z_0, z_1, z_2, z_3) \in Q^3$, i.e., $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$. Using expression (4.3), the moment map of the $SO(3)$-action on Q^3 is computed to be:

$$\mu : Q^3 \to \mathfrak{so}(3)^*, \qquad \mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_1\bar{z_2}), \mathrm{Im}(z_2\bar{z_3}), \mathrm{Im}(z_3\bar{z_1})).$$

To see how we obtained it, let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be the infinitesimal generators of the $SO(3)$-action. Equation (4.3) implies that:

$$\begin{aligned} \mu_{A_3}(z) &= \frac{1}{2}u'(|z|^2)\langle A_3 z, iz\rangle = \frac{1}{2}u'(|z|^2)\left\langle \begin{pmatrix} 0 \\ -z_2 \\ z_1 \\ 0 \end{pmatrix}, \begin{pmatrix} iz_0 \\ iz_1 \\ iz_2 \\ iz_3 \end{pmatrix} \right\rangle \\ &= \frac{1}{2}u'(|z|^2)(i\bar{z}_1 z_2 - iz_1\bar{z}_2) = u'(|z|^2)\operatorname{Im}(z_1\bar{z}_2). \end{aligned}$$

We get the other two components of the moment map similarly.

Since $Z(\mathfrak{so}(3)^*) = \{0\}$, it follows from Corollary 4.3 that any $SO(3)$-invariant special Lagrangian 3-fold in Q^3 lies in the level set $\mu^{-1}(0)$, so $\operatorname{Im}(z_1\bar{z}_2) = \operatorname{Im}(z_2\bar{z}_3) = \operatorname{Im}(z_3\bar{z}_1) = 0$. By applying an appropriate rotation, we can assume that $x = (\cot t, \sin t, 0, 0), t \in [0, 2\pi)$. Hence, the level set $\mu^{-1}(0)$ is Dimension 4 and it is given by:

$$\begin{aligned} \mu^{-1}(0) = \{(gx, g\xi) | g \in SO(3), x(t) = (\cos t, \sin t, 0, 0), \\ \xi(t) = \rho(-\sin t, \cos t, 0, 0), t \in [0, \pi), \rho \geq 0\}, \end{aligned}$$

or equivalently, using the identification of T^*S^3 with Q,

$$\left\{ g.z(t,\rho) : g \in SO(3), z(t,\rho) = \begin{pmatrix} \cos(\tau) \\ \sin(\tau) \\ 0 \\ 0 \end{pmatrix}, \rho \geq 0, t \in [0, 2\pi) \right\},$$

where $\tau = t + i\rho$ lies in the $[0, \pi)$ vertical strip of the complex plane.

We now look for curves $\gamma(s)$ in the $\tau = (t, \rho)$ plane which after applying the $SO(3)$-action give rise to special Lagrangians of the form $L = g.\gamma(s)$. If

$$p = \gamma(s) = \begin{pmatrix} \cos(\tau(s)) \\ \sin(\tau(s)) \\ 0 \\ 0 \end{pmatrix}$$

is a point in $\mu^{-1}(0)$, then

$$A_3 p = \begin{pmatrix} 0 \\ 0 \\ \sin(\tau) \\ 0 \end{pmatrix}, \qquad A_2 p = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sin(\tau) \end{pmatrix} \quad \text{and} \quad A_1 p = 0,$$

where A_1, A_2, A_3 are the infinitesimal generators of $\mathfrak{so}(3)$ as defined above.

The tangent plane at p to L is spanned by the vectors $\langle X_1 = A_1 p, X_2 = A_2 p, X = \dot{\gamma}(s)\rangle$ at p, where

$$\dot{\gamma}(s) = \begin{pmatrix} -\sin(\tau)\dot{\tau} \\ \cos(\tau)\dot{\tau} \\ 0 \\ 0 \end{pmatrix}.$$

L is invariant under the $SO(3)$-flow and $\omega_{St}|_L = 0$, since it lies in $\mu^{-1}(0)$. Therefore, L is Lagrangian. One can also verify this directly with formula (3.6). Now, we will impose the condition that L is special Lagrangian, i.e., $\operatorname{Im}\Omega_{St} = 0$ should hold.

Using equation (3.7), we compute $\Omega_{St}(X_1, X_2, X_3) = (dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(\gamma(s), X_1, X_2, \dot{\gamma}(s))$:

$$\begin{vmatrix} \cos(\tau) & 0 & 0 & -\sin(\tau)\dot{\tau} \\ \sin(\tau) & 0 & 0 & \cos(\tau)\dot{\tau} \\ 0 & \sin(\tau) & 0 & 0 \\ 0 & 0 & -\sin(\tau) & 0 \end{vmatrix} = -\sin^2(\tau)\dot{\tau}.$$

Integrating, the condition $\operatorname{Im}\Omega_{St} = 0$ becomes:

$$\operatorname{Im}\bigl(2\tau - \sin(2\tau)\bigr) = c,$$

which is equivalent to

$$2\rho - \cos(2t)\sinh(2\rho) = c, \tag{5.5}$$

where c is any real constant. □

REMARK 1. Notice that $\rho = 0$ which gives $S^3 \subset T^*S^3$ is indeed a solution to equation (5.5).

REMARK 2. The result we obtained in the $n = 3$ case can be generalized to obtain a family of $SO(n)$-invariant special Lagrangians in T^*S^n. This family was also obtained by Anciaux [An] using different methods.

The $SO(3)$-invariant special Lagrangian L can also be written intrinsically as follows. Choose (z_0, z_1, z_2, z_3) coordinates on the complex quadric Q^3.

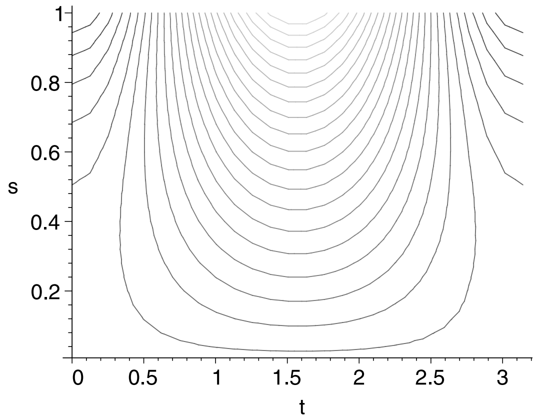

FIGURE 2. Phase portrait for the equation $2s - \cos(2t) \times \sinh(2s) = c$.

Then L is given by the equations:

$$\begin{aligned} u'(|z|^2)\,\mathrm{Im}(z_1\bar{z}_2) &= 0,\\ u'(|z|^2)\,\mathrm{Im}(z_2\bar{z}_3) &= 0,\\ \mathrm{Im}\big(\arccos(z_0) - z_0\sqrt{1-z_0^2}\big) &= c, \end{aligned} \tag{5.6}$$

where c is a constant. This family does not foliate the deformed conifold.

Asymptotic behavior. In what follows, we will study the equation $2\rho - \cos(2t)\sinh(2\rho) = c$ in the (t,ρ)-plane (see Figure 2) and describe the asymptotic behavior of the special Lagrangian 3-folds obtained in Theorem 5.4.

When $\rho \to \infty$, equation (5.5) becomes $\cos 2t = 0$, so $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. We analyze only the case $t = \frac{\pi}{4}$, the other three cases being similar.

(a) $t = \frac{\pi}{4}$. The unit vector $\frac{z}{|z|}$ is:

$$\frac{1}{\sqrt{2\cosh(2\rho)}}\begin{pmatrix}\cosh\rho - i\sinh\rho\\ \cosh\rho + i\sinh\rho\\ 0\\ 0\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}1-i\\ 1+i\\ 0\\ 0\end{pmatrix} \in Q_0 \quad \text{as } \rho \to \infty.$$

Applying the $SO(3)$-action in the limit, one gets a surface Σ, diffeomorphic to S^2 and described in coordinates as:

$$\Sigma = \left\{ \frac{1}{2}\begin{pmatrix}1-i\\ (1+i)\cos\varphi\cos\theta\\ (1+i)\cos\varphi\sin\theta\\ (1+i)\sin\varphi\end{pmatrix}, \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \theta \in [0, 2\pi) \right\} \subset Q_0.$$

The cone on Σ is special Lagrangian in the conifold Q_0, endowed with the Ricci-flat metric found by Candelas and de la Ossa [CO]. To see this, we will first show that the cone $C(\Sigma)$ is Lagrangian. The moment map of the $SO(3)$-action on the cone is:

$$\mu_0 : Q_0 \to \mathbb{R}^3, \qquad \mu_0(z) = u'_{cone}(|z|^2)(\mathrm{Im}(z_1\bar{z}_2), \mathrm{Im}(z_2\bar{z}_3), \mathrm{Im}(z_3\bar{z}_1)),$$

where $u_{cone}(r^2)$ is the potential function for the conifold given in Section 3.2. Since the cone on Σ is seen to lie in $\mu_0^{-1}(0,0)$, it follows that it is Lagrangian.

Next, we show that the cone is SL, i.e., $\mathrm{Im}\,\Omega_{cone}|_{C(\Sigma)} \equiv 0$. For this, we compute Ω_{cone} on three tangent vectors Y_1, Y_2, Y_3 to the cone $C(\Sigma)$. One of them is the position vector and the other two vectors are $A_3 z$ and $A_2 z$. The unit vector normal to the cone is given by $w = \bar{z} = \frac{1}{2}(1+i, 1-i, 0, 0)$ and we compute:

$$\begin{aligned}\mathrm{Im}\,\Omega_{cone}(Y_1, Y_2, Y_3) &= \mathrm{Im}(\Omega_0)(\bar{z}, Y_1, Y_2, Y_3)\\ &= \mathrm{Im}\,\frac{1}{8}\begin{vmatrix} 1-i & 0 & 0 & 1+i \\ 1+i & 0 & 0 & 1-i \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{vmatrix} = 0.\end{aligned}$$

Hence, the cone on Σ is special Lagrangian.

The other cases can be dealt with similarly. The case $t = \frac{5\pi}{4}$ yields the cone for $t = \frac{\pi}{4}$ rotated by π (the opposite cone). The cases $t = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ give a SL double cone on S^2, which is the previous double cone rotated by $\frac{\pi}{2}$.

The above analysis shows the following proposition.

Proposition 5.5. *For generic $c \in \mathbb{R}$, the $SO(3)$-invariant special Lagrangian L_c in the complex quadric has two components, each of them asymptotic to a special Lagrangian submanifold with four ends, each end being a cone on S^2 in the conifold, diffeomorphic to $S^2 \times (0, \infty)$.*

Remark. In the particular case of the Eguchi–Hanson metric ($n = 2$), the subgroups $SO(2)$ and T^1 of $SO(3)$ coincide and equations (5.2) & (5.6) give the same $SO(2)$-invariant SL 3-fold. In coordinates $z = (z_0, z_1, z_2)$ on Q^2, these special Lagrangians are given by the equations:

$$\begin{aligned}\frac{|z|}{\sqrt{|z|^2+1}}\,\mathrm{Im}(z_1\bar{z}_2) &= c_1,\\ \mathrm{Im}(z_0) &= c_2,\end{aligned}$$

where c_1, c_2 are any constants.

5.4. The general $SO(n)$-case. One can generalize our method to higher dimensions and recover the $SO(n)$-invariant family of special Lagrangians that Anciaux obtained in [An]. Computing the imaginary part of the holomorphic

n-form Ω_{St} to check the special Lagrangian condition yields

$$\operatorname{Im}(\sin^{n-1}(\tau)\dot{\tau}) = 0.$$

Now, let $F(\tau)$ be the function $\operatorname{Im}(\int_0^\tau \sin^{n-1}(\sigma)\,d\sigma)$. Combining with the moment map conditions, the $SO(n)$-invariant special Lagrangians are given by the following set of equations:

$$\begin{aligned} u'(|z|^2)\operatorname{Im}(z_1\bar{z}_j) &= 0, \qquad 2 \le j \le n, \\ \operatorname{Im}(F(\arccos(z_0))) &= c, \end{aligned} \tag{5.7}$$

where and c constant and the function u satisfies (3.2) for the given dimension.

6. Special Lagrangian 3-folds in the resolved conifold

Following the original paper of Candelas–de la Ossa [CO], we will first give a brief description of the resolved conifold and its Calabi–Yau structure.

Let Q_0 be conifold defined by the equation $\sum_{i=0}^{3} z_i^2 = 0$. One can resolve the singularity of the conifold. Making the linear change of variables:

$$\begin{pmatrix} X \\ Y \\ U \\ V \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -i & -1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{6.1}$$

the conifold equation becomes:

$$XY - UV = 0. \tag{6.2}$$

We note that the change of variables matrix is in $U(4)$, so the Euclidean length is still preserved:

$$r^2 = \sum_{i=0}^{3} |z_i|^2 = |X|^2 + |Y|^2 + |U|^2 + |V|^2. \tag{6.3}$$

The $SO(4,\mathbb{C})$ action on the z variables is now conjugated to an action on the new variables via $g \mapsto \tilde{g} = PgP^{-1}$. Let

$$W = \begin{pmatrix} X & U \\ V & Y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} z_0 - iz_1 & z_3 - iz_2 \\ -z_3 - iz_2 & z_0 + iz_1 \end{pmatrix}. \tag{6.4}$$

To resolve the conifold, one has to replace the equation (6.2) by the pair of equations:

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{6.5}$$

where $[\lambda_1 : \lambda_2]$ are homogeneous coordinates on $\mathbb{C}P^1 = S^2$. In other words, the resolved conifold M is the complex 3-fold defined by:

$$\begin{aligned}\{(X,Y,U,V,[\lambda_1 : \lambda_2]) \in \mathbb{C}^4 \times \mathbb{C}P^1 |\\ XY-UV=0, X\lambda_1+U\lambda_2=0, V\lambda_1+Y\lambda_2=0\}.\end{aligned} \tag{6.6}$$

Each point $(X,Y,U,V) \in Q_0$, except the origin, determines a unique point

$$[\lambda_1 : \lambda_2] = [-U : X] = [-Y : V] \in \mathbb{C}P^1$$

and the singularity at the origin is replaced by a copy of $\mathbb{C}P^1$. Outside the origin, the resolved conifold is topologically the same as the conifold.

We will also use inhomogeneous coordinates $\lambda_+ = \frac{\lambda_2}{\lambda_1}$ in the coordinate patch H_+ where $\lambda_1 \neq 0$ and $\lambda_- = \frac{\lambda_1}{\lambda_2}$ in the coordinate patch H_-, where $\lambda_2 \neq 0$. In H_+, we can take (U,Y,λ_+), and in H_-, we can take (X,V,λ_-) as coordinates for the conifold. On the intersection $H_+ \cap H_-$, we have:

$$(X,V,\lambda_-) = (-\lambda_+U, -\lambda_+Y, \lambda_+^{-1}).$$

In [CO], Candelas and de la Ossa showed that the resolved conifold is in fact an $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{C}P^1 \simeq S^2$ with fibre $\mathbb{C}^2$. The radius in the fibre is given by

$$r^2 = \operatorname{tr}(W^*W) = (1+|\lambda_+|^2)(|U|^2+|Y|^2) = (1+|\lambda_-|^2)(|X|^2+|V|^2).$$

Asymptotically, as $r \to \infty$, the resolved conifold approaches the singular conifold Q_0.

The Ricci-flat metric on M found in [CO] is:

$$g_{rc} = F_a'(r^2)\operatorname{tr}(dW^*\,dW) + F_a''(r^2)|\operatorname{tr}(W^*\,dW)|^2 + 4a^2 g_{S^2} \tag{6.7}$$

where F_a is a function of r^2 satisfying an appropriate differential equation and g_{S^2} is the Fubini–Study metric on S^2 with area π. In the patch H_+, g_{S^2} is given by:

$$g_{S^2} = \frac{|d\lambda_+|^2}{(1+|\lambda_+|^2)^2}.$$

The resolution parameter a ($a=0$ for the conifold) measures the size of the bolt $\mathbb{C}P^1$. The differential equation satisfied by the function F_a is given by imposing the Ricci-flat condition and is given given by:

$$F_a'(r^2) = r^{-2}(-2a^2 + 4a^4 N^{-\frac{1}{3}} + N^{\frac{1}{3}}), \tag{6.8}$$

where $N(r) = \frac{1}{2}(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4})$.

For the conifold Q_0 with $a=0$, we have $F_0' = r^{-\frac{2}{3}}$.

The Kähler form $\omega_{rc}(v,w) = g_{rc}(Jv,w)$ on the resolved conifold can be expressed as a sum of two terms $\omega_{rc} = d\alpha_{rc} + 4a^2\omega_{S^2}$, where the one-form α_{rc} is given by:

$$\begin{aligned}\alpha_{rc} &= F_a'(r^2)\operatorname{Im}\operatorname{tr}(W^*\,dW)\\ &= F_a'(r^2)\operatorname{Im}(\bar{X}\,dX + \bar{Y}\,dY + \bar{U}\,dU + \bar{V}\,dV)\end{aligned} \tag{6.9}$$

and ω_{S^2} is the standard Kähler form of area π on S^2. It can be expressed as $\omega_{S^2} = d\alpha_\pm$ in the two patches $H_\pm$ on S^2 where:

$$\alpha_\pm = \frac{1}{2}\mathrm{Im}\frac{\lambda_\pm d\overline{\lambda}_\pm}{1+|\lambda_\pm|^2} \quad \text{on } H_\pm.$$

One can see that α_{rc} is invariant under the action of $SO(4,\mathbb{R}) = SO(4)$. To show that $\alpha_\pm$ is also $SO(4)$-invariant, we now compute the action of $SO(4)$ on $\mathbb{C}P^1$. The following three matrices acting on the z-coordinates generate the $\mathfrak{so}(4)$ as a Lie algebra:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6.10}$$
$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Conjugating by the change of coordinate matrix P (6.1), we obtain the action on the X, Y, U, V-coordinates:

$$\tilde{B}_1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tag{6.11}$$
$$\tilde{A}_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

In H_+, $\lambda_+ = -\frac{X}{U} = -\frac{V}{Y}$, and so $d\lambda_+ = -\frac{dX}{U} + \frac{X\,dU}{U^2}$. The action of $\tilde{B}_1$ on λ_+ is given by evaluating $d\lambda_+$ on the vector field corresponding to $\tilde{B}_1$. We obtain: $\tilde{B}_1.\lambda_+ = -i\lambda_+$. Similarly, on H_- we compute that $\tilde{B}_1.\lambda_- = i\lambda_-$. The action of $\tilde{B}_2$ is the same as for $\tilde{B}_1$ and the action of $\tilde{A}_3$ is given by:

$$\tilde{A}_3.\lambda_+ = \frac{1}{2}(1+\lambda_+^2), \qquad \tilde{A}_3.\lambda_- = -\frac{1}{2}(1+\lambda_-^2).$$

A computation now shows that the form $\alpha_\pm$ is invariant under the action of these three generators and hence under the $SO(4)$-action. Therefore, the Kähler form and the metric are $SO(4)$-invariant. The holomorphic volume 3-form on M has the form:

$$\Omega_{rc} = dU \wedge dY \wedge d\lambda_+ = dV \wedge dX \wedge d\lambda_- \tag{6.12}$$

in local coordinates and it is also invariant under the action of $SO(4)$. We will now describe the cohomogeneity one special Lagrangians in the resolved conifold M.

6.1. T^2-invariant special Lagrangians. Under the coordinate transform (6.1), the action of the maximal torus T^2 of $SO(4)$ on the z-coordinates corresponds to the diagonal matrix diag $(e^{-i\theta_1}, e^{i\theta_1}, e^{i\theta_2}, e^{-i\theta_2})$ with respect to the (X, Y, U, V)-coordinates. The T^2-action on the patch H_+ with coordinates (U, Y, λ) is given by:

$$g.(U, Y, \lambda_+) = \left(e^{i\theta_2}U, e^{i\theta_1}Y, e^{-i(\theta_1+\theta_2)}\lambda_+\right),$$

and on the patch H_- with coordinates (X, V, μ) is given by:

$$g.(X, V, \lambda_-) = \left(e^{-i\theta_1}X, e^{-i\theta_2}V, e^{i(\theta_1+\theta_2)}\lambda_-\right).$$

The next result gives the T^2-invariant SL 3-folds of the resolved conifold.

THEOREM 6.1. *The special Lagrangian 3-folds in the resolved conifold M described by equation (6.6), which are invariant under the action of the maximal torus T^2 of $SO(4)$ are given by:*

$$\begin{aligned} \frac{1}{2}F_a'(r^2)(|X|^2 - |Y|^2) + 4a^2\frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} &= c_1, \\ \frac{1}{2}F_a'(r^2)(|V|^2 - |U|^2) + 4a^2\frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} &= c_2, \\ \mathrm{Im}(XY) &= c_3, \end{aligned} \tag{6.13}$$

where F_a' is given by (6.8) and c_1, c_2 and c_3 are real constants.

Proof. Using the two infinitesimal generators:

$$\tilde{B}_1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

of the T^2-action on the resolved conifold, the moment map is given by:

$$\mu : M \to (\mathfrak{t}^2)^* \simeq \mathbb{R}^2,$$
$$\mu = \left(\frac{1}{2}F_a'(r^2)(|X|^2 - |Y|^2) + 4a^2\mu_{S^2}, \frac{1}{2}F_a'(r^2)(|V|^2 - |U|^2) + 4a^2\mu_{S^2}\right),$$

where $\mu_{S^2}(\lambda_1, \lambda_2) = \frac{|\lambda_2|^2}{|\lambda_1|^2+|\lambda_2|^2}$ is the moment map of the standard S^1-action on $\mathbb{C}P^1$.

Since $(\mathfrak{t}^2)^* = \mathbb{R}^2$ is Abelian, it follows from Proposition 4.2 that any T^2-invariant special Lagrangian 3-fold L in M lies in a level set $\mu^{-1}(c)$, where

$c=(c_1,c_2)\in\mathbb{R}^2$. The first two equations enforce this condition and ensures that the submanifold is Lagrangian. Now, we impose the special Lagrangian condition at a given point $p=(U,Y,\lambda_+)$. On H_+, we compute Ω_{rc} on the three tangent vectors $Y_1=\tilde{B}_1p=(0,iY,-i\lambda_+), Y_2=\tilde{B}_2p=(iU,0,-i\lambda_+)$, and $Y_3=\dot{p}=(\dot{U},\dot{Y},\dot{\lambda}_+)$:

$$\begin{aligned}\Omega_{rc}(Y_1,Y_2,Y_3) &= (dU\wedge dY\wedge d\lambda_+)(Y_1,Y_2,Y_3)\\ &= \begin{vmatrix} 0 & iU & \dot{U}\\ iY & 0 & \dot{Y}\\ -i\lambda_+ & -i\lambda_+ & \dot{\lambda}_+\end{vmatrix}\\ &= \dot{U}Y\lambda_+ + U\dot{Y}\lambda_+ + UY\dot{\lambda}_+ = (UY\lambda_+)^{\cdot}.\end{aligned} \tag{6.14}$$

Integrating the condition $\operatorname{Im}\Omega_{rc}=0$, we obtain $\operatorname{Im}(UY\lambda_+)=c$ for $c\in\mathbb{R}$. Using $\lambda_+=-\frac{X}{U}$, we finally obtain the third equation of the theorem. Note that in the other coordinate patch H_-, we will obtain $\operatorname{Im}(VX\lambda_-)=c$ which is the same equation since $\lambda_-=-\frac{Y}{V}$. □

Remark 1. Equations (6.13) are linearly independent at a generic point, so the above family foliates the resolved conifold. The generic orbit is $T^2\times\mathbb{R}$, where T^2 is an orbit of the maximal torus in $SO(4)$. For the special values $c_1=c_2=c_3=0$, the SL intersect the $\mathbb{C}P^1$ in circles.

Remark 2. Asymptotically, as $r\to\infty$ (or equivalently as $a\to 0$), these special Lagrangians approach the special Lagrangian cone on T^2 in the conifold described by the equations:

$$\begin{aligned}|X|^2-|Y|^2&=0,\\ |V|^2-|U|^2&=0,\\ \operatorname{Im}(XY)&=0,\end{aligned}$$

which is the same asymptotic cone found in the case of the deformed conifold in Section 5.2. In z coordinates on the conifold, the equations become: $\operatorname{Im}(z_0\bar{z_1})=\operatorname{Im}(z_2\bar{z_3})=\operatorname{Im}(z_0^2+z_1^2)=0$.

6.2. ***SO*(3)-invariant special Lagrangians.** In what follows, we describe the $SO(3)$-invariant special Lagrangian 3-folds in M.

Theorem 6.2. *The $SO(3)$-invariant special Lagrangian submanifolds in the resolved conifold given by equation (6.6) are:*

$$L_c=\{\tilde{g}.p\in M|g\in SO(3), p=(0,Y,0,0), \operatorname{Re}(Y^2)=c\}$$

where $\tilde{g}$ is the conjugated action on the (X,Y,U,V)-coordinates.

Proof. Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be the infinitesimal generators of $SO(3)$. Using the change of coordinate matrix P (6.1), the action on the (X,Y,U,V)-variables is given by:

$$\tilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad \tilde{A}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & i & -i \\ -i & i & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix},$$

$$\tilde{A}_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

The moment map with respect to these generators is $\mu : M \to \mathfrak{so}(3)^*$ given by:

$$\mu = \left(\frac{1}{2}(|U|^2 - |V|^2), \frac{1}{2}\operatorname{Im}\big((U-V)(\bar{Y}-\bar{X})\big), \frac{1}{2}\operatorname{Im}\big((Y-X)(\bar{U}+\bar{V})\big)\right) + \mu_{S^2},$$

where

$$\mu_{S^2}(\lambda_+) = \left(\frac{|\lambda_+|^2}{1+|\lambda_+|^2}, -\frac{\operatorname{Re}(\lambda_+ - |\lambda_+|^2\overline{\lambda_+})}{2(1+|\lambda_+|^2)}, -\frac{\operatorname{Im}(\lambda_+ + |\lambda_+|^2\overline{\lambda_+})}{2(1+|\lambda_+|^2)}\right)$$

on the patch coordinate patch H_+. A similar expression can be computed on H_-.

Let $p = (X,Y,U,V) \in M$. Since $Z(\mathfrak{so}(3)^*) = 0$, we need to start at a point in $\mu^{-1}(0)$. We make the simplifying assumption that $U = V = 0$. Using the fact that $XY = UV$ on the resolved conifold, we can assume that $X = 0$ or $Y = 0$. If we choose $Y = 0$, then the point $(X,0,0,0) \notin \mu^{-1}(0)$ since $\mu(X,0,0,0)$ has the first component equal to 1. Hence, we must choose $p = (0,Y,0,0)$, where $Y \neq 0$. Since $\lambda_+ = -\frac{V}{Y} = 0$, it follows that $p \in \mu^{-1}(0)$. We now look for curves $Y(s)$ in the complex Y-plane, which after applying the $SO(3)$-action give rise to special Lagrangians of the form $L = \tilde{g}.Y(s)$, $g \in SO(3)$.

The tangent plane at p to L is spanned by the vectors:

$$\{v_1 = \tilde{A}_2 p = (PA_2P^*)p, v_2 = \tilde{A}_3 = (PA_3P^*)p, v_3 = \dot{p}\},$$

where P is the change of coordinate matrix in (6.1) and the dot denotes differentiation with respect to the parameter s. The generic orbit of the $SO(3)$ action is an S^2, so the three infinitesimals generators are linearly dependent (note that $PA_1P^*p = 0$). Calculations yield that the tangent space of L at the point p is spanned by the vectors:

$$\left\{ v_1 = \frac{iY}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, v_2 = \frac{Y}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ \dot{Y} \\ 0 \\ 0 \end{pmatrix} \right\}.$$

L is invariant under the $SO(3)$-action and $\omega_{rc}|_L = 0$, since L lies in the zero level set of the moment map. We now impose the special Lagrangian condition $\operatorname{Im}\Omega_{rc}|_L = 0$. Working on the patch H_+ with coordinates (U, Y, λ_+), the holomorphic volume form is given from (6.12):

$$\Omega_{rc} = dU \wedge dY \wedge d\lambda_+.$$

Since $\lambda_+ = -\frac{V}{Y}$, we can instead use coordinates (U, Y, V), getting:

$$\begin{aligned}
\Omega_{rc} &= dU \wedge dY \wedge d\lambda_+ \\
&= -dU \wedge dY \wedge \frac{V\,dY - Y\,dV}{Y^2} = -\frac{1}{Y}\,dU \wedge dY \wedge dV, \\
\Omega_{rc}(v_1, v_2, v_3) &= -\frac{1}{Y} dU \wedge dY \wedge dV(v_1, v_2, v_3) \\
&= -\frac{1}{Y}\begin{vmatrix} \frac{iY}{2} & -\frac{Y}{2} & 0 \\ 0 & 0 & \dot{Y} \\ -\frac{iY}{2} & -\frac{Y}{2} & 0 \end{vmatrix} = -\frac{i}{2} Y\dot{Y}.
\end{aligned}$$

Integrating, the condition $\operatorname{Im}\,\Omega_{rc} = 0$ becomes $\operatorname{Re}(Y^2) = c$. Letting $Y = u + iv$, we get a family of hyperbolas: $u^2 - v^2 = c$. ☐

The special Lagrangian L_c is topologically $S^2 \times \mathbb{R}$ and has two components, each asymptotic to the special Lagrangian cones on S^2 given by setting $c = 0$ (obtained from the lines $u + v = 0$ and $u - v = 0$) in the conifold. This case is reminiscent of the flat case studied in [HL].

Remark 1. L_c does not intersect the zero-section (the bolt) S^2 of the resolved conifold since $r^2 = |Y|^2$ is preserved under $SO(3)$.

Remark 2. Note that the $SO(3)$-invariant special Lagrangian 3-folds in both the resolved and the deformed conifold have the same limiting SL cone.

Remark 3. In this paper, we described only the special Lagrangian 3-folds with phase 1, but a similar analysis will yield formulas for special Lagrangians with a fixed phase θ.

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