MATHEMATICAL MODELS FOR SMALL DEFORMATIONS OF STRINGS

by

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Lecture given at Faculdade de Matemáticas UFPA

(Belém – March 2008)

FIXED ENDS

Let us consider a stretched string which in rest position is represented by the segment $[\alpha_0, \beta_0]$ in the x-axis of an orthogonal system of coordenate in a plane $x0u$, with $0 < \alpha_0 < \beta_0$. We suppose small vibrations and the vibrations on the plane $x0u$, which we call vertical vibrations or vertical deformations of the string.

We suppose τ_0 the tension in the string at rest position $[\alpha_0, \beta_0]$, that is, τ_0 is the force per unite of area of the cross section of the string. At time $t > 0$ the point (x, t), $\alpha_0 < x < \beta_0$, of the string, belongs to a plane curve $\Gamma(t)$ which equation is $u = u(x, t)$. The tension is variable and at $\Gamma(t)$ it is τ, which is the force by unit of area of the cross section of $\Gamma(t)$. The tension is a vector $\vec{\tau}$ with modulus τ. Let be $\gamma_0 = \beta_0 - \alpha_0$ and S the lenght of the curve deformation $\Gamma(t)$ at time t. The variation of the tension is $\tau - \tau_0$ and the variation of the lenght of deformation is $(S - \gamma_0)/\gamma_0$, the mean deformation. The Hooke's law says that $\tau - \tau_0$ is a linear function of $(S - \gamma_0)/\gamma_0$, that is

$$\tau - \tau_0 = k \, \frac{S - \gamma_0}{\gamma_0} \cdot \tag{1.1}$$

In general k is constant when the string is homogeneous, that is, its density, mass per unit of lenght, is constant. Suppose that it is not homogeneous, that is, k depends of $\alpha_0 < x < \beta_0$ and of the time $t > 0$. Thus the Hooke's law is

$$\tau - \tau_0 = k(x, t) \, \frac{S - \gamma_0}{\gamma_0} \tag{1.2}$$

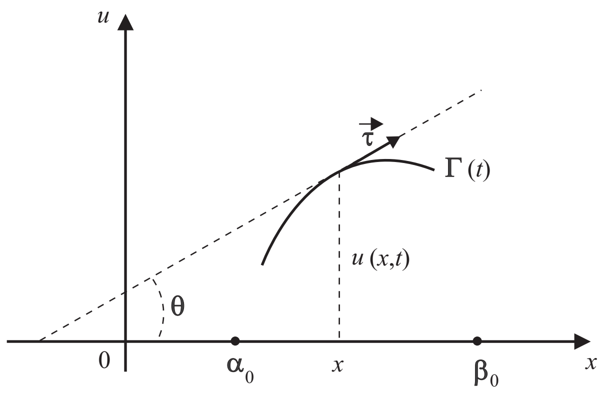

Figure 1

The tension τ in the point (x, u), of the deformation curve $\Gamma(t)$, is a vector $\vec{\tau}$ which has the direction of the tangent to $\Gamma(t)$, at the point (x, u), with modulus τ. We suppose $\Gamma(t)$ regular and $\Gamma(\alpha_0) = \Gamma(\beta_0) = 0$, the string has fixed ends. See Fig. 1.

Let θ be the angle of the direction $0x$ with the vector $\vec{\tau}$. The components of $\vec{\tau}$ are:

$$\tau \text{ sen } \theta \qquad \text{and} \qquad \tau \cos \theta. \tag{1.3}$$

By hypothesis we have small vertical deformations of the string $[\alpha_0, \beta_0]$. Thus, we don't need to consider the horizontal component, $\tau \cos \theta$, which is "very small". Thus we don't consider, in the present argument, the horizontal component of $\vec{\tau}$.

Let $d(x, t)$ be the density of the string at points x at time t, that is the mass per unity of length. The variations of the tension $\vec{\tau}$ gives origin to a force on $\Gamma(t)$ and, by Newton's law, we have:

$$\frac{\partial}{\partial t}\left(\gamma_0\, d(x,t)\,\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(\tau \text{ sen } \theta\right), \tag{1.4}$$

$\gamma_0\, d(x,t)$ is the mass of the string.

We supposed small vibrations or small deformations what means that the gradient of deformations is small, that is, we must have:

$$\left|\frac{\partial u}{\partial x}\right| \ll 1. \tag{1.5}$$

As a consequence of (1.5) we have $\text{sen}\ \theta \approx \text{tg}\ \theta = \dfrac{\partial u}{\partial x}\cdot$

From (1.2), the material is non homogeneous, the tension $\tau - \tau_0$ depends on x, t. We obtain, from (1.4)

$$\frac{\partial}{\partial x}\,(\tau\ \text{sen}\ \theta) = \frac{\partial \tau}{\partial x}\ \text{sen}\ \theta + \tau\,\frac{\partial\, \text{sen}\ \theta}{\partial x}\,,$$

or

$$\frac{\partial}{\partial x}\,(\tau\ \text{sen}\ \theta) = \frac{\partial \tau}{\partial x}\,\frac{\partial u}{\partial x} + \tau\,\frac{\partial^2 u}{\partial x^2}\,\cdot \tag{1.6}$$

Analysis of $\dfrac{\partial \tau}{\partial x}$

In fact, the length of the arc $\Gamma(t)$ is

$$S = \int_{\alpha_0}^{\beta_0} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx. \tag{1.7}$$

From the hypothesis (1.5) it follows that the series representation for the function $\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx$ is uniformly convergent. Then it is reasonable to consider the approximation

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} \approx 1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2.$$

Then, from (1.7), we obtain

$$S = \int_{\alpha_0}^{\beta_0} \left[1 + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 \right] dx$$

or

$$S - \gamma_0 = \frac{1}{2} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx. \tag{1.8}$$

From(1.2) and (1.8) we have

$$\tau - \tau_0 = \frac{k(x,t)}{2\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx, \tag{1.9}$$

then

$$\frac{\partial \tau}{\partial x} = \frac{1}{2\gamma_0} \frac{\partial k(x,t)}{\partial x} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx. \tag{1.10}$$

From (1.6) and (1.10) we get

$$\begin{aligned} \frac{\partial}{\partial x} (\tau \text{ sen } \theta) = & \left[\frac{1}{2\gamma_0} \frac{\partial k(x,t)}{\partial x} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \right] \frac{\partial u}{\partial x} + \\ & + \left[\tau_0 + \frac{k(x,t)}{2\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \right] \frac{\partial^2 u}{\partial x^2} \end{aligned} \tag{1.11}$$

Substituting (1.11) in (1.4) we get the partial differential equation:

$$\begin{aligned} \frac{\partial}{\partial t} \left(\gamma_0 \, d(x,t) \, \frac{\partial u}{\partial t} \right) - & \left[\frac{1}{2\gamma_0} \frac{\partial k(x,t)}{\partial x} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \right] \frac{\partial u}{\partial x} - \\ & - \left[\tau_0 + \frac{k(x,t)}{2\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \right] \frac{\partial^2 u}{\partial x^2} = 0. \end{aligned} \tag{1.12}$$

Thus, (1.12) is the mathematical model for the physical problem of small vertical vibrations of a stretched string fixed at the ends $(\alpha_0, 0)$, $(\beta_0, 0)$, when the material of the string is not homogeneous.

We change the notation in order to formulate the mathematical problem for (1.12). In fact, set

$$\left|\begin{aligned} &\gamma_0\, d(x,t) = \rho(x,t), \quad \text{with} \quad \rho(x,t) \geq \rho_0 > 0 \\ &M(x,t,\lambda) = \tau_0 + \frac{k(x,t)}{2\gamma_0}\,\lambda \\ &N(x,t,\lambda) = \frac{1}{2\gamma_0}\,\frac{\partial\, k(x,t)}{\partial x}\,\lambda = \frac{\partial}{\partial x}\, M(x,t,\lambda). \end{aligned}\right. \tag{1.13}$$

Thus the model (1.12) can be written in the form:

$$\frac{\partial}{\partial t}\left(\rho(x,t)\,\frac{\partial u}{\partial t}\right) - M\left(x, t, \int_{\alpha_0}^{\beta_0}\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} - \\ - N\left(x, t, \int_{\alpha_0}^{\beta_0}\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial u}{\partial x} = 0. \tag{1.14}$$

Particular Cases

- Suppose the material of the string is homogeneous, that is, $k(x,t) = k$ constant for $\alpha_0 \leq x \leq \beta_0$ and $t \geq 0$. Thus $\dfrac{\partial k}{\partial x} = 0$, k is constant with respect to x and we suppose it is also constant with respect to t. The density $d(x,t)$ is also constant and we represent $d(x,t)\gamma_0$ by the constant ρ. The model (1.12) reduces to

$$\rho\,\frac{\partial^2 u}{\partial t^2} - \left[\tau_0 + \frac{k}{2\gamma_0}\int_{\alpha_0}^{\beta_0}\left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.15}$$

This model was obtained by G. KIRCHHOFF – Vorlesungen über mechanik, Tauber - Leipzig 1883. It is called Kirchhoff's model.

To observe that in the Kirchhoff's model the string is homogeneous but the tension is variable, with the time, that is, by (1.9) the tension at time t is given by

$$\tau(t) = \tau_0 + \frac{k}{2\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx, \tag{1.16}$$

where τ_0 is the tension of the string $[\alpha_0, \beta_0]$ in the rest position. The term

$$\frac{k}{2\gamma_0} \int_{\alpha_0}^{\beta_0} \left(\frac{\partial u}{\partial x} \right)^2 dx \tag{1.17}$$

is the contribution when the tension is variable.

• Suppose now we have homogeneous material and the tension a constant τ_0, for each time t. In this case, we have not the contribution (1.17) and $\tau = \tau_0$ for all t. The model (1.15) of Kirchhoff reduces to

$$\rho \frac{\partial^2 u}{\partial t^2} - \tau_0 \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.18}$$

or

$$\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}, \tag{1.19}$$

$C^2 = \frac{\tau_0}{\rho} \cdot$

The model (1.19) was obtained by Jean D'Alembert in 1741, cf. J. D'ALEMBERT – Recherches sur les vibrations des cordes sonores - Opuscules Mathématiques, Tome Premier (1741) pp. 1-65 - Acad. Fran. des Sciences, Paris, France.

We could say that (1.19) *was the first partial differential equation describing problems of physics.*

The method above developed is when the string has fixed ends $(\alpha_0, 0)$, $(\beta_0, 0)$.

Suppose the ends of the string are variable. Let us consider the case homogeneous.

VARIABLE ENDS

We suppose the homogeneous case but with variable tension. We consider the ends variable with $t > 0$, that is,

$$0 < \alpha(t) \leq \alpha_0 < x < \beta_0 \leq \beta(t).$$

At time $t > 0$ we represent the string by $[\alpha(t), \beta(t)]$, with $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$, see Figure 2. Thus, the lenght of the string is $\gamma(t) = \beta(t) - \alpha(t)$ at $t > 0$ and $\gamma_0 = \beta_0 - \alpha_0$ at $t = 0$.

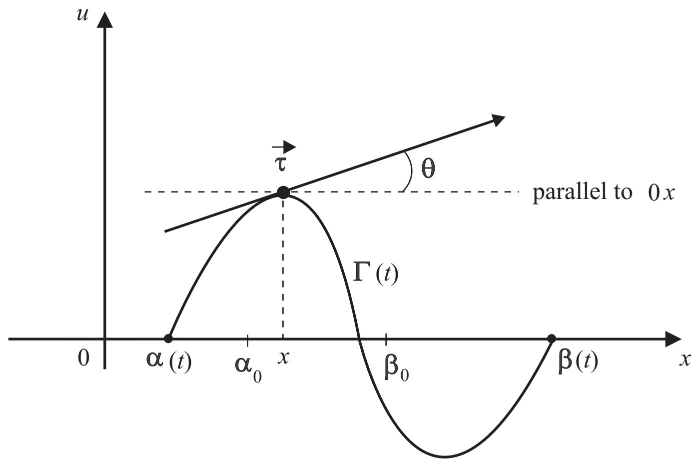

Figure 2

We employ the notation:

τ_0 the tension at $[\alpha_0, \beta_0]$, rest position.

$\vec{\tau}(t)$ vector tension at the curve $\Gamma(t)$, deformation of $[\alpha(t), \beta(t)]$, at time t.

$\tau(t)$ modulus of the vector $\vec{\tau}(t)$ which has the direction of the tangent vector at $\Gamma(t)$.

$\widehat{\tau}(t)$ tension in the deformation $[\alpha(t), \beta(t)]$ of $[\alpha_0, \beta_0]$ at the time $t > 0$.

The components of $\vec{\tau}(t)$ at time t, are:

$\tau(t)$ sen θ vertical

$\tau(t)$ cos θ horizontal

By θ we represent the angle of the direction $0x$ with the tangent vector to $\Gamma(t)$ at time t, as in Figure 2.

By $u(x,t)$ we represent the deformations at time t of x in $[\alpha(t), \beta(t)]$.

By hypothesis of small deformations we consider only the vertical component.

The variations of this component are $\dfrac{\partial}{\partial x}(\tau(t) \text{ sen } \theta)$ and, by Newton's law, we have:

$$\frac{\partial}{\partial x}\left(\tau(t) \text{ sen } \theta\right) = m\,\frac{\partial^2 u}{\partial t^2} \tag{1.20}$$

m the mass of the string. We suppose the deformations are very small so that the density of $[\alpha_0, \beta_0]$, $[\alpha(t), \beta(t)]$, $\Gamma(t)$ are approximately the same. Thus $m = \rho\,\gamma_0$ is constant.

Analysis of the tension $\tau(t)$.

By Hooke's law we obtain:

- deformation of $[\alpha_0, \beta_0]$ into $[\alpha(t), \beta(t)]$

$$\widehat{\tau}(t) - \tau_0 = k\,\frac{\gamma(t) - \gamma_0}{\gamma_0} \tag{1.21}$$

- deformation of $[\alpha(t), \beta(t)]$ into $\Gamma(t)$

$$\tau(t) - \widehat{\tau}(t) = k\,\frac{S(t) - \gamma(t)}{\gamma(t)}\,, \tag{1.22}$$

$S(t)$ is the lenght of the curve $\Gamma(t)$.

We know that

$$\begin{aligned} S(t) &= \int_{\alpha(t)}^{\beta(t)} \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}\, dx = \\ &= \int_{\alpha(t)}^{\beta(t)} \left[1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right] dx = \\ &= \gamma(t) + \frac{1}{2}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned}$$

This approximation can be done because it is supposed small deformation, that is, small gradient of deformations, $\left|\dfrac{\partial u}{\partial x}\right| \ll 1$.

Thus, we have

$$S(t) - \gamma(t) = \frac{1}{2}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx. \tag{1.23}$$

Substituting (1.23) in (1.22) we obtain:

$$\tau(t) - \widehat{\tau}(t) = \frac{k}{2\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx. \tag{1.24}$$

From (1.21) and (1.24) we obtain the tension $\tau(t)$ on $\Gamma(t)$ given by:

$$\tau(t) = \tau_0 + k\,\frac{\gamma(t) - \gamma_0}{\gamma_0} + \frac{1}{2\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx. \tag{1.25}$$

We go back to the equilibrium equation (1.20) we have

$$\frac{\partial}{\partial x}(\tau(t)\,\text{sen}\;\theta) = \tau(t)\,\frac{\partial}{\partial x}\;\text{sen}\;\theta = \tau(t)\,\frac{\partial}{\partial x}\;\text{tg}\;\theta = \tau(t)\,\frac{\partial^2 u}{\partial x^2}$$

and substituting in (1.20) we obtain

$$m\,\frac{\partial^2 u}{\partial t^2} = \tau(t)\,\frac{\partial^2 u}{\partial x^2}\,. \tag{1.26}$$

By (1.25) and (1.26) we have the mathematical model when the ends are moving

$$\frac{\partial^2 u}{\partial t^2} - \left[\frac{\tau_0}{m} + \frac{k}{m\nu_0}\big(\gamma(t)-\gamma_0\big) + \frac{1}{2m\nu(t)}\int_{\alpha(t)}^{\beta(t)}\left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{po^2 u}{\partial t^2} = 0. \tag{1.27}$$

Thus, (1.27) is the mathematical model for vertical vibrations of a stretched string, homogeneous, when the ends are variable with the time t and variable tension.

Particular Cases

- If $\alpha(t) = \alpha_0$, $\beta(t) = \beta_0$, for all $t > 0$, we have $\gamma(t) = \gamma_0$, $t \geq 0$, then (1.27) reduces to

$$\frac{\partial^2 u}{\partial t^2} - \left[\frac{\tau_0}{m} + \frac{1}{2m\nu_0}\int_{\alpha_0}^{\beta_0}\left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{\partial^2 u}{\partial x^2} = 0,$$

which is the KIRCHHOFF model.

- If the tension is constant, $\tau(t) = \tau_0$ for all $t > 0$, the perturbation in Kirchhoff's model is zero and we have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\tau_0}{m}\,\frac{\partial^2 u}{\partial x^2} = 0,$$

which is the D'ALEMBERT's model.

We acknowledge Professor I-Shih Liu of IM-UFRJ for stimulating conversations when we investigated the variable ends.

MATHEMATICAL PROBLEMS

When we deduce mathematical models, from physical phenomenon, we usually consider many approximations. Thus, we must formulate certains mathematical problems about the model and prove that these mathematical problems formulated are well posed in the sense of JACQUES HADAMARD. It means that the problem has a non trivial solution, this solutions is unique and it depends continuously of the datum of the problem. Otherwise, the problem is called non well posed. In the following we will propose, for the models obtained above, same well posed problem. For example, let us fixe our attention for D'Alembert (1.18). Set $C^2 = \dfrac{\tau_0}{\rho}$ as was done.

Initial Boundary Value Problem. Find $u(x,t)$, from $\alpha_0 < x < \beta_0$, $0 < t < T$, $T > 0$, into the real numbers $\mathbb{R}$, satisfying:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - C^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \quad \text{in} \quad \alpha_0 < x < \beta_0\,,\ 0 < t < T,\ T > 0 \\ u(\alpha_0, t) = u(\beta_0, t) = 0 \quad \text{for} \quad 0 < t < T \\ u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \alpha_0 > x < \beta_0 \end{array}\right. \tag{1.28}$$

The condition $(1.28)_2$ says that the string has fixed ends. The $(1.28)_3$ gives conditions on the rest position $u(x,0)$ and initial velocity $\dfrac{\partial u}{\partial t}(x,0)$. They are known function $u_0(x)$ and $u_1(x)$. With convenient choice of u_0, u_1 we can prove that the boundary value problem (1.28) is well posed.

Cauchy Problem. It consists to investigate the well poseness of the problem – find $u(x,t)$, but for $-\infty < x < +\infty$, and $0 < t < \infty$, solution of the initial value problem

$$\left| \begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - C^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad -\infty < x < +\infty,\ t > 0 \\ u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(t),\ -\infty < x < +\infty. \end{array} \right. \tag{1.29}$$

Let us remember how D'Alembert solved the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

when we set $C = 1$. He wrote

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right).$$

We look for $u(x,t)$, $\alpha_0 \le x \le \beta_0$, $0 \le t \le T$ and we suppose it regular. Thus, its differential du is given by:

$$du = q\,dt + p\,dx,$$

with

$$q = \frac{\partial u}{\partial t} \quad \text{and} \quad p = \frac{\partial u}{\partial x}\cdot$$

Now let us consider the differential form

$$w = p\,dt + q\,dx,$$

with $p = \dfrac{\partial u}{\partial x}$, $\quad q = \dfrac{\partial u}{\partial t}$, $\quad u$ solution of D'Alembert equation, that is,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$$

or

$$\frac{\partial}{\partial x}\, p = \frac{\partial}{\partial t}\, q,$$

what says that w is a exact differential form. Then w is a differential of a function $v(x, t)$, that is,

$$d\, v = p\, dt + q\, dx.$$

We obtain

$$d(u + v) = (p + q) d(x + t)$$

$$d(u - v) = (p - q) d(x - t)$$

Then D'Alembert says that

$$u + v = \phi(x + t)$$

$$u - v = \psi(x - t).$$

Thus the solution $u(x, t)$ of the equation is

$$u(x, t) = \frac{1}{2}\, [\phi(x + t) + \psi(x - t)] \tag{1.30}$$

ϕ, ψ "arbitrary" function, second D'Alembert.

The solution of the Cauchy problem (1.29) is

$$u(x, t) = \frac{1}{2}\, [u_0(x + t) + u_0(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} u_1(s)\, ds. \tag{1.31}$$

For details, we can see L.A. Medeiros - N.G. Andrade - Iniciação às Equações Diferenciais Parciais, LTC, Rio de Janeiro, RJ, 1978. □

Now, we propose the problems (1.28), initial boundary value problem, and (1.29), Cauchy problem, for Kirchhoff's operator, for the case of fixed ends.

Initial Boundary Value Problem. Find $u(x,t)$ from $\alpha_0 < x < \beta_0$, $0 < t < T, \quad T > 0$, to the real numbers $\mathbb{R}$, satisfying:

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - \left[\dfrac{\tau_0}{\rho} + \dfrac{k}{2\rho\gamma_0}\displaystyle\int_{\alpha_0}^{\beta_0}\left(\dfrac{\partial u}{\partial x}\right)^2 dx\right]\dfrac{\partial^2 u}{\partial x^2} = 0 \\ \text{in} \quad \alpha_0 < x < \beta_0, \quad 0 < t < T, \quad T > 0 \\ u(\alpha_0, t) = u(\beta_0, t) = 0 \quad \text{for} \quad 0 \le t \le T \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \quad \alpha_0 < x < \beta_0 . \end{array}\right. \tag{1.32}$$

The boundary value problem (1.32) is non linear and consequently we have not chance to obtain solutions as for (1.28). The technique to solve the problem (1.32) is more elaborate. In general, we obtain local solution in t, that is, there exists $T_0 > 0$ so that the solution exists on $[0, T_0)$. The technique of weak solutions applying Sobolev's spaces is employed.

Cauchy Problem. It consists to find $u(x,t), \quad \infty < x < +\infty, \quad 0 < t < \infty$ with values in the real numbers, solution of the initial value problem

$$
\left|
\begin{array}{l}
\dfrac{\partial^2 u}{\partial t^2} - \left(\dfrac{\tau_0}{\rho} + \dfrac{k}{2\rho\gamma_0} \displaystyle\int_{-\infty}^{+\infty} \left(\dfrac{\partial u}{\partial x} \right)^2 \, dx \right) \dfrac{\partial^2 u}{\partial x^2} = 0 \\
\text{for} \qquad -\infty < x < +\infty, \quad t > 0 \\
u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \quad -\infty < x < +\infty.
\end{array}
\right. \tag{1.33}
$$

This is a problem formulated in the unbounded domain $-\infty < x < +\infty$, $0 < t < \infty$. The problem (1.32) is formulated in the bounded domain $\alpha_0 < x < \beta_0$, $0 < t < T$. When we investigate (1.32) and (1.33) by techniques of functional analysis, we have compactness in (1.32) but not in (1.33). This make a difference in the methods in classical analysis, in the basic courses of partial differential equation. We employ Fourier series in the bounded domain and Fourier transforms in the Cauchy problem.

The nonlinear boundary value and Cauchy problem, respectively (1.32) and (1.33), can be formulated in general. Let us formulate the initial boundary value problem (1.32). In fact, let Ω be a bounded open set of $\mathbb{R}^n$ with boundary Γ, regular. The nonlinearity we represent by $M(\lambda)$. Here is a generalization of $\dfrac{\tau_0}{\rho} + \dfrac{k}{2\rho\nu_0}\lambda$ which, by hypothesis, does not depends of x and t. We consider the cylinder $Q = \Omega \times (0,T)$, $T > 0$, with lateral boundary $\Sigma = \Gamma \times (0,T)$. A point of $\mathbb{R}^n$ is represented by $x = (x_1, \dots, x_n)$ with x_i real numbers. Thus, we look for a real function $u(x,t)$, $(x,t) \in Q$ solution of

$$\left|\begin{array}{l} \dfrac{\partial^2 u}{\partial t^2} - M\left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u = f \quad \text{in} \quad Q \\ u = 0 \quad \text{on} \quad \Sigma, \\ u(x,0) = u_0(x), \quad \dfrac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \Omega. \end{array}\right. \tag{1.34}$$

We represent by Δ an ∇, respectively the Laplace operator in $\mathbb{R}^n$ and the gradient.

The Cauchy problem in this general case is formulate as (1.32).

For results and abstract formulation of this problem, see: J.L. LIONS - On some questions in boundary value problems of mathematical physics - Contemporary developments in continuum mechanics and partial differential equations - ed. M. de La Penha and L.A. Medeiros, North Holland, Amsterdam, (1978), pp. 285-346. □

Variable Ends

Now let us consider the Kirchhoff's model (1.27) when the ends of the string move with the time, that is

$$\frac{\partial^2 u}{\partial t^2} - \left[\frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t)-\gamma_0}{\gamma_0} + \frac{1}{2m\gamma(t)}\int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x}\right)^2 dx\right]\frac{\partial^2 u}{\partial x^2} = 0.$$

We have $0 < \alpha(t) < \beta(t)$, $\gamma(t) = \beta(t) - \alpha(t)$ and $\gamma_0 = \beta_0 - \alpha_0\,,\ \ \alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$.

Let us define $\widehat{Q}$ the noncylindrical domain $\widehat{Q} = \{(x,t) \in \mathbb{R}^2; \alpha(t) < x < \beta(t)$, for $0 < t < T\}$. The boundary of $\widehat{Q}$ is

$$\widehat{\Sigma} = \bigcup_{0<t<T} \{(\alpha(t), \beta(t), t\}.$$

Set

$$a(t) = \frac{\tau_0}{m} + \frac{k}{m}\,\frac{\gamma(t) - \gamma_0}{\gamma_0}\,,$$
$$b(t) = \frac{k}{2m\gamma(t)}\,,$$

and

$$\widehat{L}\,u(x,t) = \frac{\partial^2 u}{\partial t^2} - \left[a(t) + b(t) \int_{\alpha(t)}^{\beta(t)} \left(\frac{\partial u}{\partial x} \right)^2 \, dx \right] \frac{\partial^2 u}{\partial x^2}\,.$$

We propose the following boundary value problem for $\widehat{L}$.

Find the real function $u(x,t)$ for $(x,t) \in \widehat{Q}$ solution of

$$\left| \begin{array}{l} \widehat{L}u(x,t) = f(x,t) \quad \text{in} \quad \widehat{Q}, \\ u(x,t) = 0 \quad \text{for} \quad (x,t) \in \widehat{\Sigma} \\ u(x,0) = u_0(x), \; \dfrac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for} \quad \alpha_0 < x < \beta_0\,. \end{array} \right. \tag{1.35}$$

With convenient hypothesis on $\alpha(t)$, $\beta(t)$, u_0, u_1, f we prove existence and uniqueness for (1.35), cf. L.A. Medeiros-J. Limaco-S.B. Menezes, Vibrations of Elastic Strings: Mathematical Aspects, J. of Computational Analysis and Applications, V.4, Nº 2, (April 2002), Part One, pp. 91-127 and Vol. 4, Nº 3, (July 2002) Part Two, pp. 211-263. □

About the model (1.14) can also be formulated as (1.32). For some results see: Tania Rabello, Maria Cristina Vieira, L.A. Medeiros, On a Perturbation of Kirchhoff Operator (submited for publication). □

INEQUALITIES

We also can formulate inequalities instead of equalities for the Kirchhoff's operator or for D'Alembert operator. We consider the case of variable ends. In fact, let us consider the notation of the initial boundary value problem (1.35).

We want to find $u(x,t)$, $u\colon \widehat{Q} \to \mathbb{R}$ solution of

$$\left| \begin{array}{l} \widehat{L}u(x,t) \geq f(x,t) \quad \text{in} \quad \widehat{Q} \\ u(x,t) = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x,0) = u_0(x), \ \dfrac{\partial u}{\partial t}(x,o) = u_1(x), \quad \text{in} \quad (\alpha_0, \beta_0) \end{array} \right. \tag{1.36}$$

In this case we consider the convex set K_t defined, for $t > 0$, by

$$K_t = \left\{ w \in H_0^1(\Omega_t). \left| \frac{\partial w}{\partial x} \right| \leq \frac{1}{\gamma(t)} \ \text{ a.e. in } \ \Omega_t \right\}.$$

To observe that $H_0^1(\Omega_t)$ is the Sobolev space on Ω_t, $\Omega_t = (\alpha(t), \beta(t))$ a section of $\widehat{Q}$ at level $t > 0$. Thus, K_t is a convex set of $H_0^1(\Omega_t)$.

With the definition of solution for (1.36), we prove that with the choice

$$u_0 \in H_0^1(\Omega_0) \cap H^2(\Omega), \qquad u_1 \in K_t \subset H_0^1(\Omega)$$

and a convenient f, we obtain a solution unique, for (1.36) defined for all $(x,t) \in \widehat{Q}$.

Remark 1. It is interesting to observe that the solution (1.35), also in the cylindrical case, there exists for $0 < t < T_0 < T$, called local solution. To obtain a global solution it is necessary to consider initial data u_0, u_1 with restrictions on their norms. Note that in the inequality (1.36) we obtain solution for all $t > 0$ with less restriction on the initial data. For detail see:

M.D.G. da Silva - L.A. Medeiros - A.C. Biazutti - Vibrations of Elastic Strings: Unilateral Problem - J. of Compt. Analysis and Applications, Vol. 8, Nº 1 (2006), pp. 53-73. $\square$