Hartman–Grobman Theorem

Gabriela Estevez

Universidade Federal do Rio de Janeiro

June 8, 2020
The Hartman–Grobman theorem or linearisation theorem is a theorem about the local behaviour of dynamical systems in a neighbourhood of the hyperbolic equilibrium point. The theorem owes its name to Philip Hartman and Vadim M. Grobman.
The Hartman–Grobman theorem or linearisation theorem is a theorem about the local behaviour of dynamical systems in a neighbourhood of the hyperbolic equilibrium point. The theorem owes its name to Philip Hartman and Vadim M. Grobman.

The theorem states that a smooth diffeomorphism $F$ is topological conjugate to $DF$, near a hyperbolic fixed point $p$, by a local homeomorphism. Hence this theorem provides invaluable information about the behaviour of orbits while they remain in a neighbourhood of the fixed point.
We say that $p$ is a **hyperbolic fixed point** of $f \in \text{Diff}^r(M)$, if $Df_p : T_pM \to T_pM$ is a hyperbolic isomorphism, that is, if $Df_p$ has no eigenvalue of modulus 1.
We say that $p$ is a **hyperbolic fixed point** of $f \in \text{Diff}^r(M)$, if $Df_p : T_pM \to T_pM$ is a hyperbolic isomorphism, that is, if $Df_p$ has no eigenvalue of modulus 1.

### Hartman–Grobman Theorem

Let $M$ be a manifold $n$–dimensional, $f \in \text{Diff}^r(M)$, $p$ be a hyperbolic fixed point of $f$ and $A = Df_p : T_pM \to T_pM$. Then there exist neighbourhoods $p \in V \subseteq M$, $0 \in U \subseteq T_pM$ and $h : V \to U$ a homeomorphism such that $h \circ f = A \circ h$. 

\[
\begin{align*}
V \subseteq M &\xrightarrow{f} M \\
U \subseteq T_pM &\xrightarrow{A} T_pM \\
\downarrow h &\quad & \quad \downarrow h \\
\downarrow &f &\quad \downarrow &
\end{align*}
\]
Proof:

Since this is a local problem, using a local chart we can assume that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism with 0 as a hyperbolic fixed point.
Proof:

Since this is a local problem, using a local chart we can assume that \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a diffeomorphism with 0 as a hyperbolic fixed point.

Since \( A = Df_0 \) is a hyperbolic isomorphism, there exists an invariant splitting \( \mathbb{R}^n = E^u \oplus E^s \) and a norm \( \| \| \) on \( \mathbb{R}^n \) in which

- \( \| A^s \| \leq a < 1 \), where \( A^s = A|_{E^s} : E^s \to E^s \),
Proof:

Since this is a local problem, using a local chart we can assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism with 0 as a hyperbolic fixed point.

Since $A = Df_0$ is a hyperbolic isomorphism, there exists an invariant splitting $\mathbb{R}^n = E^u \oplus E^s$ and a norm $\| \cdot \|$ on $\mathbb{R}^n$ in which

- $\|A^s\| \leq a < 1$, where $A^s = A|_{E^s} : E^s \to E^s$,
- $\|(A^u)^{-1}\| \leq a < 1$, where $A^u = A|_{E^u} : E^u \to E^u$. 
Let $C^0_b(\mathbb{R}^n)$ be the Banach space of bounded continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the uniform norm: $\|v\| = \sup\{\|v(x)\| : x \in \mathbb{R}^n\}$. Since $\mathbb{R}^n = E_u \oplus E_s$, we have $C^0_b(\mathbb{R}^n) = C^0_b(\mathbb{R}^n, E_s) \oplus C^0_b(\mathbb{R}^n, E_u)$.
Let $C^0_b(\mathbb{R}^n)$ be the Banach space of bounded continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the uniform norm: $\|v\| = \sup\{\|v(x)\| : x \in \mathbb{R}^n\}$.

Since $\mathbb{R}^n = E^u \oplus E^s$, we have $C^0_b(\mathbb{R}^n) = C^0_b(\mathbb{R}^n, E^s) \oplus C^0_b(\mathbb{R}^n, E^u)$. 
Let $C^0_b(\mathbb{R}^n)$ be the Banach space of bounded continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the uniform norm: $\|v\| = \sup\{\|v(x)\| : x \in \mathbb{R}^n\}$.

Since $\mathbb{R}^n = E^u \oplus E^s$, we have $C^0_b(\mathbb{R}^n) = C^0_b(\mathbb{R}^n, E^s) \oplus C^0_b(\mathbb{R}^n, E^u)$.

For all $v \in C^0_b(\mathbb{R}^n)$ we have $v = v^s + v^u$, with $v^s = \pi_s \circ v$ and $v^u = \pi_u \circ v$ obtained from the natural projections $\pi_s : E^s \oplus E^u \to E^s$ and $\pi_u : E^s \oplus E^u \to E^u$. 

Gabriela Estevez (UFRJ)
Lemma 1

There exists $\varepsilon > 0$ such that, if $\phi_1, \phi_2 \in C^0_b(\mathbb{R}^n)$ have Lipschitz constant less than or equal to $\varepsilon$, then $A + \phi_1$ and $A + \phi_2$ are conjugate.
Lemma 1

There exists $\varepsilon > 0$ such that, if $\phi_1, \phi_2 \in C^0_b(\mathbb{R}^n)$ have Lipschitz constant less than or equal to $\varepsilon$, then $A + \phi_1$ and $A + \phi_2$ are conjugate.

Lemma 2

Given $\varepsilon > 0$, there exist a neighbourhood $U$ of 0 and an extension of $f|_U$ to $\mathbb{R}^n$ of the form $A + \phi$, where $\phi \in C^0_b(\mathbb{R}^n)$ has Lipschitz constant at most $\varepsilon$. 
Let $\varepsilon > 0$ be as in Lemma 1. By Lemma 2 there exists a neighbourhood $U$, of 0, and an extension of $f|_U$ of the form $A + \phi$ where the Lipschitz constant of $\phi$ is at most $\varepsilon$. 

Proof of Hartman-Grobman Theorem

Let $\varepsilon > 0$ be as in Lemma 1. By Lemma 2 there exists a neighbourhood $U$, of 0, and an extension of $f|_U$ of the form $A + \phi$ where the Lipschitz constant of $\phi$ is at most $\varepsilon$.

By Lemma 1, there exists a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h \circ A = (A + \phi) \circ h$. Thus $h \circ A = f \circ h$, on a neighbourhood of 0 as required. ■
So, we only need to prove Lemma 1 and Lemma 2.
So, we only need to prove Lemma 1 and Lemma 2.

We start with the proof of Lemma 1.
Proof of Lemma 1

We must find a homeomorphism \( h : \mathbb{R}^n \to \mathbb{R}^n \) that satisfies
\[
h \circ (A + \phi_1) = (A + \phi_2) \circ h.
\]
Proof of Lemma 1

We must find a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ that satisfies
$h \circ (A + \phi_1) = (A + \phi_2) \circ h$.

Let us try a solution of the form $h = \text{Id} + u$, with $u \in C^0_b(\mathbb{R}^n)$.
Proof of Lemma 1

We must find a homeomorphism \( h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) that satisfies
\[
h \circ (A + \phi_1) = (A + \phi_2) \circ h.
\]

Let us try a solution of the form \( h = \text{Id} + u \), with \( u \in C^0_b(\mathbb{R}^n) \). Then we need
\[
(Id + u) \circ (A + \phi_1) = (A + \phi_2) \circ (Id + u).
\]
Proof of Lemma 1

We must find a homeomorphism \( h : \mathbb{R}^n \to \mathbb{R}^n \) that satisfies \( h \circ (A + \phi_1) = (A + \phi_2) \circ h \).

Let us try a solution of the form \( h = Id + u \), with \( u \in C^0_b(\mathbb{R}^n) \). Then we need

\[
(Id + u) \circ (A + \phi_1) = (A + \phi_2) \circ (Id + u).
\]

or equivalently,

\[
Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u). \tag{1}
\]
Proof of Lemma 1

We must find a homeomorphism \( h : \mathbb{R}^n \to \mathbb{R}^n \) that satisfies
\[
h \circ (A + \phi_1) = (A + \phi_2) \circ h.
\]

Let us try a solution of the form \( h = Id + u \), with \( u \in C^0_b(\mathbb{R}^n) \). Then we need
\[
(Id + u) \circ (A + \phi_1) = (A + \phi_2) \circ (Id + u).
\]
or equivalently,
\[
Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u). \tag{1}
\]

Let us show that there exists a unique \( u \in C^0_b(\mathbb{R}^n) \) that satisfies equation (1).
Consider the linear operator $L \in \mathcal{L}(\mathbb{R}^n)$ given by $L(u) = Au - u(A + \phi_1)$. 

Claim: $L$ is invertible and $\|L^{-1}\| < \|A^{-1}\| - a$. 

Gabriela Estevez (UFRJ)
Consider the linear operator $L \in \mathcal{L}(\mathbb{R}^n)$ given by
$L(u) = Au - u(A + \phi_1)$.

Then equation (1) becomes,

$$L(u) = \phi_1 - \phi_2(Id + u).$$
Consider the linear operator $L \in \mathcal{L}(\mathbb{R}^n)$ given by 

$$L(u) = Au - u(A + \phi_1).$$

Then equation (1) becomes,

$$L(u) = \phi_1 - \phi_2(Id + u).$$

**Claim:**

$L$ is invertible and $\|L^{-1}\| < \frac{\|A^{-1}\|}{1-a}$. 
Consider the map \( g : C^0_b(\mathbb{R}^n) \to C^0_b(\mathbb{R}^n) \) given by
\[
g(u) = L^{-1} (\phi_1 - \phi_2(Id + u)).
\]
Consider the map $g : C^0_b(\mathbb{R}^n) \rightarrow C^0_b(\mathbb{R}^n)$ given by
\[ g(u) = L^{-1} (\phi_1 - \phi_2(Id + u)). \]

Let $u_1, u_2 \in C^0_b(\mathbb{R}^n)$, then
\[
\|g(u_1) - g(u_2)\| = \|L^{-1} (\phi_1 - \phi_2(Id + u_1)) - L^{-1} (\phi_1 - \phi_2(Id + u_2))\| \\
\leq \|L^{-1} (\phi_2(Id + u_2) - \phi_2(Id + u_1))\| \\
\leq \|L^{-1}\| \|\phi_2(Id + u_2) - \phi_2(Id + u_1)\| \\
\leq \frac{\|A^{-1}\|}{1 - a} \varepsilon \|u_2 - u_1\|.
\]
Consider the map \( g : C^0_b(\mathbb{R}^n) \to C^0_b(\mathbb{R}^n) \) given by

\[
g(u) = L^{-1}(\phi_1 - \phi_2(Id + u)).
\]

Let \( u_1, u_2 \in C^0_b(\mathbb{R}^n) \), then

\[
\|g(u_1) - g(u_2)\| = \|L^{-1}(\phi_1 - \phi_2(Id + u_1)) - L^{-1}(\phi_1 - \phi_2(Id + u_2))\|
\leq \|L^{-1}(\phi_2(Id + u_2) - \phi_2(Id + u_1))\|
\leq \|L^{-1}\| \|\phi_2(Id + u_2) - \phi_2(Id + u_1)\|
\leq \frac{\|A^{-1}\|}{1 - a} \varepsilon \|u_2 - u_1\|.
\]

If \( \varepsilon \) is small enough to make \( \frac{\|A^{-1}\|}{(1-a)} \varepsilon < 1 \), then \( g \) is a contraction and so has a unique fixed point \( u \in C^0_b(\mathbb{R}^n) \).
i.e., $u = g(u) = L^{-1}(\phi_1 - \phi_2(Id + u))$, or equivalently, $L(u) = Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u)$. And hence, $u$ is solution of (1). Finally, we need to prove that $Id + u$ is a homeomorphism. Since the homeomorphism that we are looking for also satisfies $(A + \phi_1) \circ h = h \circ (A + \phi_2)$, we can repeat the same method for $h = Id + v$. We get that the equation $Av - v(A + \phi_2) = \phi_2 - \phi_1(Id + v)$, (2) has a unique solution $v \in C_0^b(R^n)$. 

Gabriela Estevez (UFRJ)
i.e., \( u = g(u) = L^{-1} (\phi_1 - \phi_2 (Id + u)) \), or equivalently,
\[
L(u) = Au - u(A + \phi_1) = \phi_1 - \phi_2 (Id + u).
\]
And hence, \( u \) is solution of (1).
i.e., \( u = g(u) = L^{-1}(\phi_1 - \phi_2(Id + u)) \), or equivalently, 
\[
L(u) = Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u).
\]
And hence, \( u \) is solution of (1).

Finally, we need to prove that \( Id + u \) is a homeomorphism.
i.e., \( u = g(u) = L^{-1}(\phi_1 - \phi_2(Id + u)) \), or equivalently,
\[
L(u) = Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u).
\]

And hence, \( u \) is solution of (1).

Finally, we need to prove that \( Id + u \) is a homeomorphism.

Since the homeomorphism that we are looking for also satisfies
\[
(A + \phi_1) \circ h = h \circ (A + \phi_2),
\]
i.e., \( u = g(u) = L^{-1}(\phi_1 - \phi_2(Id + u)) \), or equivalently, \( L(u) = Au - u(A + \phi_1) = \phi_1 - \phi_2(Id + u) \).

And hence, \( u \) is solution of (1).

Finally, we need to prove that \( Id + u \) is a homeomorphism.

Since the homeomorphism that we are looking for also satisfies \((A + \phi_1) \circ h = h \circ (A + \phi_2)\), we can repeat the same method for \( h = Id + v \). We get that the equation

\[
Av - v(A + \phi_2) = \phi_2 - \phi_1(Id + v),
\]

has a unique solution \( v \in C^0_b(\mathbb{R}^n) \).
We claim that 
\((ld + u)(ld + v) = (ld + v)(ld + u) = ld\).
We claim that \((ld + u)(ld + v) = (ld + v)(ld + u) = ld\).

In fact, from equations

\[(ld + u)(A + \phi_1) = (A + \phi_2)(ld + u)\]

and

\[(A + \phi_1)(ld + v) = (ld + v)(A + \phi_2)\]

we have,

\[(ld + u)(ld + v)(A + \phi_2) = (ld + u)(A + \phi_1)(ld + v)\]

\[= (A + \phi_2)(ld + u)(ld + v).\]
We write \((ld + u)(ld + v) = ld + v + u(ld + v) = ld + w\), where \(w = v + u(ld + v) \in C^0_b(\mathbb{R}^n)\).
We write \((ld + u)(ld + v) = ld + v + u(ld + v) = ld + w\), where 
\[w = v + u(ld + v) \in \mathcal{C}_b^0(\mathbb{R}^n).\]

Since \((ld + w)(A + \phi_2) = (A + \phi_2)(ld + w)\) has unique solution and 
\(ld(A + \phi_2) = (A + \phi_2)ld\), then \(ld = ld + w = (ld + u)(ld + v)\).
We write \((ld + u)(ld + v) = ld + v + u(ld + v) = ld + w\), where \(w = v + u(ld + v) \in C^0_b(\mathbb{R}^n)\).

Since \((ld + w)(A + \phi_2) = (A + \phi_2)(ld + w)\) has unique solution and \(ld(A + \phi_2) = (A + \phi_2)ld\), then \(ld = ld + w = (ld + u)(ld + v)\).

Similarly we have \((ld + v)(ld + u) = ld\). This shows that \(ld + u\) is a homeomorphism which conjugates \(A + \phi_1\) and \(A + \phi_2\), proving the lemma. ■
Now, we will prove Lemma 2:
Now, we will prove Lemma 2:

**Lemma 2**

Given $\varepsilon > 0$, there exist a neighbourhood $U$ of 0 and an extension of $f|_U$ to $\mathbb{R}^n$ of the form $A + \phi$, where $\phi \in C^0_b(\mathbb{R}^n)$ has Lipschitz constant at most $\varepsilon$. 
Proof of Lemma 2

Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^\infty \) function such that:

1. \( \alpha(t) = 0, \) if \( t \geq 1; \)

And let \( \psi \in L(\mathbb{R}^n, \mathbb{R}^n) \) such that \( f = A + \psi, \) where \( \psi(0) = 0 \) and \( D\psi_0 = 0. \)

Let \( r > 0 \) such that \( \|D\psi_x\| < \epsilon/2 \) for \( x \in B_r(0). \)
Proof of Lemma 2

Let $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ be a $C^\infty$ function such that:

1. $\alpha(t) = 0$, if $t \geq 1$;
2. $\alpha(t) = 1$, if $t < 1/2$;
Proof of Lemma 2

Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^\infty \) function such that:

1. \( \alpha(t) = 0, \) if \( t \geq 1; \)
2. \( \alpha(t) = 1, \) if \( t < 1/2; \)
3. \( |\alpha'(t)| < K, \) with \( K > 2, \) for all \( t \in \mathbb{R}. \)
Proof of Lemma 2

Let \( \alpha : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that:

1. \( \alpha(t) = 0 \), if \( t \geq 1 \);
2. \( \alpha(t) = 1 \), if \( t < 1/2 \);
3. \( |\alpha'(t)| < K \), with \( K > 2 \), for all \( t \in \mathbb{R} \).
Proof of Lemma 2

Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^\infty \) function such that:

1. \( \alpha(t) = 0 \), if \( t \geq 1 \);
2. \( \alpha(t) = 1 \), if \( t < 1/2 \);
3. \( |\alpha'(t)| < K \), with \( K > 2 \), for all \( t \in \mathbb{R} \).

And let \( \psi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) such that \( f = A + \psi \), where \( \psi(0) = 0 \) and \( D\psi_0 = 0 \).
Proof of Lemma 2

Let \( \alpha : \mathbb{R} \to \mathbb{R} \) be a \( C^\infty \) function such that:

1. \( \alpha(t) = 0 \), if \( t \geq 1 \);
2. \( \alpha(t) = 1 \), if \( t < 1/2 \);
3. \( |\alpha'(t)| < K \), with \( K > 2 \), for all \( t \in \mathbb{R} \).

And let \( \psi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n) \) such that \( f = A + \psi \), where \( \psi(0) = 0 \) and \( D\psi_0 = 0 \).

Let \( r > 0 \) such that \( \|D\psi_x\| < \varepsilon/2K \) for \( x \in B_r(0) \).
Define $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\phi(x) = \alpha \left( \frac{\|x\|}{r} \right) \psi(x)$. 

Note that $\phi(0) = 0$ and $\phi(x) = 0$ if $\|x\| > r$.

Since $\phi(x) = \psi(x)$ for $\|x\| < r/2$ (in this case $\alpha(\|x\|) = 1$), then $A + \phi$ is an extension of $A + \psi = f$ in $B_{r/2}(0)$.

We only need to prove that $\phi \in C^0_b(\mathbb{R}^n)$ has Lipschitz constant at most $\epsilon > 0$. 
Define $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\phi(x) = \alpha \left( \frac{\|x\|}{r} \right) \psi(x)$.

Note that $\phi(0) = 0$ and $\phi(x) = 0$ if $\|x\| > r$. 
Define $\phi : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi(x) = \alpha \left( \frac{\|x\|}{r} \right) \psi(x)$.

Note that $\phi(0) = 0$ and $\phi(x) = 0$ if $\|x\| > r$.

Since $\phi(x) = \psi(x)$ for $\|x\| < r/2$ (in this case $\alpha(\|x\|) = 1$), then $A + \phi$ is an extension of $A + \psi = f$ in $B_{r/2}(0)$. 

We only need to prove that $\phi \in C^0_b(\mathbb{R}^n)$ has Lipschitz constant at most $\varepsilon > 0$. 
Define $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\phi(x) = \alpha \left( \frac{\|x\|}{r} \right) \psi(x)$.

Note that $\phi(0) = 0$ and $\phi(x) = 0$ if $\|x\| > r$.

Since $\phi(x) = \psi(x)$ for $\|x\| < r/2$ (in this case $\alpha(\|x\|) = 1$), then $A + \phi$ is an extension of $A + \psi = f$ in $B_{r/2}(0)$.

We only need to prove that $\phi \in C^0_b(\mathbb{R}^n)$ has Lipschitz constant at most $\varepsilon > 0$. 
On the other hand, if $x, y \in B_r(0)$ then

$$
\| \phi(x) - \phi(y) \| = \left\| \left[ \alpha \left( \frac{\|x\|}{r} \right) \psi(x) - \alpha \left( \frac{\|y\|}{r} \right) \psi(y) \right] \right\|
$$

$$
\leq \left| \alpha \left( \frac{\|x\|}{r} \right) - \alpha \left( \frac{\|y\|}{r} \right) \right| \| \psi(x) \|
$$

$$
+ \left| \alpha \left( \frac{\|y\|}{r} \right) \right| \| \psi(x) - \psi(y) \|
$$

$$
\leq K \frac{r}{r} \| x - y \| \| \psi(x) - \psi(0) \| + \frac{\varepsilon}{2K} \| x - y \|
$$

$$
\leq K \frac{r}{r} \| x - y \| \frac{\varepsilon}{2K} \| x \| + \frac{\varepsilon}{2K} \| x - y \|
$$

$$
\leq \frac{\varepsilon}{2} \| x - y \| + \frac{\varepsilon}{4} \| x - y \|
$$

$$
\leq \varepsilon \| x - y \|.
$$
If \( x \in B_r(0) \) and \( y \notin B_r(0) \), from the previous inequalities we get
\[
\| \phi(x) - \phi(y) \| \leq \varepsilon \| x - y \|.
\]
If $x \in B_r(0)$ and $y \notin B_r(0)$, from the previous inequalities we get
\[ \| \phi(x) - \phi(y) \| \leq \varepsilon \| x - y \|. \]

And if $x, y \notin B_r(0)$ then $\alpha(\| x \| / r) = 0 = \alpha(\| y \| / r)$, and hence
\[ \| \phi(x) - \phi(y) \| = 0 \leq \varepsilon \| x - y \|. \]
If \(x \in B_r(0)\) and \(y \notin B_r(0)\), from the previous inequalities we get
\[
\|\phi(x) - \phi(y)\| \leq \varepsilon \|x - y\|.
\]

And if \(x, y \notin B_r(0)\) then \(\alpha(\|x\|/r) = 0 = \alpha(\|y\|/r)\), and hence
\[
\|\phi(x) - \phi(y)\| = 0 \leq \varepsilon \|x - y\|.
\]

Thus \(\phi\) has Lipschitz constant at most \(\varepsilon\). ■
What are the conditions to obtain a differentiable conjugacy at the hyperbolic fixed point?
2003 Misha Guysinsky, Boris Hasselblatt, and Victoria Rayskin, proved that for a $C^\infty$ diffeomorphism $F$, the local homeomorphism is differentiable at the fixed point.
1. 2003 Misha Guysinsky, Boris Hasselblatt, and Victoria Rayskin, proved that for a $C^\infty$ diffeomorphism $F$, the local homeomorphism is differentiable at the fixed point.

2. 2017 Wenmeng Zhang, Kening Lu and Weinian Zhang: if $F$ is a $C^1$ diffeomorphism and $DF$ is $\alpha$-Holder continuous at the fixed point then the local homeomorphism $h$ is differentiable at the fixed point. They also gave a counterexample showing that the regularity condition on $F$ can not be lowered to $C^1$. 
References I


