Global existence and decay for the semilinear thermoelastic contact problem

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Received 9 August 2000; received in revised form 16 August 2000; accepted 4 April 2002

Abstract

In this paper a class of semilinear thermoelastic contact problems is considered and the existence and exponential decay of the weak solutions are obtained. © 2002 Elsevier Science (USA). All rights reserved.

MSC: 35B40; 35L70; 73B30

Keywords: Contact problem; Global existence of weak solution; Semilinear thermoelastic problem

1. Introduction

In this paper, we study the small longitudinal deformation along the x-axis of a one-dimensional semilinear thermoelastic rod, when the body fixed at \( x = 0 \) and unilaterally constrained at \( x = 1 \). We suppose that the expansion and contraction are due to thermal effects and body forces. Problems involving thermoelastic contact arise naturally in many situations (see \([2,4]\)). Particularly, those involving industrial processes when two or more materials may come in contact or may lose contact as a result of

\*This work was supported by NNSF of China No. 10001018, Natural Science Foundation of Jiangsu Province No. BK2001108, Scientific Research Foundation for Returned Overseas Chinese Scholar of Jiangsu Education Commission and a grant from CNPq, Brazil.

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thermoelastic expansion or contraction. Such thermoelastic phenomenon can be divided into three parts: static, quasistatic and full dynamics.

The quasistatic and static cases with various boundary conditions have been widely studied in [1–4,7,13–15], both numerically and theoretically. Various kinds of existence, uniqueness and stability results are established. These papers contain a variety of linear and nonlinear boundary conditions but in each case the problem involves both a single temperature and a single displacement, so that reformulation leads to one nonlinear equation for a single temperature.

By contrast, the fully dynamic problem is different from that of the quasistatic case. The quasistatic system can be viewed as a mixed elliptic–parabolic type, while the dynamic case is a mixed hyperbolic–parabolic type. This latter case is more complicated. There are few results which only concern the existence. In [5,12], the authors consider the linear equations with contact conditions (Signorin’s contact conditions). In [5], the authors considered unilaterally constrained at \( x = 1 \), only the existence of weak solution was obtained. In [12], the authors considered the case of two rods, both existence and exponential decay of weak solution were obtained.

We study the case in which the obstacle can be deformed so it is possible that there exists a penetration. We assume that there is friction in the interaction between the bar and obstacle, see Fig. 1.

In these conditions the displacement \( u \) can satisfy either \( u < \alpha \) or \( u > \alpha \). The corresponding equations for this situation is given by

\[
\begin{align*}
\ddot{u} - \eta \dot{u}_x + m \dot{\theta}_x + N_1(u, \theta) &= f(x) \quad \text{in} \ (0, 1) \times (0, T), \\
\dot{\theta} - k \theta_x + m u_t + N_2(u, \theta) &= g(x) \quad \text{in} \ (0, 1) \times (0, T).
\end{align*}
\]  

The initial conditions are given by

\[
\begin{align*}
u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & \theta(x, 0) &= \theta_0(x).
\end{align*}
\]  

Fig. 1.
The boundary conditions on the contact side are given by
\[ Z\frac{\partial u_x}{\partial x}(1, t) - m\frac{\partial \theta(1, t)}{\partial t} = -d[(u(1, t) - z)^+]^\mu \]
\[ - b[(u(1, t) - z)^+]^\nu u_x(1, t), \]
\[ k\frac{\partial \theta_x}{\partial t}(1, t) = -\beta\theta(1, t), \] (1.4)
while at the end \( x = 0 \) we have
\[ u(0, t) = 0, \quad \theta_x(0, t) = 0. \] (1.5)

Here \( \eta, m, k, d, b, \beta \) are positive constants, \( \mu \geq 1, \nu \geq 0 \). For the sake of simplicity, we assume that \( f \) and \( g \) are given functions depending only on \( x \).

The nonlinear terms \( N_1(u, \theta) \) and \( N_2(u, \theta) \) satisfy:
\[ N_1(u, \theta) = N_{11}(u) + N_{12}(\theta), \]
\[ N_{11} \in C^1(\mathbb{R}), \quad N_{12} \in C^1_b(\mathbb{R}), \quad N_{11}(u)u \geq 0, \quad 0 \leq N_{12}'(\theta) \leq M_1, \]
\[ N_{12}(0) = 0, \] (1.7)
\[ N_2(u, \theta) = N_{21}(u) + N_{22}(\theta), \]
\[ N_{21}(u) \in C^0,1(\mathbb{R}), \quad N_{22}(\theta) \in C^1_b(\mathbb{R}), \quad N_{22}'(\theta) \geq 0, \]
\[ |N_{11}(u)| \leq M_0|u|, \quad N_{11}(u)u \geq v \int_0^u N_{11}(s) \, ds \quad (v > 0 \text{ is a constant}), \]
\[ |N_{21}(u)| \leq M_2|u|, \quad |N_{22}(\theta)| \leq M_3|\theta|, \] (1.8)
where \( M_i \) (\( i = 0, \ldots, 4 \)) are positive constants.

In this paper, we will show the existence of weak solutions for problem (1.1)–(1.6) under conditions (1.7) and (1.8). Moreover, we obtain the exponential decay for (1.1)–(1.6) under conditions (1.7)–(1.10) provided \( M_1 \) and \( M_2 \) are small constants when \( f = g = 0 \). To show the existence of solution we use the monotonicity method, compactness method and some technical arguments. In [6], the authors obtained the uniqueness for the contact problem of thermoviscoelastic problem using the standard method, but the uniqueness is an open problem for thermoelastic contact problem [10,11]. The exponential decay of the weak solution for semilinear thermoelastic contact problem is very interesting and also very difficult. First, we obtain the exponential decay for the strong solution of (1.1)–(1.6) by the multiplier method and constructing generalized Lyapunov functional, then we obtain the exponential decay for the weak solution of (1.1)–(1.6) by lower semicontinuity of the norm. To our knowledge, the present paper is the first attempt to investigate the exponential decay of weak solution for semilinear thermoelastic contact problems.

The organization of the remaining parts of this paper is as follows. In Section 2 we prove the existence of strong solutions to system (1.1)–(1.6) under conditions (1.7) and (1.8) and the continuous dependence of the
strong solution of (1.1)–(1.6) in $H^1 \times L^2 \times L^2$. In Section 3 we show the existence of weak strong solution and to (1.1)–(1.6) provided (1.7)–(1.8) hold. In Section 4, we prove the exponential decay of weak solution to (1.1)–(1.6) provided (1.7)–(1.10) hold and when $M_2$ and $M_2$ are a small positive constant and $f = g = 0$.

2. Existence of strong solution for (1.1)–(1.6)

In this section, we show the existence of strong solution to the semilinear thermoelastic contact problem given by (1.1)–(1.6) under conditions (1.7)–(1.8). We denote the norm of $L^2(0,1)$ by $\| \cdot \|$.

**Theorem 2.1.** Let us take $f, g \in L^2(0,1)$, $\alpha > 0$. Suppose that $N_1(u, \theta)$ and $N_2(u, \theta)$ satisfy (1.7) and (1.8) and $(u_0, u_1, \theta_0) \in H^2(0,1) \times H^1(0,1) \times H^2(0,1)$ are compatible with the boundary conditions (1.4)–(1.6). Then for any $T > 0$, there exists a unique solution $(u, \theta)$ of (1.1)–(1.6) satisfying

$$\partial_t^j u \in L^\infty(0,T; H^{2-j}(0,1)), \quad j = 0,1,2,$$

$$\partial_t^j \theta \in L^\infty(0,T; H^{2-2j}(0,1)),$$

$$\theta_{xt} \in L^\infty(0,T; L^2(0,1)).$$

Moreover, we have

$$E(t; u, \theta) + \int_0^T \int_0^1 |\partial_x \theta|^2 \, dx \, dt + \int_0^T |\theta(1,t)|^2 \, dt$$

$$+ \int_0^T [(u(1,t) - z)^+] |u_t(t)|^2 \, dt$$

$$\leq C \left( \int_0^1 (|f|^2 + |g|^2) \, dx + E(0; u, \theta) \right), \quad (2.1)$$

$$||u^1 - u^2||_x^2 + ||u^1_t - u^2_t||^2 + ||\theta^1 - \theta^2||^2$$

$$\leq \bar{C} \left( ||u_0^1 - u_0^2||_x^2 + ||u^1_0 - u^2_0||^2 + ||\theta^1_0 - \theta^2_0||^2 \right), \quad (2.2)$$

where

$$E(t; u, \theta) = \int_0^1 \left( |u_t|^2 + \eta |u_x|^2 + |\theta|^2 + \int_0^u N_{11}(s) \, ds \right) \, dx$$

$$+ \frac{d}{\mu + 1} [(u(1,t) - z)^+]^{\mu+1},$$

$(u^1, u^1_t, \theta^1)$ and $(u^2, u^2_t, \theta^2)$ are two solutions of (1.1)–(1.6) with initial values $(u_0^1, u_1^1, \theta_0^1)$ and $(u_0^2, u_1^2, \theta_0^2)$, respectively, $C$ is a positive constant depending on $T$ and $\bar{C}$ is a constant depending only on $T$ and $\| (u_0^1, u_1^1, \theta_0^1) \|_{H^1 \times L^2 \times L^2}$. 

**Proof.** We only give the proof of (2.1) and the continuous dependence (2.2) of strong solutions of (1.1)–(1.6) in $H^1(0,1) \times L^2(0,1) \times L^2(0,1)$, the proof of the global existence in $H^2(0,1) \times H^1(0,1) \times H^2(0,1)$ is standard (by Faedo–Galerkin method [9]), we could refer to [12].

Multiplying (1.1) resp. (1.2) by $u_t$ and $\theta$ resp., and performing an integrating by parts over $[0,1]$, we get

$$
\frac{1}{2} \frac{d}{dt} E(t; u, \theta) = -k \int_0^1 |\theta_x|^2 dx - b[(u(1,t) - x)^+][u_t(1,t)]^2 - \beta |\theta(1,t)|^2
$$

$$
+ \int_0^1 f u_t dx + \int_0^1 g \theta dx - \int_0^1 N_{12}(\theta) u_t dx
$$

$$
- \int_0^1 N_{21}(u) \theta dx - \int_0^1 N_{22}(\theta) \theta dx.
$$

By (1.7)–(1.8) and the Cauchy–Schwarz's inequality, we have

$$
\frac{d}{dt} E(t; u, \theta) + k \int_0^1 |\theta_x|^2 dx + \beta |\theta(1,t)|^2
$$

$$
\leq \int_0^1 (|f|^2 + |g|^2) dx + C E(t; u, \theta),
$$

where $C$ is a positive constant. Using Gronwall’s inequality, (2.1) follows.

Next, we prove the continuous dependence of the strong solution of (1.1)–(1.6) in $H^1(0,1) \times L^2(0,1) \times L^2(0,1)$. Let $(u^1, u^1_t, \theta^1)$ and $(u^2, u^2_t, \theta^2)$ be two solutions to the system (1.1)–(1.6) with the initial value $(u_0^1, u_0^1_t, \theta_0^1)$ and $(u_0^2, u_0^2_t, \theta_0^2)$, respectively. Let us denote by $U = u^1 - u^2$, $\Theta = \theta^1 - \theta^2$, then $(U, \Theta)$ satisfies:

$$
U_t - \eta U_{xx} + m \Theta_x + N_1(u^1_t, \theta^1) - N_1(u^2_t, \theta^2) = 0
$$

in $(0,1) \times (0,T)$, (2.4)

$$
\Theta_t - k \Theta_{xx} + m U_{xt} + N_2(u^1_t, \theta^1) - N_2(u^2_t, \theta^2) = 0
$$

in $(0,1) \times (0,T)$. (2.5)

The initial conditions are

$$
U(x,0) = u_0^1 - u_0^2, \quad U_t(x,0) = u_0^1_t - u_0^2_t, \quad \Theta(x,0) = \theta_0^1 - \theta_0^2.
$$

(2.6)

The boundary conditions on the contact side are given by

$$
\eta U_x(1,t) - m \Theta(1,t) = -d \{(u^1(1,t) - x)^+\} - (u^2(1,t) - x)^+\}
$$

$$
- b \{(u^1(1,t) - x)^+\} u^1_t(1,t)
$$

$$
- (u^2(1,t) - x)^+\} u^2_t(1,t),
$$

(2.7)

$$
k \Theta_x(1,t) = -\beta \Theta(1,t),
$$

(2.8)
while at the end \( x = 0 \) we have

\[
U(0, t) = 0, \quad \Theta_x(0, t) = 0. \tag{2.9}
\]

Now, we establish a lemma which will be used very frequently in what follows.

**Lemma 2.2.** If \( u(x) \in K_0 = \{ u \in H^1(0, 1): u(0) = 0 \} \), then

\[
\int_0^1 |u(x)|^2 \, dx \leq 4 \int_0^1 |u_x(x)|^2 \, dx, \quad \|u(x)\|_{C[0,1]}^2 \leq 4 \int_0^1 |u_x(x)|^2 \, dx.
\]

**Proof.** Since \( u(x) \in K_0 \), the proof can be obtained using \( u^2(x) = 2 \int_0^x u(x)u_x(x) \, dx \).

In order to obtain the continuous dependence of the strong solution of (1.1)–(1.6) in \( H^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \) we give the following lemma about the boundary regularity for the solution of wave equation in one dimension.

**Lemma 2.3.** Let \( q(x) \in C^1[l_1, l_2] \) and \( f_i \in L^2(0, T; L^2(l_1, l_2)) \). Then for any solution \( v \) such that \( \partial_t^2 v \in L^2(0, T; H^2(l_1, l_2))(j = 0, 1, 2) \) of the equation

\[
v_{tt} - \eta v_{xx} = f_i,
\]

we have

\[
-\frac{d}{dt} \int_{l_1}^{l_2} q(x)v_t v_x \, dx = -\frac{q(x)}{2}(v_t^2(x, t) + \eta v_x^2(x, t))|_{l_1}^{l_2} + \frac{1}{2} \int_{l_1}^{l_2} q'(x)(v_t^2 + \eta v_x^2) \, dx - \int_{l_1}^{l_2} q(x)v_x f_1 \, dx.
\]

To show the above identity, we multiply the equation by \( qv_x \) then we use integrations by parts.

Now, first applying Lemma 2.3 to Eq. (1.1), that is \( (q, l_1, l_2, f_i) = (x, 0, 1, f - N_1(u, \theta) - m\theta_x) \), we get

\[
u_t(1, t)^2 + \eta u_x(1, t)^2 \leq 2 \frac{d}{dt} \int_0^1 xu_t u_x \, dx
\]

\[
+ C_1 \int_0^1 (u_t^2 + u_x^2 + \theta^2 + \theta_x^2 + f_t^2) \, dx, \quad t \in [0, T].
\]

Lemma 2.2 and inequality (2.1) are used here, where \( C_1 \) is a constant that depends on the coefficients in (1.1). Integrating the above inequality over
[0, T] we get
\[ \int_0^T (u_t^2(1, t) + u_x^2(1, t)) \, dt \leq C_2, \]  
(2.10)
where \( C_2 \) is a constant which depends on \( T \) and \( \| (u_0, u_1, \theta_0) \|_{H^1 \times L^2 \times L^2} \).

Then, applying Lemma 2.3 to (2.4), we have \( q(x) = x, \ l_1 = 0, \ l_2 = 1 \) and \( f_1 = -m \Theta_x - N_1(u^1, \theta^1) + N_1(u^2, \theta^2) \) we get
\[ \frac{1}{2} (U_t^2(1, t) + \eta U_x^2(1, t)) \leq \frac{d}{dt} \int_0^1 x U_x \, dx \]
\[ + C_3 \int_0^1 (U_t^2 + U_x^2 + \Theta^2) \, dx \]
\[ + \varepsilon \int_0^1 \Theta_x^2 \, dx, \]  
(2.11)
where \( C_3 \) is a constant which depends on \( T \) and \( \| (u_0, u_1, \theta_0) \|_{H^1 \times L^2 \times L^2} \) and \( \varepsilon \).

Multiply (2.4) resp. (2.5) by \( U_t \) and \( \Theta \) resp., and integrating over \( (0, 1) \), recalling the boundary conditions, and adding the resulting equalities, we get
\[ \frac{1}{2} \frac{d}{dt} \Phi(t) + k \int_0^1 \Theta_x^2 + \beta |\Theta(1, t)|^2 \leq C_4 \Phi(t) + \frac{1}{4} |U_t(1, t)|^2 \]
\[ + C |u_t^1(1, t)|^2 \int_0^1 \Theta_x^2 \, dx, \]
where \( \Phi(t) = \int_0^1 (U_t^2 + U_x^2 + \Theta^2) \, dx \) and where \( C_4 \) is a constant which depends on \( T \) and \( \| (u_0, u_1, \theta_0) \|_{H^1 \times L^2 \times L^2} \). By Gronwall’s inequality and (2.10), we have
\[ \Phi(t) + 2k \int_0^t \int_0^1 \Theta_x^2 \, dx \leq \frac{1}{2} e^{C(t)} \int_0^t |U_t(1, s)|^2 \, ds. \]
By (2.11) we have
\[ \Phi(t) + 2k \int_0^t \int_0^1 \Theta_x^2 \, dx \leq \frac{1}{2} e^{2C(t)} \Phi(t) + C_5 e^{2C(t)} \int_0^t \Phi(s) \, ds \]
\[ + e^{2C(t)} \varepsilon \int_0^t \int_0^1 \Theta_x^2 \, dx, \]
where \( C(t) = C_6 (t + \int_0^1 |u_t^1(1, s)|^2 \, ds) \) and \( C_5 \) and \( C_6 \) are constants which depend on \( T \) and \( \| (u_0, u_1, \theta_0) \|_{H^1 \times L^2 \times L^2} \). Now, let \( \varepsilon \) and \( t_0 \) be small enough, then there is \( \lambda > 0 \) such that
\[ \frac{1}{2} e^{2C(t)} \leq 1 - \lambda, \quad e^{2C(t)} \varepsilon \leq 2k. \]
So, we get
\[ \lambda \Phi(t) \leq C_5 e^{2C(t)} \int_0^t \Phi(s) \, ds, \quad t \in [0, t_0]. \]
By Gronwall’s inequality, (2.2) is proved on $[0,t_0]$. Repeating the above procedure step by step, (2.2) is obtained on $[0,T]$ for any $T > 0$.

The uniqueness of the strong solution of (1.1)–(1.6) can be obtained by (2.2) easily. By now, the proof of Theorem 2.1 is completed. □

Remark 1. From the proof, we can see that the constant $C$ in (2.2) depends only on $T$ and the right-hand side of (2.1), that is, it depends only on $T$ and the norm of $\| (u_0,u_1,\theta_0) \|_{H^1 \times L^2 \times L^2}$. Here $f$ and $g$ are independent of $t$ only for simplicity.

Remark 2. From Theorem 2.1, we have

$$[(u(1,t) - z)^+] \leq C, \quad \text{for } t \in [0,T],$$

$$\int_0^T [(u(1,t) - z)^+] |u_t(1,t)|^2 \, dt \leq C,$$  \hspace{1cm} (2.12)

and it will be used in the next section.

3. Existence of weak solutions for (1.1)–(1.6)

In this section, we show the existence of weak solutions to the semilinear thermoelastic contact problem given by (1.1)–(1.6) under conditions (1.7)–(1.8).

Theorem 3.1. Let us take $f, g \in L^2(0,1)$, $\alpha > 0$. Suppose that $N_1(u,\theta)$ and $N_2(u,\theta)$ satisfy (1.7) and (1.8) and $(u_0,u_1,\theta_0) \in K_0 \times L^2(0,1) \times L^2(0,1)$, then for any $T > 0$, system (1.1)–(1.6) has at least one solution $(u,\theta)$ satisfying:

$$u \in L^\infty(0,T;K_0), \quad u_t \in L^\infty(0,T;L^2(0,1)),$$

$$\theta \in L^\infty(0,T;L^2(0,1)) \cap L^2(0,T;H^1(0,1)),$$

where $K_0 = \{ u : u \in H^1(0,1), u(0) = 0 \}$.

Proof. We use regularization technique to prove the existence of weak solution for (1.1)–(1.6). First, we regularize the initial value, that is, there exist $(u^n_0,u^n_1,\theta^n_0) \in H^2(0,1) \times H^1(0,1) \times H^2(0,1)$ which are compatible with the boundary conditions (1.4)–(1.6) and satisfy

$$\| u^n_0 - u_0 \|_{H^1} \to 0, \quad \| u^n_1 - u_1 \|_{L^2} \to 0, \quad \| \theta^n_0 - \theta_0 \|_{L^2} \to 0 \quad \text{as } n \to \infty, \quad (3.1)$$

then we have the regularized sequence $(u^n, \theta^n_0, \theta^n_1)$ of solution for (1.1)–(1.6) with initial value $(u^n_0, u^n_1, \theta^n_0)$ by Theorem 2.1.

By (2.1), we may extract a subsequence of $(u^n, \theta^n_0, \theta^n_1)$, which we still denote in the same way, such that when $n \to \infty$

$$u^n \rightharpoonup u \quad \text{weak-} \star \quad \text{in } L^\infty(0,T;K_0),$$

$$u^n_t \rightharpoonup u_t \quad \text{weak-} \star \quad \text{in } L^\infty(0,T;L^2(0,1)),$$
\[
\theta^n \to \theta \text{ weak-} \star \text{ in } L^\infty(0, T; L^2(0, 1)),
\]
\[
\theta^n_t \to \theta_t \text{ weakly in } L^2(0, T; H^{-1}(0, 1)),
\]
\[
\theta^n \to \theta \text{ weak-} \star \text{ in } L^\infty(0, T; L^2(0, 1)),
\]
\[
\theta^n(1, t) \to \theta(1, t) \text{ weakly in } L^2(0, T),
\]
\[
(u^n(1, t) - x)^+ \to \chi \text{ weakly in } L^p(0, T) \quad \forall 1 < p < \infty.
\]

By the above property and [8, Lemma 1.4], we have
\[
u^n \to u \text{ in } C([0, T], L^2(0, 1)) \quad \text{as } n \to \infty.
\]
Therefore, it follows that,
\[
u^n \to u, \quad \text{a.e. } (u^n(1, t) - x)^+ \to (u(1, t) - x)^+ \quad \text{a.e. as } n \to \infty. \quad (3.2)
\]
By (2.12), we know
\[
[(u^n(1, t) - x)^+]^m \to \chi_1 \quad \text{weakly in } L^p(0, T) \quad \forall 1 < p < \infty \quad \text{as } n \to \infty.
\]
Using [9, Lemma 1.3], and (3.2), we have
\[
[(u^n(1, t) - x)^+]^m - [(u(1, t) - x)^+]^m
\]
weakly— in \(L^p(0, T) \quad \forall 1 < p < \infty \quad \text{as } n \to \infty. \quad (3.3)
\]
Now, let us consider the weak convergence of \((u^n(1, t) - x)^+\)\(\nu^n_t(1, t)\). By Theorem 2.1 and (2.12), we have
\[
\int_0^T \left[ \frac{d}{dt} [(u^n(1, t) - x)^+]^{l+1} \right]^2 \, dt \leq C,
\]
which implies that
\[
\frac{d}{dt} [(u^n(1, t) - x)^+]^{l+1} \to \chi_2 \quad \text{in } L^2(0, T) \text{ and in } \mathcal{D}'(0, T).
\]
Therefore,
\[
\lim_{n \to \infty} \int_0^T \frac{d}{dt} [(u^n(1, t) - x)^+]^{l+1} v \, dt = \int_0^T \chi_2 v \, dt
\]
for any \(v \in C_0^\infty(0, T)\). On the other hand
\[
\int_0^T \frac{d}{dt} [(u^n(1, t) - x)^+]^{l+1} v \, dt = - \int_0^T [u^n(1, t) - x)^+]^{l+1} v_t \, dt.
\]
Using similar arguments to those used to get (3.3), we have
\[
[(u^n(1, t) - x)^+]^{l+1} - [(u(1, t) - x)^+]^{l+1}
\]
weakly in \(L^p(0, T) \quad \forall p \in (1, \infty)\).
So, we get
\[
- \lim_{n \to \infty} \int_0^T [(u^n(1, t) - x)^+]^{l+1} v_t \, dt = - \int_0^T [(u(1, t) - x)^+]^{l+1} v_t \, dt
\]
\[
= \int_0^T \frac{d}{dt} [(u(1, t) - x)^+]^{l+1} v \, dt.
\]
Hence
\[ \chi_2 = \frac{d}{dt} [(u(t, t) - \alpha)^+]^{1+1}. \]

So we have
\[ [(u^n(t, t) - \alpha)^+]^{1+1}u^n_t(t, t) - [(u(t, t) - \alpha)^+]^{1+1}u_t(t, t), \ n \to \infty \]
weakly in $L^2(0, T)$. \hfill (3.4)

With the same arguments as used to obtain (3.3), we can easily see that
\[ N_{11}(u^n) \to N_{11}(u), \quad N_{21}(u^n) \to N_{21}(u) \text{ weak- */} \]
in $L^\infty(0, T; L^p(0, 1)), \ p \in (1, \infty)$. Since $\theta^n \in L^\infty(0, T; L^2(0, 1)), \theta^n_t \in L^2(0, T; H^{-1}(0, 1))$, by Lemma 1.4 of [8], we have
\[ \theta^n \to \theta, \quad \text{in } C([0, T]; H^{-\gamma}(0, 1)), \ 0 < \gamma < 1, \]
when $n \to \infty$, we have
\[ \left| \int_0^1 (\theta^n - \theta) \zeta \ dx \right| \to 0 \quad \forall \zeta \in H^\gamma(0, 1). \hfill (3.5) \]

About $N_{12}(\theta^n)$, by (1.7) and (2.1), we know
\[ N_{12}(\theta^n) \to \chi_3 \text{ weak- */ in } L^\infty(0, T; L^2(0, 1)) \text{ as } n \to \infty. \]

Since $0 \leq N'_{12} \leq M_1$, and by (3.5) we arrived at
\[ \left| \int_0^1 (N_{12}(\theta^n) - N_{12}(\theta)) \zeta \ dx \right| = \left| \int_0^1 N'_{12}(\zeta)(\theta^n - \theta) \zeta \ dx \right| \leq M_1 \left| \int_0^1 (\theta^n - \theta) \zeta \ dx \right| \to 0 \]
as $n \to \infty$. Where $\zeta$ is between $\theta^n$ and $\theta$. By the uniqueness, we conclude that
\[ N_{12}(\theta^n) \to N_{12}(\theta), \text{ weak- */ in } L^\infty(0, T; L^2(0, 1)). \hfill (3.6) \]

As for $N_{22}$, by (1.8) and (2.1), we know
\[ N_{22}(\theta^n) \to \chi_4, \text{ weak- */ in } L^\infty(0, T; L^2(0, 1)). \]

Next we will use the monotonicity method [9] to obtain $N_{22}(\theta^n) \to N_{22}(\theta)$. Note that
\[ X^n = \int_0^T \int_0^1 (N_{22}(\theta^n) - N_{22}(v), \theta^n - v) \ dx \ dt \geq 0 \]
\[ \forall v \in L^2(0, T; H^1(0, 1)) \cap L^\infty(0, T; L^2(0, 1)) \]
\[
\int_0^T \int_0^1 N_{22}(\theta^n) \theta^n \, dx \, dt = \frac{1}{2} \int_0^1 |\theta^n|^2 - \frac{1}{2} \int_0^1 |\theta^n(T)|^2 \\
+ \int_0^T \int_0^1 f \theta^n \, dx \, dt \\
- \int_0^T \int_0^1 N_{21}(u^n) \theta^n \, dx \, dt \\
- k \int_0^T \int_0^1 |\theta^n|^2 \, dx \, dt - \beta |\theta^n(1, t)|^2.
\]

Using the lower semicontinuity of the norm, we have

\[
\int_0^1 |\theta(T)|^2 \leq \liminf_{n \to \infty} \int_0^1 |\theta^n(T)|^2,
\]

\[
\beta \int_0^T |\theta(1, t)|^2 \, dt \leq \liminf_{n \to \infty} \beta \int_0^T |\theta^n(1, t)|^2 \, dt,
\]

\[
k \int_0^T \int_0^1 |\theta_x|^2 \, dx \, dt \leq \liminf_{n \to \infty} k \int_0^T \int_0^1 |\theta^n_x|^2 \, dx \, dt,
\]

and since \(N_{21}\) is a Lipschitz continuous function, we have

\(N_{21}(u^n) \to N_{21}(u), \quad \text{(as } n \to \infty) \text{ in } C([0, T], H^n(0, 1)), \quad 0 < v < 1.\)

Therefore

\[
\lim_{n \to \infty} \int_0^T \int_0^1 N_{21}(u^n) \theta^n \, dx \, dt = \int_0^T \int_0^1 N_{21}(u) \theta \, dx \, dt.
\]

From the above discussion, we obtain

\[
0 \leq \limsup_{n \to \infty} X^n \leq \frac{1}{2} \int_0^1 |\theta|^2 - \frac{1}{2} \int_0^1 |\theta(T)|^2 + \int_0^T \int_0^1 f \theta \, dx \, dt \\
- \int_0^T \int_0^1 N_{21}(u) \theta \, dx \, dt - \beta |\theta(1, t)|^2 \\
- k \int_0^T \int_0^1 |\theta_x|^2 \, dx \, dt + \int_0^T \int_0^1 N_{22}(v) \, dx \, dt \\
- \int_0^T \int_0^1 N_{22}(v) \theta \, dx \, dt - \int_0^T \int_0^1 \chi_4 \, dx \, dt,
\]

which implies that

\[
\int_0^T \int_0^1 (\chi_4 - N_{22}(v))(\theta - v) \, dx \, dt \geq 0.
\]
By the monotonicity of \( N_{22} \), we find
\[
\chi_4 = N_{22}(\theta).
\]
Letting \( n \to \infty \) in the following approximate system:
\[
\begin{align*}
    u''_{tt} - \eta u''_{xx} + m\theta'' + N_1(u^p, \theta^p) &= f(x) \quad \text{in } (0, 1) \times (0, T), \\
    \theta'_t - k\theta'' + m\theta''_{xx} + N_2(u^p, \theta^p) &= g(x) \quad \text{in } (0, 1) \times (0, T), \\
    u^p(x, 0) &= u^p_0(x), \quad u''_{tt}(x, 0) = u''_{tt}^p(x), \quad \theta'_t(x, 0) = \theta'_0(x), \\
    \eta u''_0(1, t) - m\theta''(1, t) &= -d[(u^p(1, t) - \alpha)^+ - b(u^p(1, t) - \alpha)^+]u''_t(1, t), \\
    k\theta''(1, t) &= -\beta\theta''(1, t), \\
    u^p(0, t) &= 0, \quad \theta''_{xx}(0, t) = 0.
\end{align*}
\]
We obtain that (1.1)–(1.6) holds in the weak sense and \((u, \theta)\) satisfies the regularity properties in Theorem 3.1, so the existence of weak solution for (1.1)–(1.6) is proved, that is, the proof of Theorem 3.1 is complete. 

4. Exponential decay of the weak solution for (1.1)–(1.6)

In this section, we prove the exponential decay of weak solution in \( H^1 \times L^2(0,1) \times L^2(0,1) \) with \( f = g = 0 \). First, we obtain the exponential decay of the strong solution for (1.1)–(1.6) in \( H^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \) when \( f = g = 0 \), then by the lower semicontinuity, we obtain the exponential decay of the weak solution for (1.1)–(1.6) in \( H^1(0, 1) \times L^2(0, 1) \times L^2(0, 1) \).

**Theorem 4.1.** Under the condition of Theorem 2.1, (1.9)–(1.10), \( l = \mu - 1, \mu \geq 1 \) and \( M_1, M_2 \) are small enough and \( f = g = 0 \), we have
\[
\eta \|u_t\|^2 + \|u_t\|^2 + \|\theta\|^2 \leq Ce^{-\gamma t}, \tag{4.1}
\]
\( \gamma \) is a positive constant which can be seen in the proof.

**Proof.** By Theorem 2.1, we know there is a unique strong solution \((u, u_t, \theta)\) for (1.1)–(1.6). Since \( f = g = 0 \), so (2.1) can be rewritten as
\[
\frac{1}{2} \frac{d}{dt} E(t; u, \theta) + k \int_0^1 |\theta_{xx}|^2 \, dx + b|u_t(1, t)|^2 \{u(1, t) - \alpha)^+ \} + \beta|\theta(1, t)|^2
\]
\[
= -\int_0^1 N_{12}(\theta)u_t \, dx - \int_0^1 N_{21}(u)\theta \, dx - \int_0^1 N_{22}(\theta)\theta \, dx. \tag{4.2}
\]
In order to obtain exponential decay, we need the following lemmas.
Lemma 4.2. Under the conditions of (1.7)–(1.10), there exists two positive constants $0 < \delta < 1$ and $M_4$ satisfying

$$2\frac{d}{dt} \int_0^1 \psi u_t \, dx \leq \int_0^1 \left( \left( \delta + \frac{8M_4^2}{m^2} \right) u_x^2 - u_t^2 \right) \, dx + \delta u_x^2(1, t)$$

$$+ M_4 \int_0^1 (\delta^2 + \theta_x^2) \, dx,$$

where $\psi(x, t) = \int_0^x \theta(y, t) \, dy$.

Proof. Integrating (1.2) over $(0, x)$ we get

$$\partial_t \psi - k \theta_x + mu_t + \int_0^x N_2(u, \theta) \, dx = 0.$$

Using the above identity and Eq. (1.1), integrating by parts and keeping in mind that $\psi(0, t) = 0$, we get

$$2\frac{d}{dt} \int_0^1 \psi u_t \, dx = \frac{2}{m} \int_0^1 (\psi u_t + \psi u_{tt}) \, dx$$

$$= \frac{2}{m} \int_0^1 \left( k \theta_x u_t - mu_t^2 + \eta \psi u_{xx} - m \psi \theta_x \right.$$ 

$$- u_t \int_0^x N_2(u, \theta) \, dx - \psi N_1(u, \theta) \bigg) \, dx$$

$$= \frac{2}{m} \int_0^1 \left( \frac{k}{m} \theta_x u_t - u_t^2 - \frac{\eta}{m} \theta u_x - \psi \theta_x \right) \, dx$$

$$+ \frac{2\eta}{m} \psi(1, t)u_x(1, t)$$

$$- \frac{2}{m} \int_0^1 u_t \int_0^x N_2(u, \theta) \, dx$$

$$- \frac{2}{m} \int_0^1 \psi N_1(u, \theta) \, dx.$$  \hspace{1cm} (4.4)

Using (1.7)–(1.9), Lemma 2.2 and the Cauchy–Schwarz’s inequality, we obtain

$$2\frac{k}{m} \int_0^1 \theta_x u_t \, dx \leq \frac{1}{4} \int_0^1 |u_t|^2 \, dx + \frac{4k^2}{m^2} \int_0^1 \theta_x^2 \, dx,$$

$$- 2\frac{\eta}{m} \int_0^1 \theta u_x \leq \frac{\delta}{2} \int_0^1 u_x^2 \, dx + \frac{2\eta^2}{m^2 \delta} \int_0^1 \theta^2 \, dx,$$

$$- 2 \int_0^1 \psi \theta_x \, dx \leq \int_0^1 (\theta^2 + \theta_x^2) \, dx,$$

$$\frac{2\eta}{m} \psi(1, t)u_x(1, t) \leq \delta u_x^2(1, t) + \frac{\eta^2}{\delta m} \int_0^1 \theta^2 \, dx,$$
\[-\frac{2}{m} \int_{0}^{1} \psi N_1(u, \theta) \, dx \leq \frac{2}{m} \int_{0}^{1} |\theta| \, dx \int_{0}^{1} |N_1(u, \theta)| \, dx \]
\[\leq \frac{2}{m} \int_{0}^{1} |\theta| \, dx \int_{0}^{1} (M_0 |u| + M_1 |\theta|) \, dx \]
\[\leq \frac{2M_1}{m} \int_{0}^{1} |\theta|^2 \, dx + \frac{8M_0^2}{\delta m^2} \int_{0}^{1} |\theta|^2 \, dx \]
\[+ \frac{\delta}{8} \int_{0}^{1} |u_x|^2 \, dx \]
\[\leq \left( \frac{2M_1}{m} + \frac{8M_0^2}{\delta m^2} \right) \int_{0}^{1} |\theta|^2 \, dx + \frac{\delta}{2} \int_{0}^{1} |u_x|^2 \, dx, \]

\[-\frac{2}{m} \int_{0}^{1} u_t \int_{0}^{x} N_2(u, \theta) \, ds \, dx \leq \frac{2}{m} \int_{0}^{1} |u_t| (M_2 |u| + M_3 |\theta|) \, dx \]
\[\leq \frac{3}{4} \int_{0}^{1} |u_t|^2 \, dx + \frac{8M_2^2}{m^2} \int_{0}^{1} u_x^2 \, dx \]
\[+ \frac{4M_3^2}{m^2} \int_{0}^{1} \theta^2 \, dx. \]

So let \( M_4 = \max \left\{ \frac{2\eta^2}{m \delta^2} + 1 + \frac{\eta^2}{m^2} + \frac{4M_1^2}{\delta m^2} + \frac{8M_0^2}{\delta m^2} + \frac{2M_0 \, \delta^2}{m^2} + 1 \right\} \) and by the above inequalities and (4.5), the proof of Lemma 4.2 is complete. \( \square \)

Applying Lemma 2.3 to Eq. (1.1), and using similar arguments as the proof of Lemma 4.2, we have

\[-\frac{d}{dt} \int_{0}^{1} xu_t u_x \, dx \leq -\frac{1}{2} \{ u_t^2(1, t) + \eta u_x^2(1, t) \} \]
\[+ M_5 \int_{0}^{1} u_x^2 \, dx + \frac{m}{2} \int_{0}^{1} \theta_x^2 \, dx + \frac{M_1}{2} \int_{0}^{1} \theta^2 \, dx, \quad (4.5) \]

where \( M_5 = \frac{m}{2} + \frac{5M_0}{2} + \frac{M_1}{2} \).

Multiplying (1.1) by \( u \) and integrating it over \((0, 1)\), using Lemma 2.2 we arrive at

\[\frac{d}{dt} \int_{0}^{1} uu_t \, dx = \int_{0}^{1} (u_t^2 - \eta u_x^2 + m \theta u_x) \, dx - \int_{0}^{1} N_1(u, \theta) u \, dx \]
\[- d[(u(1, t) - x)^+] u(1, t) - bu_t (1, t) [(u(1, t) - x)^+]^{l+1} \]
\[
\begin{align*}
&\leq \int_0^1 \left( u_t^2 - \frac{\eta}{2} u_x^2 + \frac{m^2 + 4M_1^2}{\eta} \theta^2 \right) dx \\
&\quad - \frac{b}{\mu + 1} \frac{d}{dt} \left[ (u(1, t) - \sigma)^+ \right]^{\mu + 1} \\
&\quad - \int_0^1 N_{11}(u) u \, dx - d[(u(1, t) - \sigma)^+]^{\mu + 1}.
\end{align*}
\] (4.6)

Therefore, using (4.2)–(4.5) and letting \( M_2 \) small enough such that \( \frac{8M_2^2}{m^2} \leq \frac{\eta}{16(1 + \eta)} \), choosing \( \delta \leq \frac{1}{8} \min \left\{ \frac{\eta}{2(1 + \eta)}, \frac{\eta^2}{M_5(1 + \eta)} \right\} \), we obtain

\[
\frac{d}{dt} \int_0^1 \left( \frac{2(1 + \eta)}{m} \eta_1 u_t - \frac{\eta}{4M_5} xu_x u_x + uu_t + \frac{b}{\mu + 1} \left[ (u(1, t) - \sigma)^+ \right]^{\mu + 1} \right) dx
\]
\[
+ \int_0^1 \left( \eta u_x^2 + \frac{\eta}{8} u_x^2 \right) dx + \int_0^1 N_{11}(u) u \, dx + d[(u(1, t) - \sigma)^+]^{\mu + 1}
\]
\[
\leq M_6 \int_0^1 \left( \theta_x^2 + \theta^2 \right) dx,
\] (4.7)

where \( M_6 = \max \left\{ M_4(1 + \eta) + \frac{\eta}{8M_5}, M_4(1 + \eta) + \frac{\eta M_1}{8M_5} + \frac{2m^2 + 4M_2^2}{\eta} \right\} \).

Let us define \( \gamma_1 = \min \{ k, \beta \} \), since

\[
\int_0^1 \theta^2 \, dx \leq 2 \int_0^1 \theta_x^2 \, dx + 2\theta^2(1, t)
\]

and by (1.9)–(1.10), (4.1) can be rewritten as

\[
\frac{d}{dt} E(t; u, \theta) + k \int_0^1 \left| \theta_x \right|^2 \, dx + 2b |u_t(1, t)|^2 [(u(1, t) - \sigma)^+]^l
\]
\[
+ \left( \frac{3\beta}{2} |\theta(1, t)| \right)^2 + \frac{\gamma_1}{8} \int_0^1 \theta^2 \, dx
\]
\[
\leq \frac{16M_1^2}{\gamma_1} \int_0^1 u_t^2 \, dx + \frac{64M_2^2}{\gamma_1} \int_0^1 u_x^2 \, dx.
\] (4.8)

Now let us define

\[
L(t) = NE(t; u, \theta) + \int_0^1 \left( \frac{2(1 + \eta)}{m} \eta_1 u_t - \frac{\eta}{4M_5} xu_x u_x + uu_t \right) dx
\]
\[
+ \frac{b}{\mu + 1} \left[ (u(1, t) - \sigma)^+ \right]^{\mu + 1}.
\] (4.9)

We choose \( N \) satisfying

\[
N \geq \max \left\{ \frac{2(1 + \eta)}{m} + \frac{\eta}{8M_5} + \frac{1}{2} \frac{1}{8M_5} + \frac{1}{8\eta} \right\},
\] (4.10)
then there exists a constant $M_7$ such that
\[ M_7^{-1} E(t; u, \theta) \leq \mathcal{L}(t) \leq M_7 E(t; u, \theta), \] \hspace{1cm} (4.11)
which easily follows from the definition of $E(t; u, \theta)$ and Lemma 2.2.

Now, multiplying (4.8) by $N$ and adding the resulting inequality into (4.6) we obtain
\[
\frac{d}{dt} \mathcal{L}(t) + N \tau (\theta^2 + \theta^2_x) dx + \int_0^1 \left( \eta u_t^2 + \frac{\eta}{8} u_{\gamma}^2 \right) dx + N b [u_t(1, t) - \gamma] + [u(1, t) - \gamma]^{\mu+1} + \int_0^1 N_{11}(u) u dx + d [u(1, t) - \gamma]^{\mu+1} \leq M_6 \int_0^1 (\theta^2 + \theta^2_x) dx + N \frac{16 M_7^2}{\gamma} \int_0^1 u_t^2 dx + \frac{64 N M_7^2}{\gamma} \int_0^1 u_{\gamma}^2 dx.
\]
Choosing $N$ large enough, $M_1$ and $M_2$ small enough such that
\[ N \geq \frac{2M_6}{\tau} \left( \tau = \min \left\{ k, \frac{\gamma_1}{8} \right\} \right), \quad M_7^2 \leq \frac{\eta \gamma_1}{64N}, \quad M_2^2 \leq \frac{\eta \gamma_1}{1024N} \] \hspace{1cm} (4.12)
combining with (1.10) we have
\[
\frac{d}{dt} \mathcal{L}(t) + \rho \left[ \left( \int_0^1 \left( \theta^2 + \theta^2_x + u_t^2 + u_{\gamma}^2 + \int_0^u N_{11}(s) ds \right) dx + \frac{b}{\mu + 1} [u(1, t) - \gamma]^{\mu+1} \right) \leq 0, \right]
\] \hspace{1cm} (4.13)
where $\rho = \min \left\{ \frac{N \gamma_1}{2}, \frac{1}{16} b, \frac{d(\mu+1)}{b} \right\}$. By (4.11), such that
\[
\frac{d}{dt} \mathcal{L}(t) + M_7^{-1} \rho \mathcal{L}(t) \leq 0. \] \hspace{1cm} (4.14)

Then, by Gronwall inequality we have
\[ \mathcal{L}(t) \leq \mathcal{L}(0) e^{-\rho M_7^{-1} t}. \]

Using (4.11), we have
\[ E(t; u, \theta) \leq M_7^2 E(0; u, \theta) e^{-\rho M_7^{-1} t}. \]

Let $\gamma = \rho M_7^{-1}$ and $C = M_7^2 E(0; u, \theta)$, then Theorem 4.1 is proved. \(\square\)

Using the lower semicontinuity of the norm in $H^1 \times L^2 \times L^2$ and as a corollary of Theorem 4.1 we have

**Corollary 4.3.** When $f = g = 0$, under the condition of Theorem 3.1, (1.9)–(1.10) and $M_1$ and $M_2$ satisfy (4.12), then the weak solution of (1.1)–(1.6) decays exponentially as $t \to \infty$, that is, we have
\[ \eta ||u||^2 + ||u_t||^2 + ||\theta||^2 \leq Ce^{-\gamma t}. \]
Acknowledgments

The authors thank the referee for reading carefully this paper and for the useful suggestions. The authors are grateful to the editors of JDE for their help.

References