Decay of solutions in nonsimple thermoelastic bars

Hugo D. Fernández Sarea, Jaime E. Muñoz Rivera, Ramón Quintanilla

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Dedicated to K.R. Rajagopal with great esteem on the occasion of his 60th birthday.

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Abstract
In this paper we investigate the asymptotic behavior of the semigroup associated to the solutions of the initial boundary value problem for a one-dimensional nonsimple thermoelastic solids. We show that the semigroup is exponentially stable but is not analytic. Moreover we show the impossibility of time localization of the solutions.

1. Introduction

The material response of materials to stimuli depends in a relevant way on its internal structure, however classical elasticity does not consider the inner structure. Thus, it has been needed to develop some new mathematical models for continuum materials where this kind of effects were taken into account. Some of them are the porous elastic media, micropolar elastic solids, materials with microstructure, nonlocal continuum and nonsimple elastic solids. They are accounting for effects related to the size of defects and the microstructures. In the present paper we consider the last ones. We recall that from a mathematical point of view, these materials are characterized by the inclusion of higher order gradients of displacement in the basic postulates. They were introduced by Green and Rivlin [5], Mindlin [13] and Toupin [19,20]. More details on the subject can be found in the current books [4,6] on non-classical elasticity theories. The interest to introduce high order derivatives consists in the fact that the possible configurations of the materials are clarified more and more finely by the values of the successive higher gradients.

We here will use the theory and the notation in the way developed by Iesan in his book [6]. When supply terms are not present the evolution equations are

\[ \rho \ddot{u}_i = T_{ji} - S_{kji}, \quad \rho T_0 \dot{\Xi} = Q_{ij}. \]

The displacement of typical particles at the point \( x \) at time \( t \) is \( u_i = u_i(t,x); \ x \in B \) (\( B \) is a bounded domain). The temperature in each point \( x \) and the time \( t \) is given by \( \theta = \theta(t,x) \). We denote by \( \rho \) the mass density, \( T_{ji} \), \( S_{kji} \) the stress and the hyper-stress, \( \Xi \) the entropy density, \( Q \) the heat flux vector and \( T_0 \) the uniform temperature at the reference configuration which is assumed strictly positive. The constitutive equations for an isotropic and centrosymmetric material become

* Corresponding author.

E-mail address: Ramon.Quintanilla@upc.edu (R. Quintanilla).

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The material is assumed homogeneous and then the constitutive parameters \( \mu, \beta, \alpha \) and \( c \) are assumed constants. \( \rho, c, \mu, K \) and \( x \) are assumed positive constants, the sign of \( \beta \) is not determined, but it is different from zero. The physical interpretation of the positivity of the mass density \( \rho \) and the heat capacity \( c \) is obvious and they are usually imposed in the mathematical and physical studies concerning thermoelasticity. Also the positive sign of the thermal conductivity \( K \) uses to be accepted. It is related to the defining property of a definite mechanical heat conductor. It is also known that when the thermal conductivity is negative we find an ill posed problem, that is a problem where the continuous dependence with respect initial data fails. Same problem happens in case that \( x \) is negative. Thus, to have a well posed problem we need to assume its positivity.

In a more general context the positivity of \( \alpha \) and \( \mu \) is related with the positivity of the internal energy and may be interpreted with the help of the theory of mechanical stability.

If we substitute the constitutive equations into the motion equation and the energy equation, we obtain the system of field equations

\[
\begin{align*}
\rho u_t - \mu u_{xx} + 2\alpha u_{xxx} - \beta \theta_x &= 0 \quad \text{in } (0, \infty) \times (0, L), \\
\alpha u_{tt} - K \alpha_{xx} - \beta u_{tt} &= 0 \quad \text{in } (0, \infty) \times (0, L),
\end{align*}
\]

where \( \kappa = K T_0^{-1} \).

The main aim of the papers is to study the problem proposed by the system \((1.5)\), with the following initial conditions

\[
\begin{align*}
u(0,x) &= u_0(x), \quad u_t(0,x) = u_1(x), \quad \theta(0,x) = \theta_0(x), \quad x \in (0, L)
\end{align*}
\]

and associated with boundary conditions of the type

\[
\begin{align*}
T_0 = \lambda e_{\tau} e_{\tau} + 2\mu + \beta \theta e_{\tau}, \quad Q_i = K \delta_j, \\
S_{ijk} = \frac{\partial}{\partial t} (K r \delta_{jk} + 2\kappa_{ijr} \delta_{jk} + \kappa_{ijr} \delta_{jk}) + 2\alpha (K r_{ij} \delta_{jk} + K_{jrk} \delta_{jk}) + 2\alpha_{ij} \kappa_{irk} \delta_{jk} + 2\alpha_{ij} \kappa_{ijk} + 2\alpha_{ij} \kappa_{ijk}.
\end{align*}
\]

Here \( e_{\tau} = \frac{1}{2} (u_{ij} + u_{ji}) \) and \( \kappa_{ijk} = u_{ijk} \). To define a problem we need to impose initial and boundary conditions. Among the different boundary conditions we can assume (see [6], p. 256), we here restrict our attention to the ones of the kind

\[
u_i = 0 \quad \text{and} \quad (u_{ij} n_j = 0 \ or \ r_{ini} n_i = 0) \quad \text{and} \quad (\theta = 0 \ or \ Q_l n_l = 0).
\]

Here \( n_i \) denotes the normal vector to the smooth boundary of the domain \( B \).

It is worth citing several papers on existence and decay results in this theory published in the recent years [7,8,15]. As the constitutive equations of nonsimple solids contain first and second order gradients, it is of interest understanding the relevance of the thermal dissipation in the global decay of the system. In particular, we would like to clarify the influence on the decay of the thermal dissipation, studying the longterm dynamics. This is the main aim of the present paper. In view of the known results for the classical theories we cannot expect to obtain a general result independent of the dimension. We recall that in classical or type III thermoelasticity (see [9,11,18,21]) the solutions of the one-dimensional problem decay to zero exponentially. However for dimension greater than one we only can expect for polynomial decay. In fact, for several geometries and physical studies concerning thermoelasticity. Also the positive sign of the thermal conductivity is related to the defining property of a definite mechanical heat conductor. It is also known that when the thermal conductivity is negative we find an ill posed problem, that is a problem where the continuous dependence with respect initial data fails. Same problem happens in case that \( x \) is negative. Thus, to have a well posed problem we need to assume its positivity. In a more general context the positivity of \( x \) and \( \mu \) is related with the positivity of the internal energy and may be interpreted with the help of the theory of mechanical stability.

If we substitute the constitutive equations into the motion equation and the energy equation, we obtain the system of field equations

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\]
posed in the general case. That is not possible. That is we will prove that the only solution which can be identically zero after a finite period of time is the null solution. This result is relevant in the sense that gives a good complement to the exponential stability of solutions.

Our system of equations are analytic functions with respect to the time variable.

negative exponential. This fact is relevant from the mechanical point of view. It implies that if we consider a thermoelastic combination of the conservative and the dissipative equations recall very much the problem determined in the case of the plate conservative equation with second order spatial derivative. The problem is to know if the dissipation mechanism determined similar problems. In the case of the nonsimple thermoelasticity we have the coupling of the usual heat conduction with a weakness of the coupling.

In this paper we show that the solutions of our problem are exponentially stable. That is the decay is controlled by an exponential decay of solutions and in Section 4 we prove the lack of the analyticity of the solutions. In the last section we show the impossibility of the localization of solutions.

2. Existence of solutions

In this section we obtain an existence result for the solutions of the problem determined by (1.5) and (1.6) and associated with boundary conditions of the type (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), (1.13) or (1.14). To this end we define the Hilbert spaces

\[
V_1 := H^2(0, L) \cap H^1_0(0, L) \quad V_2 := H^2_0(0, L),
\]

\[
\mathcal{L}_j := \left\{ w \in L^2(0, L) : \int_0^L w dx = 0 \right\}, \quad \mathcal{L}_j := L^2(0, L) \quad \text{for } j = 2, 3, 4.
\]

Therefore the phase space will be

\[
\mathcal{H} = \mathcal{H}_{ij} = V_1 \times L^2(0, L) \times \mathcal{L}_j, \quad i = 1, 2, \quad j = 1, 2, 3, 4.
\]

In these Hilbert space we define the inner product

\[
(U, U')_{\mathcal{H}_{ij}} = \frac{1}{2} \int_0^L \left( \rho u \bar{\nu}' + \mu u_x \bar{u}_x' + \alpha u_{xx} \bar{u}_{xx}' + c \theta \Theta' \right) dx,
\]

where \( U = (u, \nu, \theta) \) and \( U' = (u', \nu', \theta') \). Now, let us to introduce the operators

\[
\mathcal{A} = \mathcal{A}_{ij} := \begin{pmatrix}
0 & 1 & 0 \\
\frac{p}{\rho} \partial_x (\cdot) - \frac{1}{\rho} \partial_{xxx} (\cdot) & 0 & \frac{p}{\rho} \partial_k (\cdot) \\
0 & \frac{p}{\rho} \partial_x (\cdot) & \frac{p}{\rho} \partial_{xx} (\cdot)
\end{pmatrix}.
\]
Let us define
\[ \mathcal{M}_1 = \{ \theta \in H^2; \; \theta_\alpha(0) = \theta_\alpha(L) = 0 \}, \quad \mathcal{M}_2 = \{ \theta \in H^2; \; \theta(0) = \theta(L) = 0 \}, \]
\[ \mathcal{M}_3 = \{ \theta \in H^2; \; \theta_\alpha(0) = 0 \}, \quad \mathcal{M}_4 = \{ \theta \in H^2; \; \theta(0) = 0 \}. \] (2.4, 2.5)

So the domain can be
\[ D(A_{ij}) = \mathcal{H}_{ij} \cap H^4 \times V_i \times M_j. \]

Using these notations, the initial-boundary value problem (1.5)–(1.7) can be rewritten as the following initial value problem
\[ \frac{d}{dt} U(t) = A_j U(t), \quad i = 1, 2, 3, 4, \]
\[ U(0) = U_0, \]
where \( U(t) = (u, u_t, \theta)' \) and \( U_0 = (u_0, u_t, \theta_0)' \), and the prime is used to denote the transpose.

In order to simplify the notation, we write \( A \) and \( \mathcal{H} \) instead of \( A_{ij} \) and \( \mathcal{H}_{ij} \) respectively for \( i = 1, 2 \) and \( j = 1, 2, 3, 4 \).

**Lemma 2.1.** The operator \( A \) defined previously, is the infinitesimal generator of a \( C_0 \)-semigroup of contractions over \( \mathcal{H} \).

**Proof.** Observe that \( \overline{D(A)} = \mathcal{H} \). We will show that \( A \) is a dissipative operator and that \( 0 \in \rho(A) \), then our conclusion will follow using the well-known Lumer–Phillips theorem (see [16]). In fact
\[ \text{Re}(AU, U)_{\mathcal{H}} = -\kappa \int_0^L |\theta_\alpha|^2 dx \leq 0, \]

therefore the operator \( A \) is dissipative. Finally, given \( F = (f, h, q) \in \mathcal{H} \), there exists a unique \( U = (u, v, \theta) \in D(A) \) such that \( AU = F \) in \( \mathcal{H} \). That is
\[ \nu = f, \]
\[ \mu u_{xx} - 2u_{xxx} + \beta \theta_\alpha = \rho h, \]
\[ \beta v_\alpha + \kappa \theta_\alpha = cq. \]

Therefore, there exists a unique solution of the problem \( AU = F \). Thus \( 0 \in \rho(A) \). The proof is complete. \( \square \)

A direct consequence of this Lemma is the following result.

**Theorem 2.2.** For any \( U_0 = (u_0, u_t, \theta_0)' \in D(A) \) there exists a unique solution \( U(t) = (u, u_t, \theta)' \) of (1.5) and (1.6) satisfying
\[ (u, u_t, \theta)' \in C([0, \infty); \mathcal{H}) \cap C^0([0, \infty); D(A)), \]

### 3. Exponential stability

Here we prove the exponential stability of the semigroup associated to the solutions of the system (1.5) and (1.6). To this end we shall use the following well-known result from semigroup theory (see e.g. [11, Theorem 1.3.2]).

**Theorem 3.1.** A semigroup of contractions \( \{e^{\lambda t}\}_{t \geq 0} \) with infinitesimal generator \( A \) on a Hilbert space \( \mathcal{H} \) with norm \( \| \cdot \|_\mathcal{H} \) is exponentially stable if and only if
\[ i \mathbb{R} \subset \rho(A) \] (3.1)

and
\[ \exists C > 0 \quad \forall \lambda \in \mathbb{R} : \| (i\lambda I - A)^{-1} \|_\mathcal{H} \leq C. \] (3.2)

To prove the exponential stability we use the following two Lemmas.

**Lemma 3.2.** \( i\mathbb{R} = \{i\lambda; \lambda \in \mathbb{R}\} \) is contained in \( \rho(A) \).

**Proof.** The operator \( A^{-1} : \mathcal{H} \to \mathcal{H} \) is compact. In fact, consider \( (F_n) \) a bounded sequence in \( \mathcal{H} \) and \( (U_n) \) the sequence in \( D(A) \) such that \( F_n = AU_n \), \( U_n = (u_n, v_n, \theta_n) \). Since \( A^{-1} \in \mathcal{L}(\mathcal{H}) \) there exists a positive constant \( C \) such that
\[ \|U_n\|_\mathcal{H} + \|A U_n\|_\mathcal{H} \leq C, \quad \forall n \in \mathbb{N}. \] (3.3)

From (3.3) we conclude that \( (u_n, v_n, \theta_n) \) is bounded in \( D(A) = D(A_{ij}) \). Since the embedding of \( H^m(0, L) \) in \( H^j(0, L) \), \( m > j \), is compact, there exists a subsequence \( (u_n, v_n, \theta_n) \) functions \( (u, v, \theta) \) such that
Lemma 3.3. The operator $C_p$ where

\[ C_p \] \hspace{1cm} \text{for a positive constant} \] implies that

\[ C_p \] with

\[ i \lambda t - \lambda = 0, \] \hspace{1cm} (3.4)

\[ i \lambda \rho u - \mu u_{xx} + \alpha u_{xxx} - \beta \theta_x = 0, \] \hspace{1cm} (3.5)

\[ i \lambda \xi \theta - \beta v_x - \kappa \theta_x = 0, \] \hspace{1cm} (3.6)

Since $(i(i \lambda - \lambda) U, U)_\Lambda = 0$ we have

\[ \int_0^L |\theta_x|^2 \, dx = 0, \] \hspace{1cm} and then $\theta = 0$. By (3.4) and (3.6) we get $u = v = 0$. Thus we have a contraction and the proof is complete. \( \square \)

Lemma 3.3. The operator $A$ defined in (2.3) satisfies

\[ \lim_{|\lambda| \to \infty} \text{sup} \| (i(i \lambda - \lambda)^{-1} \|_{L(H)} < \infty. \]}

Proof. Given $\lambda \in \mathbb{R}$ and $F = (f, g, h) \in \mathcal{H}$, there exists a unique $U = (u, v, \theta) \in D(A)$, such that $(i(i \lambda - \lambda) U = F$, that is,

\[ i \lambda u - v = f \text{ in } V_i, \] \hspace{1cm} (3.7)

\[ i \lambda \rho u - \mu u_{xx} + \alpha u_{xxx} - \beta \theta_x = \rho g \text{ in } L^2, \] \hspace{1cm} (3.8)

\[ i \lambda \xi \theta - \beta v_x - \kappa \theta_x = ch \text{ in } L_i, \] \hspace{1cm} (3.9)

with $i = 1, 2$ and $f = 1, 2, 3, 4$. Also note that

\[ \text{Re}((i(i \lambda - \lambda) U, U)_\Lambda = \kappa \int_0^L |\theta_x|^2 \, dx = \text{Re}(F, U)_\Lambda \] \hspace{1cm} and then

\[ \int_0^L |\theta_x|^2 \, dx \leq C \|F\|_{\Lambda} \|U\|_{\Lambda}, \] \hspace{1cm} (3.10)

for a positive constant $C$. Multiplying (3.8) by $u$ in $L^2(0, L)$ and using (3.7) we obtain

\[ \mu \int_0^L |u_x|^2 \, dx + \alpha \int_0^L |u_{xx}|^2 \, dx = \beta \int_0^L \theta_x u_x \, dx + \rho \int_0^L |v|^2 \, dx + \rho \int_0^L |f|^2 \, dx + \rho \int_0^L |g|^2 \, dx. \] \hspace{1cm} (3.11)

which, using (3.10) implies that

\[ \frac{\mu}{2} \int_0^L |u_x|^2 \, dx + \alpha \int_0^L |u_{xx}|^2 \, dx \leq \rho \|v\|^2 + C \|U\|_{\Lambda} \|F\|_{\Lambda}, \] \hspace{1cm} (3.12)

for a positive constant $C$.

The next step is to estimate $\|v\|_{L^2}$. To this end we define the functions $\phi$, $\omega$, $z$ and $y$ as solutions of the following problems:

\[ - \phi_{xx} = \nu \text{ in } [0, L] \] \hspace{1cm} with $\phi_x(0) = \phi_x(L) = 0$, \hspace{1cm} (3.13)

\[ - \omega_{xx} = \theta \text{ in } [0, L] \] \hspace{1cm} with $\omega_x(0) = \omega_x(L) = 0$, \hspace{1cm} (3.14)

\[ - z_{xx} = h \text{ in } [0, L] \] \hspace{1cm} with $z(0) = z(L) = 0$. \hspace{1cm} (3.15)

\[ - y_{xx} = g \text{ in } [0, L] \] \hspace{1cm} with $y(0) = y(L) = 0$. \hspace{1cm} (3.16)

Note that, using Poincaré’s inequality, we have

\[ \|z_x\|_{L^2} \leq C_p \|h\|_{L^2}, \] \hspace{1cm} (3.17)

\[ \|y_x\|_{L^2} \leq C_p \|g\|_{L^2}, \] \hspace{1cm} (3.18)

where $C_p > 0$ is the Poincaré’s constant. Then, in order to estimate $\|v\|_{L^2}$ we multiply (3.9) by $\phi_x$ in $L^2(0, L)$ to obtain

\[ i \lambda \xi (\theta, \phi_x)_{L^2} - k(\theta_{xx}, \phi_x)_{L^2} - \beta (v_x, \phi_x)_{L^2} = c(h, \phi_x)_{L^2}, \] \hspace{1cm} (3.19)

That is, the subsequence $(A^{-1}F_i)$ converges in $\mathcal{H}$.

Suppose that there exists $\lambda \in \mathbb{R}$, $\lambda \neq 0$, such that $i \lambda$ is the spectrum of $A$. Since $A^{-1}$ is compact, then $i \lambda$ must be an eigenvalue of $A$. Therefore, there is a vector $U, U \neq 0$, such that $(i(i \lambda - \lambda) U = 0$ in $\mathcal{H}$ or equivalently

\[ i \lambda t - \lambda = 0, \] \hspace{1cm} (3.4)

\[ i \lambda \rho u - \mu u_{xx} + \alpha u_{xxx} - \beta \theta_x = 0, \] \hspace{1cm} (3.5)

\[ i \lambda \xi \theta - \beta v_x - \kappa \theta_x = 0, \] \hspace{1cm} (3.6)
Now we estimate term by term. First, using (3.13) and (3.12) we obtain the identities

$$I_1 = -i\nu C(\omega, \phi_x)_{i^2} = i\nu C(\omega, \phi_x)_{i^2} = c(\omega, i\nu)_{i^2},$$

so, we replace $-i\nu$ given by (3.8). Then, using (3.13) and (3.15) we obtain

$$I_1 = \frac{c}{\rho} (\omega, \mu u_x - \alpha u_{xx} + \beta \rho_x + \rho g)_{i^2} = \frac{c\mu}{\rho} (\theta, u_{x})_{i^2} - \frac{cZ}{\rho} [\partial u_{xx}]_{x^2} + \frac{cZ}{\rho} (\phi_x, u_{x})_{i^2} + \frac{c\beta}{\rho} ||\phi||^2 - c(\omega, \gamma, y_{x})_{i^2}.$$  

(3.19)

Also, using (3.12) we deduce

$$I_2 = k(\theta, v)_{i^2} \quad \text{and} \quad I_3 = \beta ||v||^2_{i^2}.$$  

(3.20)

Finally, using (3.14) we have

$$I_4 = -c(z, \phi_x)_{i^2} = -c(z, \phi_x)_{i^2}.$$  

(3.21)

Then, substituting (3.19)-(3.21) into (3.18) results

$$\beta ||v||^2_{i^2} = \frac{cZ}{\rho} [\partial u_{xx}]_{x^2} + \frac{c\beta}{\rho} ||\phi||^2 + \frac{c\mu}{\rho} (\theta, u_{x})_{i^2} + \frac{cZ}{\rho} (\phi_x, u_{x})_{i^2} + c(\omega, \gamma, y_{x})_{i^2} - c(\omega, \gamma, y_{x})_{i^2} + k(\theta, v)_{i^2} - c(z, v)_{i^2},$$

which, using Sobolev’s embedding and inequalities (3.26), (3.17) and (3.10), we obtain

$$3 ||v||^2_{i^2} \leq \epsilon ||u_{xx}(0)||^2 + \epsilon ||u_{xx}(L)||^2 + C_c ||U||_{\infty} ||F||_{\infty} + \frac{\mu}{8} ||u_{x}||^2_{i^2} + \frac{\gamma}{4} ||u_{xx}||^2_{i^2},$$

(3.22)

for all $\epsilon > 0$ with $C_c > 0$.

Here, note that in the case $i = 1$ the boundary terms in (3.22) do not appears. The problem is to estimate that terms in the case $i = 2$ and $j = 1, 2, 3, 4$.

To overcome this problem we define the linear real function $q(x) : [0, L] \rightarrow \mathbb{R}$ given by

$$q(x) = \frac{2}{L} x + 1, \quad \text{for all } x \in [0, L].$$

(3.23)

So, multiplying (3.8) by $q(x)u_x$ in $L^2(0,L)$ results

$$\left\{ \begin{array}{ccc}
\dot{u}_x \mu (v, qu_x)_{i^2} - \mu (u_{xx}, qu_{x})_{i^2} + \alpha (u_{xx}, qu_{x})_{i^2} - \beta (u_x, qu_x)_{i^2} = \rho (g, qu_x)_{i^2}.
\end{array} \right.$$  

(3.24)

Using (3.7) we deduce that

$$K_1 = \rho (v, -q u_x - q f_x^1)_{i^2} = -\rho (v, q u_x)_{i^2} - \rho (v, q f_x^1)_{i^2}.$$  

Taking the real part, using the definition of $q(x)$ and the boundary conditions to $i = 2$, we obtain

$$Re(K_1) = -\frac{\mu}{L} ||v||^2_{i^2} - \rho Re(v, q f_x^1)_{i^2}.$$  

(3.25)

Analogously, taking the real part we have

$$Re(K_2) = -\frac{\mu}{L} ||u_{x}||^2_{i^2}.$$  

(3.26)

Finally, integrating by parts and using the properties of $q(x)$ we have

$$Re(K_3) = \frac{\alpha}{2} ||u_{xx}(0)||^2 + \frac{\alpha}{2} ||u_{xx}(L)||^2 - \frac{3\alpha}{L} ||u_{xx}||^2_i.$$  

(3.27)

Then, substituting (3.25)-(3.27) into (3.24) we have

$$\frac{\alpha}{2} ||u_{xx}(0)||^2 + \frac{\alpha}{2} ||u_{xx}(L)||^2 = \rho \frac{\mu}{L} ||v||^2_{i^2} - \frac{\mu}{L} ||u_{x}||^2_{i^2} + \frac{3\alpha}{L} ||u_{xx}||^2_{i} + \rho Re(v, q f_x^1)_{i^2} + \beta Re(\theta, qu_x)_{i^2} + Re(f^2, qu_x)_{i^2},$$

which implies

$$||u_{xx}(0)||^2 + ||u_{xx}(L)||^2 \leq C_1 \left[ ||v||^2_{i^2} ||u_{x}||^2_{i^2} + ||u_{xx}||^2_{i^2} + C_2 ||U||_{\infty} ||F||_{\infty} \right].$$  

(3.28)

Therefore, choosing $\epsilon > 0$ in (3.22) such that

$$\epsilon \leq \min \left\{ \frac{1}{C_1}, \frac{\mu}{8C_1}, \frac{\alpha}{4C_1} \right\}.$$
we obtain, substituting (3.28) into (3.22), that
\[ 2\|v\|_{2}^{2} \leq C_{r}\|U\|_{s_{r}}\|F\|_{s_{r}} + \mu \|u_{n}\|_{2}^{2} + \frac{\kappa}{2}\|u_{\alpha}\|_{2}^{2}. \]  
(3.29)

So, combining (3.10), (3.11) and (3.29), there exists a constant \( M > 0 \) independent of \( \lambda \) and \( F \in \mathcal{H} \) such that
\[ \|U\|_{s_{r}} \leq C\|F\|_{s_{r}}, \]
which implies condition (3.2). Thus, the proof is complete. \( \square \)

4. A lack of analyticity

The aim of this section is to show that the semigroup associated to system (1.5) is not analytic in general. To this end we use the following characterization of analytic semigroups. For the proof see [11].

**Theorem 4.1.** Let \( \rho(A) \) be the resolvent set of the linear operator \( A \). Then, a semigroup of contractions \( \{e^{ta}\}_{t \geq 0} \) in a Hilbert space \( \mathcal{H} \) with norm \( \| \cdot \|_{\mathcal{H}} \) is of analytic type if and only if
\[ i\mathbb{R} \subset \rho(A) \quad (i := \sqrt{-1}) \]
and
\[ \lim_{|\lambda| \to \infty} \sup \|\lambda(i\lambda - A)^{-1}\|_{L(\mathcal{H})} < \infty. \]  
(4.2)

We first consider the case of boundary conditions (1.7), this is
\[ u(t, 0) = u(t, L) = u_{\alpha}(t, 0) = u_{\alpha}(t, L) = \theta_{x}(t, 0) = \theta_{x}(t, L) = 0 \quad \text{in} \quad (0, \infty). \]
The main result of this section is formulated in the following theorem

**Theorem 4.2.** The semigroup \( S_{A_{11}}(t) \), defined on the Hilbert space \( \mathcal{H}_{11} \), is not analytic.

**Proof.** Applying Theorem 4.1, it is sufficient to show that there is a sequence \( (\lambda_{n}) \) of real numbers and a bounded sequence \( (F_{n}) \) in \( \mathcal{H}_{11} \) such that \( \lambda_{n} \to \infty \) as \( n \to \infty \) and
\[ \lim_{n \to \infty} \|\lambda_{n}(i\lambda_{n}I - A_{11})^{-1}\|_{\mathcal{H}_{11}} = \infty. \]

In fact, for each \( n \in \mathbb{N} \), we consider \( F_{n} = (0, \sin(\frac{\pi n}{L})x, 0) \) and let us the vector \( U_{n} = (u_{n}, v_{n}, \theta_{n}) \) belongs to \( D(A_{11}) \) be the unique solution of the resolvent equation \( i\lambda I - A_{11})U_{n} = F_{n} \), with \( \lambda \in i\mathbb{R} \), or equivalently
\[ i\lambda u_{n} - v_{n} = 0 \quad \text{in} \quad H^{2} \cap H_{0}^{1}(0, L), \]
\[ i\lambda \rho v_{n} - \mu u_{\alpha} + 2\mu u_{\max} - \beta \theta_{x} = \sin \left(\frac{n\pi}{L}x\right) \quad \text{in} \quad L^{2}(0, L), \]
\[ i\lambda \theta_{n} - \beta \theta_{x} + \kappa \theta_{\max} = 0 \quad \text{in} \quad L^{1}. \]

Due to boundary conditions (1.7), the solutions of the above system are the form
\[ u_{n} = A_{n} \sin \left(\frac{n\pi}{L}x\right), \quad \theta_{n} = B_{n} \cos \left(\frac{n\pi}{L}x\right). \]

Then we obtain
\[ -\lambda^{2} \rho A_{n} + \mu \left(\frac{n\pi}{L}\right)^{2} A_{n} + \alpha \left(\frac{n\pi}{L}\right)^{4} A_{n} + \beta \left(\frac{n\pi}{L}\right) B_{n} = 1, \]
\[ i\lambda \beta B_{n} - i\beta \lambda \left(\frac{n\pi}{L}\right) A_{n} + \kappa \left(\frac{n\pi}{L}\right)^{2} B_{n} = 0. \]

We take \( \lambda = \lambda_{n} = \frac{n\pi}{L} \sqrt{\mu + \alpha(\frac{n\pi}{L})^{2}} \) and get
\[ B_{n} = \frac{L}{\beta \pi n}, \quad A_{n} = \left(-c + i \frac{\kappa \rho \pi n}{\sqrt{\mu L^{2} + \alpha \pi^{2} n^{2}}} \right) \frac{L^{2}}{\beta^{2} \pi^{2} n^{2}}. \]

Since
\[ \|U_{n}\|_{s_{r}}^{2} = \rho \int_{0}^{L} |v_{n}|^{2} dx + \mu \int_{0}^{L} |u_{\alpha}|^{2} dx + \alpha \int_{0}^{L} |u_{\max}|^{2} dx + c \int_{0}^{L} |\theta_{x}|^{2} dx + \alpha \int_{0}^{L} |u_{\max}|^{2} dx \]
\[ = \frac{2L}{2\beta^{2}} \left(c^{2} + \frac{\kappa^{2} \rho^{2} \pi^{2} n^{2}}{\mu L^{2} + \alpha \pi^{2} n^{2}} \right), \quad \forall n \in \mathbb{N}, \]
we have
\[ \lim_{n \to \infty} \| \lambda_n U_n \|_{H^1_{\text{loc}}} = \infty, \]
which complete the proof of the Theorem. □

**Remark 4.3.** In the case of boundary conditions (1.8), (1.9), . . . , (1.13) or (1.14), we also expect that the semigroup is not analytic. Note that, in these cases, we cannot apply the procedure used previously, because we do not have appropriate solutions (satisfying boundary conditions) to solve system (4.3). Explicitly speaking, the non-analyticity of the semigroup associated to system (1.5) with boundary conditions (1.8)–(1.14) is an open problem.

5. Impossibility of localization

The aim of this section is to prove the impossibility of time localization of solutions of the system of the nonsimple thermoelasticity with the boundary conditions (1.7). To prove the impossibility of solutions we will show the uniqueness of solutions of the backward in time problem. Thus, it will be suitable to recall that the system of equations which govern the backward in time problem is:

\[ \begin{align*}
\rho u_{tt} - \mu u_{xx} - z u_{xxx} - \beta \theta_x &= 0 & \text{in} & \quad (0, \infty) \times (0, L), \\
\kappa \theta_t + \kappa \theta_{xx} - \beta u_{tt} &= 0 & \text{in} & \quad (0, \infty) \times (0, L).
\end{align*} \tag{5.1} \]

We will study the problem determined by system (5.1), with boundary conditions (1.7) and with the null initial conditions

\[ u(0, \cdot) = 0, \quad u_t(0, \cdot) = 0, \quad \theta(0, \cdot) = 0. \tag{5.2} \]

**Lemma 5.1.** Let us assume that the conditions hold. Let \( (u, \theta) \) be a solution of the problem determined by the system (5.1), the initial conditions (5.2) and the boundary conditions (1.7). Then \( u = \theta = 0 \).

**Proof.** The energy conservation gives

\[ E_1(t) = \frac{1}{2} \int_0^L (\mu |u_t|^2 + z |u_{xx}|^2 + \rho |u_t|^2 + c \theta^2) dx = \int_0^t \int_0^L \kappa \theta_t^2 dx ds. \]

If we multiply the first equation of (5.1) by \( u_t \) and the second one by \(-\theta\), we obtain

\[ E_2(t) = \frac{1}{2} \int_0^L (\mu |u_t|^2 + z |u_{xx}|^2 + \rho |u_t|^2 - c \theta^2) dx = \int_0^t \int_0^L (2 \beta u_t \theta_x - \kappa \theta_t^2) dx ds. \]

We need a third equality. To obtain it, we use the Lagrange identity method. For a fixed \( t \in (0, T) \), we use the identities

\[ \frac{\partial}{\partial s} (\rho \dot{u}(s)\dot{u}(2t-s)) = \rho \ddot{u}(s)\dot{u}(2t-s) - \rho \dot{u}(s)\ddot{u}(2t-s), \]
\[ \frac{\partial}{\partial s} (c \dot{\theta}(s)\dot{\theta}(2t-s)) = c \ddot{\theta}(s)\dot{\theta}(2t-s) - c \dot{\theta}(s)\ddot{\theta}(2t-s). \]

From the field equations, the boundary conditions and the null initial conditions, we obtain that the following equality

\[ \int_0^L (\mu |u_t|^2 + z |u_{xx}|^2 + c \theta^2) dx = \int_0^L \rho |u_t|^2 dx. \]

Thus, we have

\[ E_2(t) = \int_0^L (\mu |u_t|^2 + z |u_{xx}|^2) dx. \]

Let \( \epsilon \) be a small, but positive constant. Let us consider \( E(t) = E_2(t) + \epsilon E_1(t) \). We note that

\[ E(t) = \frac{1}{2} \int_0^t \int_0^L (\epsilon (\rho |u_t|^2 + c \theta^2) + (2 + \epsilon) (\mu |u_t|^2 + z |u_{xx}|^2)) dx, \tag{5.3} \]

is a positive function and it defines a measure on the solution. We take \( \epsilon \) strictly less than one, but greater than zero. As

\[ E(t) = -(1 - \epsilon) \int_0^t \int_0^L \kappa |\theta_t|^2 dx ds + 2 \int_0^t \int_0^L \beta u_t \theta_x dx ds, \]

we have:

\[ \frac{dE}{dt} = -(1 - \epsilon) \int_0^t \int_0^L \kappa |\theta_t|^2 dx ds + 2 \int_0^t \beta u_t \theta_x ds. \tag{5.4} \]
The A-G inequality implies that
\[
\int_0^L \beta_1 u_1 \theta_1 dx \leq K_1 \int_0^L \rho \theta u_1 dx + \epsilon_1 \int_0^L \kappa |\theta_1|^2 dx,
\]
where \(\epsilon_1\) is as small as we want, but positive and \(K_1\) can be calculated in terms of the constitutive coefficients and \(\epsilon_1\).

If we take \(\epsilon_1 \leq 1 - \epsilon\), there exists a positive constant \(C\) such that
\[
\frac{dE}{dt} \leq C \int_0^L (\rho \theta u_1 + c\theta^2) dx.
\]

We obtain that the estimate
\[
\frac{dE}{dt} \leq CE(t),
\]
is satisfied for every \(t \geq 0\). This inequality implies that \(E(t) \leq E(0) \exp(C t)\) for every \(t\) greater than zero. As we assume null initial conditions we see that \(E(t) \equiv 0\) for every \(t \geq 0\). If we take into account the definition of \(E(t)\), it follows that \(u \equiv 0, \theta \equiv 0\) for every \(t \geq 0\) and then in view of the initial conditions, it follows that the only solution to our problem is the null solution.

Thus, we can state the following: \(\Box\)

**Theorem 5.2.** Let \((u, \theta)\) be a solution of the problem determined by the system (1.5), the initial conditions (1.6) and the boundary conditions (1.7) such that \(u = \theta = 0\) after a finite time \(t_0 > 0\). Then \(u = \theta = 0\) for every \(t \geq 0\).

**Remark 5.3.** The analysis of this section also works for the other boundary conditions considered previously.

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**References**