UNIFORM RATES OF DECAY OF SOLUTIONS FOR A NON LINEAR LATTICE WITH MEMORY

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Abstract. In this paper we study a nonlinear lattice with memory and show that the problem is globally well posed, furthermore we find uniform rates of decay of the total energy. Our main result shows that the memory effect is strong enough to produce a uniform rate of decay. That is, if the relaxation function decays exponentially then, the corresponding solution also decays exponentially. When the relaxation kernel decays polynomially then, the solution also decays polynomially as time goes to infinity.

1. Introduction

Memory, i.e. history dependent behavior of a physical system, is a generic phenomenon which appears when a dynamical system is connected with another one simulating environment. The phenomenon always occurs when the external system possess a nonzero characteristic temporal scale determining its respond on excitations originated by the dynamical system under considerations. The respective examples are well known in general electrodynamics, theory of elasticity, optics [1], etc. If a dynamical system is nonlinear, then the situation becomes much more complicate because the lack of applicability of the superposition principle and, as a consequence, appearance of a new temporal scale dictated by the nonlinearity. (Notice that the coupling by itself is already a nonlinearity if one considers an extended system consisting of the dynamical system and the environment.) If in addition the dynamical system is discrete, i.e. has the smallest spatial scale, while describing the dynamics, one has to take into account one more temporal scale induced by the discreteness. In the present paper we address to mathematical aspects of global existence of a solution as well as the asymptotic behavior as time approaches infinity. We concentrate on a simple discrete model simulating the above physical phenomena. More specifically, the effect of the memory can be originated, in particular, by the effect of coupling of a given nonlinear chain (say Fermi-Pasta-Ulam chain) with an external bath modeled by a system of damped oscillators. In other words we consider a nonlinear chain with inter-site interactions and described by the Hamiltonian

\[ H_1 = \sum_{n=1}^{N} \left( \frac{P_n^2}{2M} + \frac{K}{2} (U_n - U_{n-1})^2 + \frac{V}{p+1} (U_n - U_{n-1})^{p+1} \right) \]

where \( N \) is a number of lattice sites and \( p \) is a positive integer, with a network of independent oscillators governed by the Hamiltonian

\[ H_2 = \sum_{n=1}^{N} \left( \frac{P_n^2}{2m} + \frac{k}{2} q_n^2 \right) \]

In the above formulas \( M \) and \( m \) are the masses of particles of the respective chains, \( K \) and \( V \) are respectively, linear and nonlinear force constants of inter-site interactions, \( k \) characterizes the
strength of the on-site potential of a linear oscillator, $U_n$ and $q_n$ are respective displacements of
the particles from the equilibrium positions, and $P_n$ and $p_n$ are their linear momenta. It will be
assumed that both chains satisfy the cyclic boundary conditions $U_{n+N} = U_n$, $q_{n+N} = q_n$. Now
consider coupling the chains in the form

$$H_c = \chi \sum_{n=1}^{N} q_n (U_n - U_{n-1})$$

where $\chi$ is the strength of the coupling. Then, the whole system is described by the Hamiltonian

$$H = H_1 + H_2 + H_c$$

Next, assuming that the oscillator network has a dissipation, one arrives to the equations of
motion

$$M \ddot{U}_n - K \Delta U_n + V [ (U_n - U_{n+1})^p + (U_n - U_{n-1})^p ] = \frac{\chi}{2} (q_n - q_{n-1})$$

$$m \ddot{q}_n + \gamma \dot{q}_n + k q_n = \frac{\chi}{2} (U_{n+1} - U_n),$$

where $\Delta$ stands for the second difference: $\Delta u_n \equiv u_{n+1} + u_{n-1} - 2u_n$, $n = 1, \ldots, N$, $\ddot{U}_n = \frac{d^2 U_n}{dt^2}$
and $\ddot{q}_n = \frac{d^2 q_n}{dt^2}$.

Let us concentrate on the so-called over-damped limit which is characterized by the relation
$m \omega \ll \gamma$, where $\omega = \sqrt{k/m}$ is the characteristic frequency of the linear oscillators. Then the
kinetic energy term $\ddot{q}_n$ can be neglected in (1.6) and expressing $q_n$ through $U_n$ from that equation
one arrives at the system of evolution equations

$$\ddot{u}_n(t') - \Delta u_n(t') - (u_n - u_{n+1})^p - (u_n - u_{n-1})^p = \int_0^{t'} g(t' - \tau) \Delta u_n(\tau) d\tau$$

where $u_n = \left( \frac{\lambda}{N} \right)^{1/2} U_n$, $t' = \omega t$

$$g(t) \equiv g_0 \exp \left( - \frac{t}{t_0} \right), \quad g_0 = \frac{\chi^2}{4 \gamma}, \quad t_0 = \frac{k}{\gamma}$$

Starting with the model described above one can easily construct several generalizations.
Let us mention only the following three: First, considering coupling of a nonlinear chain with
several different chains of the overdamped oscillators (having different force, $k_j$, and dissipative
constants $\gamma_j$, where $j = 1, 2, \ldots, N$, with $N$ being a number of overdamped chains) one arrives
to (1.7) with the kernel

$$g(t) \equiv \sum_{j=1}^{N} g_j \exp \left( - \frac{t}{t_j} \right)$$

This model simulates a situation where the environment of the dynamical system possesses $N$
different characteristic scales $t_j$.

Another generalization of the model can be achieved by considering different coupling. In
particular, on-site coupling $H_c = \sum_{n=1}^{N} U_n q_n$ results in the following evolution equation

$$\ddot{u}_n(t') - \Delta u_n(t') - (u_n - u_{n+1})^p - (u_n - u_{n-1})^p = \int_0^{t'} g(t' - \tau) u_n(\tau) d\tau$$

where $g(t)$ is given by (1.8).
The remaining part of this work is organized as follows. In section 2 we establish the local and global existence of solutions. In section 3 we prove the exponential decay of the total energy provided the relaxation function decays exponentially to zero as time goes to infinity. Finally, in section 4 we deduce the polynomial decay as $t \to \infty$, provided the relaxation function also decays polynomially with the same rate. Here we use similar ideas as in [3, 5, 6].

We would like to emphasize that the dissipative mechanism we consider in model (2.1) allow us to get not only the exponential decay but also the global existence of solutions for any initial data. This fact differs from the continuous model, where the corresponding type of dissipation allows only to obtain global solutions provided the initial data is small (in a suitable norm). The exponential decay was obtained because of the linearity of the dissipative mechanism. If we consider nonlinear dissipative memory effect, for example quadratic or cubic ones, we probably lost the exponential decay, getting only a polynomial decay.

2. Statement of the problem

Let us consider in detail model (1.7). For sake of convenience we rewrite it in the form

$$\ddot{u}_n - \Delta u_n + g * \Delta u_n - \lambda_p [(u_{n-1} - u_n)^p + (u_{n+1} - u_n)^p] = 0$$

where

$$g * f = \int_0^t g(t - \tau) f(\tau) \, d\tau$$

$p$ is an odd integer, $\lambda_p > 0$ and the prime standing for the dimensionless time is omitted. The consideration will be restricted to the cyclic boundary conditions, i.e. we will require that

$$\begin{cases}
  u_m(t) = u_{m+N}(t), & m = 0, 1, \cdots, N - 1, \quad \forall t \in \mathbb{R}^+
  u_n(0) = a_n, & \dot{u}_n(0) = b_n, & n = 1, \cdots, N,
\end{cases}$$

where $a_n$ and $b_n$ are given real numbers. We will also assume that the displacement as well as the velocity of the center of mass of the lattice at time $t = 0$ are zero. That is

$$\sum_{n=1}^N a_n = 0 \quad \text{and} \quad \sum_{n=1}^N b_n = 0$$

To show the global existence result of the nonlinear problem (2.1), (2.2) we rewrite the solutions in terms of the generalized solution of the following linear Lattice.

$$\ddot{v}_n - \Delta v_n = 0$$

$$v_n(0) = a_n, \quad \dot{v}_n(0) = b_n$$

$$v_n(t) = v_{n+N}(t), \quad n = 0, 2, \cdots, N - 1, \quad \forall t \geq 0$$

As in [2] we have the following result.

**Lemma 2.1.** Let us assume that the displacement and the velocity of the center of mass of the lattice at time $t = 0$ are zero, i.e. (2.3) holds, and let us denote by $v_n$ the solution of (2.4)-(2.6). Then, we have that

a)  $I(t) := \sum_{n=1}^N v_n(t) = 0, \quad \forall t > 0,$

b)  $\sum_{n=1}^N |v_n(t)|^2 \leq 4N^2 \sum_{n=1}^N |v_{n+1}(t) - v_n(t)|^2 \quad \forall t \geq 0.$
Proof.- See [2] (Lemma 2.1)

Note that we can write explicitly the solution of equation (2.4)–(2.6) as

\[
v_n(t) = \sum_{m=1}^{N} \left\{ \tilde{G}(n-m,t)a_m + G(n-m,t)b_m \right\}
\]

where \(G(n,t)\) is the Green function given by

\[
G(n,t) = \frac{1}{N} \sum_{k=1}^{N} \frac{\sin(\omega_k t)}{\omega_k} \cos(2\pi \frac{kn}{N})
\]

where \(\omega_k = 2\sin\left(\frac{k\pi}{N}\right)\) represents the dispersion relation.

3. Existence Result

In this section for sake of completeness we briefly describe the main steps how to obtain global existence and uniqueness of solution to problem (2.1)–(2.2) in a natural function space. The details are quite similar to those given in [2] when was considered an stronger type of dissipative mechanism.

Let \(T > 0\) and consider the linear space \(X(T)\) consisting of all functions \(U\) of the form

\[
U(t) = (u_1(t), \ldots, u_n(t))
\]

such that

\[
(3.1) \quad u_n \in C^2([0,T), \mathbb{R}), \quad u_n(t) = u_{n+N}(t), \quad \forall t \in [0,T), \quad u_n(0) = a_n, \quad \dot{u}_n(0) = b_n.
\]

with \(a_n\) and \(b_n\) satisfying (2.3) and

\[
(3.2) \quad \sup_{0 \leq t < T} \sum_{n=1}^{N} \left( |\ddot{u}_n| + |u_{n+1} - u_n| + |u_n|^2 \right) < \infty
\]

Given an element \(U \in X(T)\) we define norm \(\|U\|_X\) as

\[
\|U\|_X^2 = \sup_{0 \leq t < T} \sum_{n=1}^{N} \left( |\ddot{u}_n|^2 + |u_{n+1} - u_n|^2 + |u_n|^2 \right)
\]

Under the above conditions the space \((X(T), \| \cdot \|_X)\) becomes a Banach space. Let \(v_n\) be the solution of (2.4)–(2.6) such that its initial value satisfies (2.3). Then the function

\[
(3.3) \quad V(t) = (v_0(t), v_1(t), \ldots, v_{N-1}(t))
\]

belongs to \(X(T)\), for any \(T > 0\). In fact multiplying equation

\[
\ddot{v}_n - \Delta v_n = 0
\]

by \(\dot{v}_n\) and adding from \(n = 1\) up to \(n = N\) we obtain that

\[
\sum_{n=1}^{N} \left( |\ddot{v}_n|^2 + |v_{n+1} - v_n|^2 \right) = \sum_{n=1}^{N} \left( |\ddot{b}_n|^2 + |a_{n+1} - a_n|^2 \right) < \infty
\]
for any $t \geq 0$ which together with Lemma 2.1 (item b) in [2] shows that $V(t) \in X(T)$. Next, we define the class of functions where the relaxation function $g$ belongs. Let $\mathcal{F}$ be the family

$$\mathcal{F} = \left\{ f \in H^2(0, \infty) \cap C^2(0, \infty); \ f(t) > 0, \ \dot{f}(t) < 0, \ 1 - \int_0^\infty f(s) \, ds > 0 \right\}$$

Here we consider additionally that the kernel satisfies

\begin{equation}
-c_0 f(t) \leq f'(t) \leq -c_1 f(t), \quad |f''(t)| \leq c_2 f(t) \quad \forall t \geq 0
\end{equation}

A typical example of $g \in \mathcal{F}$ are the exponentials functions (see e.g. (1.8)), or linear combinations of exponential functions with different characteristic scales $T_j > 0$ (see e.g. (1.9)).

**Theorem 3.1 (Local Existence).** Let us suppose that $g \in \mathcal{F}$, then there exists $T > 0$ and a unique function $U(t) = (u_0(t), u_1(t), \cdots, u_{N-1}(t)) \in X(T)$ defined in $[0, T]$ satisfying

$$u_n(t) = v_n(t) + \sum_{m=1}^{N} \int_0^t G(n - m, t - s)Q(u_m(s), g) \, ds$$

where $G(n, t)$ is given by (2.7), $Q(u_m(s), g)$ is given by

$$Q(u_m(s), g) = \lambda_p \left[ |u_{m-1}(s) - u_m(s)|^p + |u_{m+1}(s) - u_m(s)|^p \right] - g \ast \Delta u_m(s)$$

and $p \geq 1$ is an odd integer.

**Proof.** Let $R > 0$ and $V(t)$ as in (3.3). Let us denote by $Y_R(T)$ the set

$$Y_R(T) = \{ Z \in X(T); \ |Z - V|_X \leq R, \ z_n(0) = v_n(0) = a_n; \ \dot{z}_n(0) = \dot{v}_n(0) = b_n \}.$$

Clearly $Y_R(T)$ is a close subset of $X(T)$. Let us consider the function

$$PZ(t) = (\hat{P}z_0, \cdots, \hat{P}z_{N-1})$$

where

$$\hat{P}z_n = v_n + \sum_{m=1}^{N} \int_0^t G(n - m, t - s)Q(u_m(s), g) \, ds$$

Observe that

$$\hat{P}z_n(0) = v_n(0) = a_n, \quad \frac{d}{dt} \hat{P}z_n(0) = \dot{v}_n(0) = b_n$$

The proof consists of two steps. First we show that $P$ maps $Y_R(T)$ into itself continuously, then we show that $P$ is a contraction in $Y_R(T)$ provided $T$ is small enough. Let $Z \in Y_R(T)$, to prove that $PZ \in Y_R(T)$ it is enough to show that $\|PZ - V\|_X < R$. Let

$$w_n = \hat{P}z_n - v_n = \sum_{m=1}^{N} \int_0^t G(n - m, t - s)Q(u_m(s), g) \, ds$$

We need to estimate the following terms

\begin{align*}
& a) \sup_{0 \leq t<T} \sum_{n=1}^{N} |w_n|^2, \quad b) \sup_{0 \leq t<T} \sum_{n=1}^{N} |w_{n+1} - w_n|^2, \quad c) \sup_{0 \leq t<T} \sum_{n=1}^{N} |w_n|^2.
\end{align*}

From now on we will denote by $C$ a positive constant which may vary from place to place. Using the estimates in [2] for the term corresponding to

$$\lambda_p \left[ |u_{m-1} - u_m|^p + |u_{m+1} - u_m|^p \right]$$


when $g = 0$, (see Theorem 2.1 in [2]), one has to obtain estimates for

$$\tilde{w}_n = \sum_{m=1}^{N} \int_0^t G(n - m, t - s) g * \Delta u_m(s) \, ds$$

Since $|G(n, t)| \leq |t|$ for any $t$, using Holder’s inequality we obtain

$$(3.5) \quad |\tilde{w}_n(t)| \leq C \sup_{0 \leq \tau < T} g(\tau) NT^3 \|U\|_X.$$  

Using the estimates for $w_n$ (with $g = 0$) given in [2] together with (3.5) we deduce that

$$\sup_{0 \leq t < T} \sum_{n=1}^{N} |w_n(t)|^2 \leq CT^4 N^2 \|U\|_X^{2p} + \sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{w}_n(t)|^2$$

$$(3.6) \quad \leq CT^4 N^3 \|U\|_X^{2p} + CT^6 N^3 \|U\|_X^2$$

which give us the estimates for item c). Now let us estimate a). To this end let us consider

$$|\tilde{w}_n(t)|^2 = \left( \frac{d}{dt} \tilde{P}_n - \dot{v}_n \right)^2$$

$$(3.7) \quad = \left| \sum_{m=1}^{N} \int_0^t \dot{G}(n - m, t - s) \left\{ \lambda_p \left[ |u_{m-1} - u_m|^p + |u_{m+1} - u_m|^p \right]^2 - g * \Delta u_m \right\} \right|^2$$

Since $|\tilde{w}_n|^2$ was already estimated in [2] (see (2.21)), remains only to obtain an estimate for

$$y_n(t) = \sum_{m=1}^{N} \int_0^t \dot{G}(n - m, t - s) g * \Delta u_m(s) \, ds$$

Since $|\dot{G}(n, t)| \leq 1$, for any $t$, then using Holder’s inequality we obtain that

$$\sum_{n=1}^{N} |y_n(t)|^2 \leq CN^2 T^4 \|U\|_X^4.$$  

which together with the estimate of (3.7) implies that

$$(3.8) \quad \sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{w}_n(t)|^2 \leq CT^2 N^3 \|U\|_X^{2p} + CT^4 N^3 \|U\|_X^2$$

which proves item a). Similar consideration shows that

$$(3.9) \quad \sup_{0 \leq t < T} \sum_{n=1}^{N} |w_{n+1} - w_n|^2 \leq CT^4 N^3 \|U\|_X^{2p} + CT^6 N^3 \|U\|_X^2$$

Finally, since $V \in Y_R(T)$ then

$$\|U\|_X \leq R + \|V\|_X \leq R + C_1$$

where

$$C_1^2 = (1 + 4N^2) \sum_{m=1}^{N} (|b_m|^2 + |a_{m+1} - a_m|^2)$$

Thus we conclude from (3.6), (3.8) and (3.9) that

$$\|PZ - V\|_X \leq R$$
provided $T$ is small enough, this shows that $P(Y_R(T)) \subset Y_R(T)$, continuously. Finally, we will show that $P$ is also a contraction for $T$ small enough.

Let $U(t)$ and $W(t)$, where $W(t) = (w_0(t), \ldots, w_{N-1}(t))$, elements of $Y_R(T)$ and let us denote by $\tilde{R}_n$

$$\tilde{R}_n = \tilde{P}u_n - \tilde{P}w_n = \sum_{m=1}^{N} \int_{0}^{t} \hat{G}(n-m, t-s) \left\{ R_m^{(1)} - R_m^{(2)} \right\} ds$$

where

$$R_m^{(1)} = \lambda_p \left[ |u_{m-1} - u_m|^p + |u_{m+1} - u_m|^p - |w_{m-1} - w_m|^p - |w_{m+1} - w_m|^p \right]$$

and

$$R_m^{(2)} = -g \ast \Delta(u_m - w_m)$$

We will show that

$$\sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{R}_n|^2, \quad \sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{R}_{n+1} - \tilde{R}_n|^2, \quad \sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{R}_n|^2.$$  

can be bounded by

$$\gamma \|U - W\|_X^2$$

for some constant $0 < \gamma < 1$ and for $T$ small enough. As in [2] we get

$$\sup_{0 \leq t < T} \sum_{n=1}^{N} \left( \sum_{m=1}^{N} \int_{0}^{t} G(n-m, t-s)R_m^{(1)} ds \right)^2 \leq CN^3 T^4 \left( \|u\|_X + \|W\|_X \right)^{2(p-1)} \|U - W\|_X^2$$

On the other hand, easily we have

$$|g \ast \Delta(u_m - w_m)| \leq \left( \sup_{0 \leq \tau < T} g \right) s \|U - W\|_X$$

Consequently

$$\sup_{0 \leq t < T} \sum_{n=1}^{N} \left( \sum_{m=1}^{N} \int_{0}^{t} G(n-m, t-s)R_m^{(2)} ds \right)^2 \leq CN^3 T^6 \left( \sup_{0 \leq \tau < T} g \right)^2 \|U - W\|_X^2$$

which together with (3.11) gives the required estimate for

$$\sup_{0 \leq t < T} \sum_{n=1}^{N} |\tilde{R}_n|^2.$$
will need an a priori estimate. Under the assumptions of Theorem 3.1 we consider $E(t)$ given by

$$E(t) = \frac{1}{2} \sum_{n=1}^{N} \left( |\dot{a}_n|^2 + \left( 1 - \int_{0}^{t} g(s) ds \right) |u_{n+1} - u_n|^2 + \frac{1}{2} g [u_{n+1} - u_n] \right) +$$

$$+ \frac{\lambda_p}{p+1} \sum_{n=1}^{N} |u_{n+1} - u_n|^{p+1}$$

where the binary operator $g[\varphi]$ is defined as

$$g[\varphi] = \int_{0}^{t} g(t - \tau) [\varphi(t) - \varphi(\tau)]^2 \ d\tau.$$

We will refer to $E(t)$ as the first order energy.

**Lemma 3.1.** Let $g, v_j \in C^1(0, T; \mathbb{R})$ for $j = 1, \ldots, N + 1$. Then, the identity

$$\sum_{n=1}^{N} \int_{0}^{t} g(t - \tau) (v_{n+1} - v_n) \ d\tau \cdot (v_{n+1} - v_n)$$

$$= -\frac{1}{2} g(t) \sum_{n=1}^{N} |(v_{n+1} - v_n)|^2 + \frac{1}{2} \sum_{n=1}^{N} g[\varphi](v_{n+1} - v_n)$$

$$- \frac{1}{2} \frac{d}{dt} \left\{ \sum_{n=1}^{N} g[\varphi](v_{n+1} - v_n) - \left( \int_{0}^{t} g(\tau) \ d\tau \right) \sum_{n=1}^{N} |(v_{n+1} - v_n)|^2 \right\}$$

holds

**Proof.-** It is easy to see that

$$\frac{d}{dt} \left\{ \sum_{n=1}^{N} g[\varphi](v_{n+1} - v_n) \right\}$$

$$= \dot{g}[\varphi](v_{n+1} - v_n) - 2 \sum_{n=1}^{N} \int_{0}^{t} g(t - \tau) (v_{n+1}(\tau) - v_n(\tau)) d\tau \cdot (v_{n+1}(t) - v_n(t))$$

$$+ 2 \int_{0}^{t} g(t - \tau) d\tau \sum_{n=1}^{N} (v_{n+1}(t) - v_n(t))(v_{n+1}(t) - v_n(t))$$

$$= \dot{g}[\varphi](v_{n+1} - v_n) - 2 \sum_{n=1}^{N} \int_{0}^{t} g(t - \tau) (v_{n+1} - v_n)(\tau) d\tau \cdot (v_{n+1} - v_n)$$

$$+ \frac{d}{dt} \left\{ \int_{0}^{t} g(\tau) d\tau \sum_{n=1}^{N} |(v_{n+1} - v_n)|^2 \right\} - g(t) \sum_{n=1}^{N} |(v_{n+1} - v_n)|^2$$

This shows our result.  \(\square\)
Lemma 3.2. Under the assumptions of Theorem 3.1, let $U(t)$ be the solution of (2.1). Then
\[
\frac{d}{dt}E(t) = \frac{1}{2}\dot{g}\Box(u_{n+1} - u_n) - \frac{1}{2}g(t)\sum_{n=1}^{N}|u_{n+1} - u_n|^2
\]
for any $0 < t < T_{\text{max}}$, where $E(t)$ is given by (3.14).

Proof. Multiplying equation (2.1) by $\dot{u}_n$ summing up over $n$ and using Lemma 3.1, our conclusion follows.

Theorem 3.2 (Global existence and Uniqueness). Let $p \geq 3$ be an odd integer, $g \in \mathcal{F}$, $\lambda_p > 0$ and the initial conditions satisfying condition (2.3). Under these conditions problem (2.1)-(2.2) has a unique global solution in the function space $X(T)$ for any $T > 0$.

Proof. Let $U$ be the maximal solution given by Theorem 3.1. Since $g \in \mathcal{F}$, Lemma 3.2 implies that the first order energy is a decreasing function with respect to the time, thus
\[
E(t) \leq E(0)
\]
In particular the above inequality implies that the expression
\[
\frac{1}{2}\sum_{n=1}^{N}\left\{|u_n|^2 + \left(1 - \int_0^t g(s)\,ds\right)|u_{n+1} - u_n|^2\right\}
\]
is bounded by a constant depending only on the initial data. Finally, the bound for
\[
\sum_{n=1}^{N}|u_n|^2
\]
follows by Lemma 2.1 in [2]. This proves that $T_{\text{max}} = \infty$. The uniqueness follows by standard arguments, using Gronwall’s inequality.

4. Exponential stability

In this section we will consider the exponential stability of the system (2.1). Let us assume that $g \in \mathcal{F}$. To facilitate our analysis, we introduce the following binary operators
\[
(g \Box h)(t) := \int_0^t g(t - s)\{h(t) - h(s)\}\,ds.
\]
It is easy to see that
\[
|g \Box h|^2 \leq \left(\int_0^t g(s)\,ds\right)g\Box h, \quad \text{for} \quad g(t) \geq 0.
\]
Now we are able to show the exponential rate of decay of the solution.

Theorem 4.1. Under the assumptions of Theorem 3.2 and with the kernel $g \in \mathcal{F}$ satisfying (3.4), the first order energy associated to the solution of (2.1)-(2.2) decays exponentially as time goes to infinity. That is, there exist positive constants $c$ and $\gamma$ that do not depend on the initial data, such that
\[
E(t) \leq cE(0)e^{-\gamma t} \quad \forall t \geq 0,
\]
where by $E(t)$ is given by (3.14).
The method we use here is based on the construction of a functional $\mathcal{L}$ for which an inequality of the form

$$\frac{d}{dt} \mathcal{L}(t) \leq -c \mathcal{L}(t)$$

holds, with $c > 0$. To construct such functional $\mathcal{L}$ we start from the energy identity. Then, we look for other functions whose derivatives introduce negative terms such as:

$$\sum_{n=1}^{N} |u_{n+1} - u_n|^2,$$

$$- \sum_{n=1}^{N} |u_{n+1} - u_n|^2,$$ etc. until we are able to construct the whole energy on the right hand side of the energy inequality. Finally, we take $\mathcal{L}$ as the summatory of such functions. Unfortunately, the above process also introduces terms without definite sign. To overcome this difficulty, we introduce a new multiplier which allows us to get the appropriate estimates. Finally, we choose carefully the coefficients of each term of $\mathcal{L}$, such that the resulting summatory satisfies the required inequality.

To show the exponential decay of the solution let us introduce the following functional

$$I(t) := \sum_{n=1}^{N} \left\{ \dot{u}_n (g \ast u_n)_{t} - \frac{1}{2} g \ast (u_{n+1} - u_n)^2 \right\}.$$

**Lemma 4.1.** Under the same assumptions as in Theorem 4.1, we have that for any $\delta > 0$ there exists $C_\delta > 0$ satisfying

$$\frac{d}{dt} I(t) \leq - (g(0) \sum_{n=1}^{N} |\dot{u}_n|^2 + \delta \sum_{n=1}^{N} |u_{n+1} - u_n|^2 + C_\delta \sum_{n=1}^{N} g \ast (u_{n+1} - u_n)$$

$$+ C_\delta g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2,$$

Here $C_\delta \to \infty$ when $\delta \to 0$.

**Proof.** Multiplying equation (2.1) by $(g \ast u_n)_{t}$ we get:

$$\sum_{n=1}^{N} \dot{u}_n (g \ast u_n)_{t} - \sum_{n=1}^{N} \Delta u_n (g \ast u_n)_{t} + \sum_{n=1}^{N} g \ast (u_{n+1} - u_n) \{(g \ast (u_{n+1} - u_n))_{t}\}$$

$$- \lambda_p [(u_{n-1} - u_n)^p + (u_{n+1} - u_n)^p] (g \ast u_n)_{t} = 0,$$

from where we have:
\[ (4.2) I_1 = \frac{d}{dt} \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t - \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t \]

\[ (4.3) = \frac{d}{dt} \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t - \sum_{n=1}^{N} \dot{u}_n(g(0) u_n + \dot{g} * u_n)_t \]

\[ (4.4) = \frac{d}{dt} \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t - g(0) \sum_{n=1}^{N} |\dot{u}_n|^2 - \sum_{n=1}^{N} \dot{u}_n \{ \dot{g}(0) u_n + \ddot{g} * u_n \} \]

\[ (4.5) = \frac{d}{dt} \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t - g(0) \sum_{n=1}^{N} |\dot{u}_n|^2 - \sum_{n=1}^{N} \dot{u}_n \{ \dot{g} \circ (u_{n+1} - u_n) + \ddot{g} \circ (u_{n+1} - u_n) \} \]

from where it follows that

\[ I_1(t) = \frac{d}{dt} \left\{ \sum_{n=1}^{N} \dot{u}_n(g * u_n)_t \right\} - g(0) \sum_{n=1}^{N} |\dot{u}_n|^2 - \sum_{n=1}^{N} \dot{u}_n \{ g' \circ (u_{n+1} - u_n) \} \]

On the other hand:

\[ I_2 = - \sum_{n=1}^{N} \Delta u_n \{ g(0) u_n + g' * u_n \} \]

\[ = \sum_{n=1}^{N} g(t)(u_{n+1} - u_n)^2 + \sum_{n=1}^{N} g' \circ u_n(u_{n+1} - u_n). \]

Which allows us to get the estimate

\[ I_2(t) \leq C_\delta g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 + C_\delta \sum_{n=1}^{N} g \delta(u_{n+1} - u_n) + \delta \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \]

where \( \delta \) is a small parameter to be fixed later and \( C_\delta \to \infty \) as \( \delta \to 0 \). Next, we analyze \( I_3 \),

\[ I_3(t) = - \sum_{n=1}^{N} g \circ (u_{n+1} - u_n) \cdot (g \circ (u_{n+1} - u_n)_t \]

\[ = - \frac{d}{dt} \sum_{n=1}^{N} |g \circ (u_{n+1} - u_n)|^2. \]

Finally, since

\[ I_4(t) = \lambda_p \sum_{n=1}^{N} [(u_{n-1} - u_n)^p - (u_{n+1} - u_n)^p](g * u_n)_t \]
Note that $I_4$ can be written as
\[
I_4(t) = \lambda_p g(0) \sum_{n=1}^{N} (u_{n+1} - u_n)^{p+1} + [(u_{n-1} - u_n)^p - (u_{n+1} - u_n)^p] (g' \circ u_n)_t
\]
\[
= \lambda_p g(t) \sum_{n=1}^{N} (u_{n+1} - u_n)^{p+1} + (u_{n+1} - u_n)^p g' \circ (u_{n+1} - u_n)
\]

Since the first order energy is bounded we get that
\[
|u_{j+1} - u_j| \leq \sqrt{\sum_{n=1}^{N} (u_{n+1} - u_n)^2} \leq \sqrt{2E(0)}
\]
Consequently
\[
\sum_{n=1}^{N} (u_{n+1} - u_n)^p g' \circ (u_{n+1} - u_n) \leq \sqrt{2E(0)}^{p-1} \sum_{n=1}^{N} |u_{n+1} - u_n| g' \circ (u_{n+1} - u_n)
\]
\[
\leq \delta \sum_{n=1}^{N} |u_{n+1} - u_n|^2 + C \delta E(0)^{p-1} g \square (u_{n+1} - u_n)
\]
Since
\[-I_1 = I_2 + I_3 + I_4\]
our conclusion follows by substitution of the relations for $I_i$, $i = 1, 2, 3$ into the above identity. The proof is now complete. \qed

**Lemma 4.2.** With the same hypotheses as Lemma 4.1, there exists positive constants $c_0$ and $C$, such that the solution of equation (2.1) satisfies,
\[
\frac{d}{dt} \sum_{n=1}^{N} u_n u_n \leq \sum_{n=1}^{N} \bar{u}_n^2 - c_0 \int_{\Omega} |u_{n+1} - u_n|^2 - \lambda_p \sum_{n=1}^{N} (u_{n+1} - u_n)^{p+1}
\]
\[
+ C \sum_{n=1}^{N} g \square (u_{n+1} - u_n).
\]

**Proof.** Let us multiply equation (2.1) by $u_n$ to get
\[
\sum_{n=1}^{N} \bar{u}_n u_n - \sum_{n=1}^{N} u_n \Delta u_n + \sum_{n=1}^{N} g \ast \Delta(u_{n+1} - u_n)u_n - \lambda_p [(u_{n-1} - u_n)^p + (u_{n+1} - u_n)^p] = 0.
\]
Note that
\[
I_1(t) = \frac{d}{dt} \sum_{n=1}^{N} u_n u_n - \sum_{n=1}^{N} |\bar{u}_n|^2.
\]
On the other hand
\[ I_2(t) = \sum_{n=1}^{N} |u_{n+1} - u_n|^2, \]
\[ I_3(t) = -\sum_{n=1}^{N} (u_{n+1} - u_n)g \ast (u_{n+1} - u_n) \]
\[ I_4(t) = -\lambda_p \sum_{n=1}^{N} (u_{n+1} - u_n)^{p+1} \]

Finally, using the fact that
\[ I_3(t) \leq \int_0^t g(s) ds |u_{n+1} - u_n|^2 - \int_0^t g(s) ds \Box (u_{n+1} - u_n) \]
our conclusion follows \[ \square \]

Let us denote by \( \mathcal{L}_{N_1}(t) \) the functional
\[ \mathcal{L}_{N_1}(t) = N_1 E(t) + I(t) + \frac{g(0)}{2} \sum_{n=1}^{N} u_n \dot{u}_n \]
where \( N_1 \) is a large positive number to be chosen later. In these conditions we get

**Proof of Theorem 4.1** From Lemma 4.1 and Lemma 4.2 we get that
\[
\frac{d}{dt} \left\{ I(t) + \frac{g(0)}{2} \sum_{n=1}^{N} \dot{u}_n \right\} \leq -\frac{g(0)}{2} \sum_{n=1}^{N} |\dot{u}_n|^2 - (C - \delta) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \\
+ C_\delta \left\{ g(\Box (u_{n+1} - u_n)) + g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}.
\]

Using Lemma ?? and the above inequality we arrive at
\[
\frac{d}{dt} \mathcal{L}_{N_1}(t) \leq -\kappa_0 \left\{ \sum_{n=1}^{N} |\dot{u}_n|^2 + \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\} \\
- (N_1 - C_\delta) \left\{ g(\Box (u_{n+1} - u_n)) + g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}.
\]

Therefore taking \( N_1 > 2C_\delta \), we get
\[
\frac{d}{dt} \mathcal{L}_{N_1}(t) \leq -\kappa_0 \left\{ \sum_{n=1}^{N} |\dot{u}_n|^2 + \sum_{n=1}^{N} |(u_{n+1} - u_n)|^2 \right\} \\
- \frac{N_1}{2} \left\{ g(\Box (u_{n+1} - u_n)) + g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}.
\]

It is easy to verify that there exist positive constant \( C_1 \) and \( C_2 \) such that
\[ (4.6) \quad C_1 E(t) \leq \mathcal{L}_{N_1}(t) \leq C_2 E(t). \]
Thus
\[
\frac{d}{dt} \mathcal{L}_{N_1}(t) \leq -\kappa_0 \mathcal{L}_{N_1}(t)
\]
From where we get that
\[
\mathcal{L}_{N_1}(t) \leq \mathcal{L}_{N_1}(0)e^{-\kappa_0 t} \quad \forall t \geq 0
\]
Using (4.6) once more, our conclusion follows. The proof is now complete \(\square\)

5. Polynomial stability

In this section we show that the first order energy decay polynomially when the kernel also decays polynomially. To do that, we will assume that the kernel \(g \in \mathcal{F}\) instead of hypothesis (3.4) satisfies the following inequalities

\[
g \in C^2([0, \infty]), \quad g(t) > 0, \quad |g''(t)| \leq cg(t)^{1+\frac{1}{p}}.
\]

\[
-\kappa_0 g(t)^{1+\frac{1}{p}} \leq g'(t) \leq \kappa_1 g(t)^{1+\frac{1}{p}}
\]

\[
\alpha := 1 - \int_0^\infty g(\tau) \, d\tau > 0
\]

Clearly (5.1)-(5.3) are sufficient conditions on \(g\) in order to prove Theorems 3.1 and 3.2. The above hypothesis on \(g\) imply that the kernel is of type \(g(t) \approx c(1 + t)^{-p}\). Under the above conditions we will show that the first order energy decays to zero with the same rate as \(g\). First we will prove the following Lemmas.

**Lemma 5.1.** Let \(m\) and \(h\) be integrable functions, \(0 \leq r < 1\) and \(q > 0\). Then, for \(t \geq 0\)
\[
\int_0^t |m(t-s)h(s)| \, ds \leq \left( \int_0^t |m(t-s)|^{1+\frac{1}{q}} |h(s)| \, ds \right)^{\frac{q}{q+1}} \left( \int_0^t |m(t-s)|^r |h(s)| \, ds \right)^{\frac{1}{q+1}}.
\]

**Proof:** In fact, let us take
\[
v(s) := |m(t-s)|^{\frac{1}{1+q}} |h(s)|^{\frac{1}{q+1}}, \quad w(s) := |m(t-s)|^r |h(s)|^{\frac{1}{q+1}}.
\]

Applying Hölder’s inequality to \(|m(s)h(s)| = v(s)w(s)\) with exponents \(\delta = \frac{q}{q+1}\) for \(v\) and \(\delta^* = q + 1\) for \(w\) our conclusion follows. \(\square\)

**Lemma 5.2.** Let \(\phi_j \in C(0, T), 1 \leq j \leq N + 1\). Then, for \(p > 1, 0 \leq r < 1\) and \(t \geq 0\), we have
\[
\left( \sum_{n=1}^N g \Box (\phi_{n+1} - \phi_n) \right)^{\frac{1+r}{1+q}} \leq 2 \left( \int_0^t g^r \, ds \sum_{n=1}^N |\phi_{n+1} - \phi_n|^2 \right)^{\frac{1}{1+r}} \sum_{n=1}^N g^{1+\frac{1}{p}} \Box (\phi_{n+1} - \phi_n),
\]

while for \(r = 0\) we get
\[
\left( \sum_{n=1}^N g \Box (\phi_{n+1} - \phi_n) \right)^{1+\frac{1}{p}} \leq 2 \left( \sum_{n=1}^N \int_0^t |\phi_{n+1} - \phi_n|^2 \, ds + t|\phi_{n+1} - \phi_n|^2 \right)^{\frac{1}{1+r}} \sum_{n=1}^N g^{1+\frac{1}{p}} \Box (\phi_{n+1} - \phi_n).
Proof: The above inequalities are a immediate consequence of Lemma 5.1 taking
\[ m(s) := g(s), \quad h(t, s) := |\phi_{n+1}(t) - \phi_n(t) - \phi_{n+1}(s) + \phi_n(s)|^2, \quad q := (1 - r)p. \]

Note that
\[ h(t, s) \leq 2|\phi_{n+1}(t) - \phi_n(t)|^2 + 2|\phi_{n+1}(s) - \phi_n(s)|^2 \]

From Lemma 5.1 we have that
\[ g^2(n+1 - n) = \int_0^t g(t - s)h(t, s)ds \]
\[ \int_0^t g(1 + \frac{1}{p}(t - s)h(t, s)ds \right)^{\frac{1}{1+(1-r)p}} \]
Recalling the definition of \( h \) and taking sommatory on \( n \) our conclusion follows. \( \Box \)

As in section 4 we have

Lemma 5.3. Under the above conditions and for \( g \in C^3 \), satisfying conditions (5.1)–(5.2), we have that for any \( \delta > 0 \) there exists \( C_\delta \) satisfying
\[ \frac{d}{dt}I(t) \leq -g(0) \sum_{n=1}^N |\tilde{u}_n|^2 + \delta \sum_{n=1}^N |u_{n+1} - u_n|^2 + C_\delta \sum_{n=1}^N g^{1+\frac{1}{p}}(u_{n+1} - u_n) \]
\[ +C_\delta g(t) \sum_{n=1}^N |u_{n+1} - u_n|^2 \]
where \( I(t) \) is as in section 4. Here \( C_\delta \to \infty \) when \( \delta \to 0 \).

Proof. Using the same procedure as in 4.1, assumption (5.2) and the inequality
\[ g' \circ (u_{n+1} - u_n) \leq \kappa_1 g^{1+\frac{1}{p}}(u_{n+1} - u_n) \]
our conclusion follows. The proof is now complete. \( \Box \)

Similarly

Lemma 5.4. With the same hypotheses as Lemma 4.1 and assuming that
\[ \int_0^\infty g^{1-\frac{1}{p}}(s)ds < \infty \]
we have that the solution of equation (2.1) satisfies,
\[ \frac{d}{dt} \sum_{n=1}^N u_n \tilde{u}_n \leq \sum_{n=1}^N |\tilde{u}_n|^2 - \alpha_0 \sum_{n=1}^N |u_{n+1} - u_n|^2 - \lambda_p \sum_{n=1}^N (u_{n+1} - u_n)^{p+1} \]
\[ +C \sum_{n=1}^N g^{1+\frac{1}{p}}(u_{n+1} - u_n). \]

Proof. The only difference with respect Lemma 4.2 is that we use the estimate
\[ g \circ (u_{n+1} - u_n) \leq \left( \int_0^t g^{1-\frac{1}{p}}ds \right)^{1/2} \left( g^{1+\frac{1}{p}}u_n \right)^{1/2} \]
from which our conclusion follows. \( \Box \)
Theorem 5.1 (Polynomial decay). Under the above notations and with assumptions (5.1)–(5.3), we have that the first order energy decays polynomially to zero, as time goes to infinity. That is, there exist a positive constant $C$ such that

$$E(t) \leq CE(0) \frac{1}{(1+t)^p}.$$ 

Proof. Using the above procedure as in the exponential case we get the inequality (for $r > 0$)

$$\frac{d}{dt} L_{N_1}(t) \leq -\kappa_0 \left\{ \sum_{n=1}^{N} |\hat{u}_n|^2 + \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}$$

$$- (N_1 - C_\delta) \left\{ g^{1+\frac{2}{p}} \Box (u_{n+1} - u_n) + g(t) \sum_{n=1}^{N} |(u_{n+1} - u_n)|^2 \right\}.$$ 

Therefore taking $N_1 > 2C_\delta$, we get

$$\frac{d}{dt} L_{N_1}(t) \leq -\kappa_0 \left\{ \sum_{n=1}^{N} |\hat{u}_n|^2 + \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}$$

$$- \frac{N}{2} \left\{ g^{1+\frac{2}{p}} \Box (u_{n+1} - u_n) + g(t) \sum_{n=1}^{N} |u_{n+1} - u_n|^2 \right\}.$$ 

It is easy to verify that there exist positive constants $C_1$ and $C_2$ such that

$$(5.4) \quad C_1 \mathcal{E}(t) \leq L_{N_1}(t) \leq C_2 \mathcal{E}(t) \quad \forall t \geq 0.$$ 

From Lemma 5.2 we arrive to

$$\frac{d}{dt} L_{N_1}(t) \leq -\kappa_1 [L_{N_1}(t)]^{\frac{1+\frac{2}{p}}{1+(1-r)p}}$$

which implies that

$$L_{N_1}(t) \leq L_{N_1}(0) \frac{1}{(1+t)^{(1-r)p}}.$$ 

Using Lemma 5.2 once more, and repeating the same above procedure for $r = 0$ we get

$$(5.6) \quad \frac{d}{dt} L_{N_1}(t) \leq -\kappa_1 [L_{N_1}(t)]^{\frac{1+\frac{2}{p}}{2}}$$

From where we get that

$$L_{N_1}(t) \leq L_{N_1}(0) \frac{1}{(1+t)^p}.$$ 

The proof is now complete \(\square\)

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REFERENCES


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