Asymptotic Behaviour in \( n \)-Dimensional Thermoelasticity

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Abstract—We study the thermoelastic system and we prove that the divergence of the displacement vector field and the thermal difference decay exponentially as time goes to infinity. Moreover, we show that the decay cannot hold in general.

Keywords—Thermoelasticity, Uniform rate of decay, Exponential decay, Initial boundary value problems.

1. INTRODUCTION

In one-dimensional thermoelasticity, thanks to the works [1–11], it is well known that the energy associated with the solution of the thermoelastic system decays exponentially as time goes to infinity. Whereas for \( n \)-dimensional materials, the situation is more complicated and there are only a few results concerning asymptotic behaviour. In general, it is not true that the total energy decays to zero as was shown in [12]. For example, for materials that occupy the whole \( \mathbb{R}^3 \), Dassios and Grillakis [13] showed that the heat difference and the curl free part of the displacement vector field decay uniformly in time like \( t^{3/2} \), while the free divergence part conserves its energy. In the special case of symmetrical solutions, when the material has a spherical shape, it was shown in [14] that the total energy decays exponentially. For bounded domain, Chiriță [15] proved the asymptotic equipartition of the mean kinetic and strain energy and that the thermal difference decays to zero, but no rate of decay was obtained. However, the question about a uniform rate of decay for bounded domains in its general form seems to be untouched. So to fill this gap, we study these points here.

The main result of this paper is to show that the curl free part of the displacement vector field, as well as the thermal difference, decays exponentially to zero as time goes to infinity, while the divergence free part conserves its energy. So, when the initial data is taken such that the divergence free part is zero, the energy decays exponentially, whereas, when the free divergence part is not zero, then the total energy does not decay to zero uniformly.

The system for isotropic thermoelastic materials is given by

\[
\begin{align*}
\ddot{u} - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \alpha \nabla \theta &= 0, & \text{in } \Omega, \\
\dot{\theta} - \kappa \Delta \theta + \alpha \text{div} u_t &= 0, & \text{in } \Omega,
\end{align*}
\] (1.1) (1.2)

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\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \theta(x,0) = \theta_0(x), \quad \text{in } \Omega, \]  
with the following boundary conditions:

\[ u(x,t) = 0, \quad \theta(x,t) = 0, \quad \text{on } \Gamma. \]  

In that follows, we will use the notation \( u_t, \dot{u}, \) and \( u_{tt}, \ddot{u} \) to indicate first- and second-order derivatives with respect to the time.

\section{2. Decomposition of the Energy and Exponential Decay}

In this section, we show that the displacement can be decomposed into two parts: a curl free part and the divergence free part. To do this, we solve system (1.1)-(1.4) over the spaces

\[ \mathcal{H}_c = \mathcal{W}_c \times \mathcal{E}_c \times L^2(\Omega), \]
\[ \mathcal{H}_d = \mathcal{W}_d \times \mathcal{E}_d, \]

where

\[ \mathcal{W}_c = \left\{ w \in [H^1_0(\Omega)]^n; \text{curl } (w) = 0 \right\}, \quad \mathcal{E}_c = \left\{ w \in [L^2(\Omega)]^n; \text{curl } (w) = 0 \right\}, \]
\[ \mathcal{W}_d = \left\{ w \in [H^1_0(\Omega)]^n; \text{div } (w) = 0 \right\}, \quad \mathcal{E}_d = \left\{ w \in [L^2(\Omega)]^n; \text{div } (w) = 0 \right\}. \]

\textbf{Remark 2.1.} If \( w \in [H^1_0(\Omega)]^n \), then there exists functions \( w^c, w^d \) in \([H^1_0(\Omega)]^n\), with \( \text{curl } w^c = 0 \) and \( \text{div } w^d = 0 \), such that

\[ w = w^c + w^d. \]  

In fact, the regularity follows immediately, and lets us shows that \( w^c = 0 \) on \( \partial \Omega \). To do this, suppose that \( w \) has compact support. Note that \( w^c = \nabla p \) and we can choose \( p \in H^1_0(\Omega) \). Let us denote by \( B \) any ball such that \( B \subset \Omega \setminus \text{support } (w) \). We will show that \( \nabla p = 0 \) over \( B \). Denoting by \( \eta \) the solution of the Dirichlet problem

\[ -\Delta \eta = f > 0, \quad \text{in } B, \quad \eta = 0, \quad \text{on } \partial B. \]

Let us extend \( \eta \) zero outside \( B \). Multiplying (2.1) by \( \nabla (\eta p) \) and using the orthogonality, we get

\[ \int_B |\nabla p|^2 \eta \, dx + \frac{1}{2} \int_B |p|^2 \, dx = 0. \]

From the maximum principle, we get that \( \eta > 0 \), hence, the above inequality implies that \( p = 0 \) over \( B \). Finally, since

\[ ||w||_{dc} = ||\text{div } w|| + ||\text{curl } w|| \]

is an equivalent norm in \([H^1_0(\Omega)]^n\), using density arguments, our conclusion follows.

So we have that

\[ [H^1_0(\Omega)]^n = \mathcal{W}_c \oplus \mathcal{W}_d, \quad \mathcal{W}_c \cap \mathcal{W}_d = \{0\}. \]

Let us denote by \( A_c \) and by \( A_d \) the following operators:

\[ A_c = \begin{pmatrix} 0 & I & 0 \\ A & 0 & -\alpha \nabla \\ 0 & -\alpha \text{div} & \kappa \Delta \end{pmatrix}, \quad A_d = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \]

with domains

\[ D(A_c) = \mathcal{W}_c \cap [H^2(\Omega)]^n \times \mathcal{W}_c \times H^1_0(\Omega) \cap H^2(\Omega), \quad D(A_d) = \mathcal{W}_c \cap [H^2(\Omega)]^n \times \mathcal{W}_c, \]
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where $A$ is the elastic operator $A = \mu \Delta + (\mu + \lambda) \nabla \cdot$. It is not difficult to show that $A_c$ and $A_d$ are infinitesimal generators of semigroups operators over $\mathcal{H}_c$ and $\mathcal{H}_d$, respectively. So, there exists only one solution $(u^c, u^d, \theta) \in D(A_d)$ of system

$$u^c_{tt} - \mu \Delta u^c - (\mu + \lambda) \nabla \cdot u^c + \alpha \nabla \theta = \nabla p, \quad \text{in } \Omega,$$

$$\theta_t - \kappa \Delta \theta + \alpha \nabla u^c = 0, \quad \text{in } \Omega,$$

$$u^c(x, 0) = u_0^c(x), \quad u^c_t(x, 0) = u_1^c(x), \quad \theta(x, 0) = \theta_0(x), \quad \text{in } \Omega,$$

$$u^c(x, t) = 0, \quad \theta(x, t) = 0, \quad \text{on } \Gamma,$$

where $\Delta p = 0$. Also, we have that there exists a solution $(u^d, u^d_t) \in D(A_d)$ for system

$$u^d_{tt} - \mu \Delta u^d = \nabla r, \quad \text{in } \Omega,$$

$$u^d(x, 0) = u_0^d(x), \quad u^d_t(x, 0) = u_1^d(x), \quad \text{in } \Omega,$$

$$u^d(x, t) = 0, \quad \text{on } \Gamma,$$

with $\Delta r = 0$ and $u_0 = u_0^c + u_0^d$, $u_1 = u_1^c + u_1^d$. Now we will prove that we can decompose $u$ solution of system (1.1),(1.2) as

$$u = u^c + u^d,$$

where $u^c$ is the solution of system (2.2)-(2.5) and $u^d$ is the solution of system (2.6)-(2.8). In fact, from Remark 2.1, $u$ can be decomposed into $u = U^c + U^d$. Substitution of this identity into equation (1.1) yields

$$\ddot{U}^c - \mu \Delta U^c - (\mu + \lambda) \nabla \cdot U^c + \alpha \nabla \theta = R, \quad \text{in } \Omega,$$

where $R = -U^d_{tt} + \mu \Delta U^d$. Since the curl of the left-hand side of the above equation is zero, we have that curl $R = 0$, therefore, there exists a function $q$ such that $R = \nabla q$, with $\Delta q = 0$. So, $U^c$ also satisfies equations (2.4)-(2.6). By the uniqueness, identity (2.9) holds. Similarly, we get $U^d = u^d$. Let us introduce the energies functions

$$E(t, u, \theta) = \int_{\Omega} |u_t|^2 + \mu |\nabla u|^2 + (\mu + \lambda) |\nabla \theta|^2 \, dx,$$

$$E_c(t, u, \theta) = \int_{\Omega} |u_t|^2 + (\mu + \lambda) |\nabla u|^2 + |\theta|^2 \, dx.$$

Under the above notations, we have the following lemma.

**LEMMA 2.1.** The energy associated with the solution of system (2.2)-(2.5) satisfies

$$\frac{d}{dt} E_c(t, u^c, \theta) = -\kappa \int_{\Omega} |\nabla \theta|^2 \, dx.$$

**PROOF.** Multiplying equation (2.2) by $u_t^c$ and equation (2.3) by $\theta$, summing up the product results, and using the fact that $u_t^c$ and $\nabla p$ are orthogonal, our identity follows.

**LEMMA 2.2.** Let us denote by $q$ a $C^2$ function such that $q = \nu$ over $\Gamma$, then we have

$$\frac{d}{dt} \left( \int_{\Omega} \hat{u}_t^c \frac{\partial q_i}{\partial x_j} \, dx + \hat{u}_t^c \frac{\partial u_t^c}{\partial x_j} \, dx \right) = -H(t) + \int_{\Omega} \hat{u}_t^c \frac{\partial q_i}{\partial x_j} \, dx - \frac{1}{2} \int_{\Omega} \text{div} \left( |\hat{u}_t^c|^2 - |\text{div} u_t^c|^2 \right) \, dx$$

$$- \int_{\Omega} \text{div} \left( (\Delta q_i) u_t^c + \nabla q_i \cdot \nabla u_t^c \right) \, dx - \alpha \int_{\Gamma} \nabla \theta \cdot \nabla (q \cdot u^c) \, dx.$$
PROOF. Let us multiply equation (2.2) by \( \nabla (q \cdot u) \) and using the orthogonality, we get

\[
\int_\Omega \bar{u}^c \cdot \nabla (q \cdot u^c) - Au^c \nabla (q \cdot u^c) + \alpha \nabla \theta \cdot \nabla (q \cdot u^c) \, dx = 0. \tag{2.10}
\]

Performing an integration by parts, and since over \( D(\mathcal{A}) \), \( Au = (2\mu + \lambda) \nabla \text{div } u \), we have

\[
\int_\Omega Au^c \cdot \nabla (q \cdot u^c) \, dx = (2\mu + \lambda) \int_\Omega \text{div } u^c \nabla (q \cdot u^c) \cdot \nu \, d\Gamma - (2\mu + \lambda) \int_\Omega \text{div } u^c \Delta (q \cdot u^c) \, dx.
\]

Since \( u^c = 0 \) on the boundary, we have

\[
I_1 = \int_\Omega \text{div } u^c \left[ q_i \frac{\partial u^c_j}{\partial x_j} \nu \right] \, d\Gamma = \int_\Gamma \text{div } u^c \left[ \nu_i \frac{\partial u^c_j}{\partial \nu} \right] \, d\Gamma = \int_\Gamma |\text{div } u^c|^2 \, d\Gamma.
\]

On the other hand,

\[
I_2 = \int_\Omega \text{div } u^c (\Delta q_i) u^c_i \, dx + \int_\Omega \text{div } u^c \frac{\partial q_i}{\partial x_j} \frac{\partial u^c_j}{\partial x_j} \, dx + \int_\Omega \text{div } u^c q_i \Delta u^c_i \, dx.
\]

Using the fact that there exist \( \phi \) such that \( u^c = \nabla \phi \), we have

\[
I_3 = \int_\Omega \Delta \phi q_i \Delta \frac{\partial \phi}{\partial x_i} \, dx = \frac{1}{2} \int_\Omega q_i \frac{\partial}{\partial x_i} |\Delta \phi|^2 \, dx = \frac{1}{2} \int_\Gamma q_i \nu_i |\text{div } u^c|^2 \, d\Gamma - \frac{1}{2} \int_\Omega \text{div } q |\text{div } u^c|^2 \, dx,
\]

from where it follows that

\[
I_2 = \int_\Omega \text{div } u^c (\Delta q_i) u^c_i \, dx + \int_\Omega \text{div } u^c q_i \frac{\partial u^c_j}{\partial x_j} \, dx + \frac{1}{2} \int_\Gamma |\text{div } u^c|^2 \, d\Gamma - \frac{1}{2} \int_\Omega \text{div } q |\text{div } u^c|^2 \, dx.
\]

Finally, we have that

\[
\int_\Omega \bar{u}^c \cdot \nabla (q \cdot u^c) \, dx = \int_\Omega \bar{u}^c_j \frac{\partial q_i}{\partial x_j} u^c_i \, dx + \int_\Omega \bar{u}^c_j \frac{\partial u^c_i}{\partial x_j} \, dx = \frac{d}{dt} \left\{ \int_\Omega \bar{u}^c_j \frac{\partial q_i}{\partial x_j} u^c_i \, dx + \int_\Omega \bar{u}^c_j \frac{\partial u^c_i}{\partial x_j} \, dx \right\} - \int_\Omega \bar{u}^c_j \frac{\partial q_i}{\partial x_j} u^c_i \, dx - \int_\Omega \bar{u}^c_j \frac{\partial u^c_i}{\partial x_j} \, dx.
\]

Using the fact that \( \text{curl } u^c = 0 \), we get

\[
\int_\Omega \bar{u}^c \cdot \nabla (q \cdot u) \, dx = \frac{d}{dt} \left\{ \int_\Omega \bar{u}^c_j \frac{\partial q_i}{\partial x_j} u^c_i \, dx + \int_\Omega \bar{u}^c_j \frac{\partial u^c_i}{\partial x_j} \, dx \right\} - \int_\Omega \bar{u}^c_j \frac{\partial q_i}{\partial x_j} u^c_i \, dx + \frac{1}{2} \int_\Gamma \text{div } q |\text{curl } u^c|^2 \, d\Gamma.
\]

Substitution of this identity and the values of \( I_1 \) and \( I_2 \) into equation (2.10) yields our conclusion.

Now, we introduce the multiplicators \( \phi \) and \( w \) given by

\[
-\Delta \phi = \text{div } u^c, \quad -\Delta w = \theta, \quad \text{in } \Omega,
\]

with

\[
\phi(x, t) = 0, \quad w(x, t) = 0, \quad \text{on } \Gamma.
\]

LEMMA 2.3. Under the above notations, we have

\[
\frac{d}{dt} \int_\Omega \theta \phi_t \, dx = -\kappa \int_\Omega \nabla \phi_t \cdot \nabla \phi_t \, dx - \alpha \int_\Omega |\nabla \phi_t|^2 \, dx - (2\mu + \lambda) \int_\Omega \theta \text{div } u^c \, dx
\]

\[
+ (2\mu + \lambda) \int_\Gamma \frac{\partial w}{\partial \nu} \text{div } u^c \, d\Gamma + \alpha \int_\Omega \nabla w \cdot \nabla \theta \, dx.
\]
PROOF. Multiplying equation (2.2) by $\phi_t$, we get

$$
\frac{d}{dt} \int_\Omega \theta \phi_t \, dx = -\kappa \int_\Omega \nabla \theta \cdot \nabla \phi_t \, dx - \alpha \int_\Omega |\nabla \phi_t|^2 \, dx + \int_\Omega \theta \phi_{tt} \, dx.
$$

To arrive at our identity, note that

$$
\int_\Omega \theta \phi_{tt} \, dx = -\int_\Omega \Delta \omega \phi_{tt} \, dx = -\int_\Omega \omega \, d\omega = \int_\Omega \nabla \omega \cdot \nabla \phi_t \, dx = \int_\Omega \nabla \omega \cdot \nabla \phi_t \, dx + \int_\Omega \phi_{tt} \, dx.
$$

from where our conclusion follows. 

**Lemma 2.4.** Let us suppose that $v \in [H_0^1(\Omega)]^n$ and $\text{curl} \, v = 0$, then there exist positive constants $c_0$ and $c_1$ for which we have

$$
c_0 \|v\|_{L^2} \leq \|\text{div} \, v\|_{H^{-1}} \leq c_1 \|v\|_{L^2}.
$$

PROOF. We only prove the left-hand side inequality. The other is immediate. In fact, by hypotheses there exist a function $p$ for which we have $v = \nabla p$. Then we can decompose $v$ into two parts:

$$
v = \nabla \phi_1 + \nabla \phi_2,
$$

where

$$
\Delta \phi_1 = \text{div} \, v, \quad \Delta \phi_2 = 0, \quad \phi_1(x) = 0, \quad \frac{\partial \phi_2}{\partial \nu} = -\frac{\partial \phi_1}{\partial \nu}, \quad \text{on} \, \Gamma.
$$

To see that identity (2.11) holds, consider the difference $F = p - (\phi_1 + \phi_2)$, then $F$ satisfies

$$
\Delta F = 0, \quad \text{in} \, \Omega, \quad \frac{\partial F}{\partial \nu} = 0, \quad \text{on} \, \Gamma.
$$

By the uniqueness of the Neumann problem, we get (2.11). We can choose $\phi_2$ such that $\int_\Omega \phi_2 \, dx = 0$. Since

$$
\|\nabla \phi_1\|_{L^2(\Omega)} \leq \|\text{div} \, v\|_{H^{-1}(\Omega)} \quad \text{and} \quad \|\nabla \phi_2\|_{L^2(\Omega)} \leq \left\| \frac{\partial \phi_1}{\partial \nu} \right\|_{H^{-1/2}(\Gamma)} \leq C \|\nabla \phi_1\|_{L^2(\Omega)}.
$$

In the last inequality, we have applied the trace theorem and Poincaré's inequality. So, we finally arrive at

$$
\|v\|_{L^2(\Omega)} = \|\nabla \phi_1 + \nabla \phi_2\|_{L^2(\Omega)} \leq C \|\nabla \phi_1\|_{L^2(\Omega)} \leq C \|\text{div} \, v\|_{H^{-1}(\Omega)},
$$

from where our conclusion follows.

The above lemma in particular implies that

$$
\|u_t^2\|_{H^{-1}(\Omega)} \leq c \|\text{div} \, u_t^2\|_{H_0^1(\Omega)} \leq C \|\nabla \phi_t\|_{L^2(\Omega)}.
$$

Now we have the conditions to show the main result of this paper.

**Theorem 2.1.** Let us suppose that the initial data $(u_0, u_1, \theta_0) \in \mathcal{H}_c$, then the solution of (1.1)-(1.4) satisfies

$$
\|\text{div} \, u_t\|_{H^{-1}(\Omega)} + \int_\Omega |\text{div} \, u|^2 + |\theta|^2 \, dx \leq C E(0, u^c, \theta) e^{-\gamma t}.
$$
Note that \( \text{div } u^c = \text{div } u \). From Lemma 2.4, the definition of \( w \), and Lemma 2.3, there exist \( C_6 \) and \( \alpha_0 > 0 \), for which we have

\[
\frac{d}{dt} \int_\Omega \theta \phi_t \, dx \leq C_6 \int_\Omega |\nabla \theta|^2 \, dx + \delta \int_\Gamma |\text{div } u^c|^2 \, d\Gamma + \delta \int_\Omega |\text{div } u^c|^2 \, dx - \alpha_0 \int_\Omega |u_1^c|^2 \, dx.
\]

From Lemma 2.2, we have that

\[
\frac{d}{dt} H(t) \leq -\frac{2\mu + \lambda}{2} \int_\Gamma |\text{div } u^c|^2 \, d\Gamma + CE(t, u^c, \theta).
\]

From the two above inequalities and taking \( \delta \) small enough, we get

\[
\frac{d}{dt} \int_\Omega u^c \cdot u^c_t \, dx = \int_\Omega |u_1^c|^2 \, dx - (2\mu + \lambda) \int_\Omega |\text{div } u^c|^2 \, dx - \alpha \int_\Omega \nabla \theta \cdot u^c \, dx
\]

\[
\leq \int_\Omega |u_1^c|^2 \, dx + C \int_\Omega |\nabla \theta|^2 \, dx - \frac{2\mu + \lambda}{2} \int_\Gamma |\text{div } u^c|^2 \, dx,
\]

from where it follows that

\[
\frac{d}{dt} \left\{ \int_\Omega \theta \phi_t \, dx + \frac{4\delta}{2\mu + \lambda} H(t) + \frac{4\delta}{2\mu + \lambda} \int_\Omega u^c \cdot u^c_t \, dx \right\}
\]

\[
\leq C \int_\Omega |\nabla \theta|^2 \, dx - \delta \int_\Gamma |\text{div } u^c|^2 \, dx - \frac{\alpha_0}{4} \int_\Omega |u_1^c|^2 \, dx - \delta \int_\Gamma |\text{div } u^c|^2 \, d\Gamma.
\]

Let us introduce the functional

\[
L(t) = NE(t, u^c, \theta) + \int_\Omega \theta \phi_t \, dx + \frac{4\delta}{2\mu + \lambda} H(t) + \frac{4\delta}{2\mu + \lambda} \int_\Omega u^c \cdot u^c_t \, dx.
\]

It is not difficult to see that, for \( N \) large enough, there exist \( c_0 \) and \( c_1 > 0 \) such that

\[
c_0 E(t, u^c, \theta) \leq L(t) \leq c_1 E(t, u^c, \theta).
\]

From the last two inequalities, we get that

\[
L(t) \leq -\gamma L(t) \Rightarrow L(t) \leq L(0)e^{-\gamma t};
\]

therefore, our conclusion follows.

Finally, we will prove that when there exists a nontrivial solution of equations (2.5)-(2.7), then the total energy does not decay to zero.

**Theorem 2.2.** Let us take initial data such that \( u^d_0 \neq 0 \) or \( u^d_1 \neq 0 \). Then the total energy associated with system (2.2) satisfies

\[
E(t) \geq \int_\Omega |u_1^d|^2 + \mu |\nabla u_0^d|^2 \, dx.
\]

**Proof.** In fact, we can decompose the displacement vector fields into two parts, given by equations (2.6)-(2.11) and (2.5)-(2.7). Note that by the orthogonality condition, we have that

\[
\int_\Omega u^c \cdot \Delta u^d \, dx = \int_\Omega \Delta u^c \cdot u^d \, dx = \int_\Omega \text{div } u^c \cdot u^d \, dx = 0,
\]

from where we obtain

\[
E(t, u, \theta) = E_c(t, u^c, \theta) + \int_\Omega |u_1^d|^2 + \mu |\nabla u_0^d|^2 \, dx \geq \int_\Omega |u_1^d|^2 + \mu |\nabla u_0^d|^2 \, dx;
\]

therefore, our conclusion follows.
REFERENCES