Global Solution and Regularizing Properties on a Class of Nonlinear Evolution Equation

Jaime E. Muñoz Rivera*

National Laboratory for Scientific Computation, Department of Research and Development,
Rua Lauro Müller 455, Botafogo Cep. 22290, Rio de Janeiro, RJ, Brazil;
and IM, Federal University of Rio de Janeiro,
Ilha da Cidade Universitária, Rio de Janeiro, RJ, Brazil

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In this paper we will consider the equation

\[ u_{tt} + M[u(t)]Au + R[u(t)]Bu = 0. \]

where

\[ [u(t)] = (u, u_t), (Au, u_t), \|u\|_2, \|Au\|_2^2, \|u_t\|_2, \|Au_t\|_2. \]

The initial value problem is proved to be locally well posed for initial data taken in \( D(A^2) \cap D(A^3) \) and globally well posed for small data. In this case we also show the exponential decay of the solution as time goes to infinity. The main result of this paper is to prove that the solution has the smoothing effect property on the initial data. This means that, if the initial data belongs to \( D(A^2) \cap D(A^3) \), then the solution \( u \) belongs to \( C^\infty([0, +\infty) \cap [D(A^k)] \) \( k \in \mathbb{N}, \) provided \( M, N, \) and \( R \) are \( C^\infty \)-function.

1. Introduction

In this paper we will study models whose solutions describes oscillations that are set up directly through the interaction between a fluid and a surface across which it is moving. We will consider, for example, models of oscillation in pipes and supersonic panel flutter. The equations of such models, over the reference configuration \([0, L]\), written in one dimensional form can be shown to be of the form:

\[ u_{tt} + u_{xxxx} + \gamma u_{xxxt} - \left( \int_0^L K |u_x|^2 + \sigma u_{xx} \right) dx \right) u_{xx} + O = 0 \]

where \( O \) denotes other terms of lower order derivatives, \( \gamma \) and \( \sigma \) are the structural viscoelastic damping coefficient and \( K > 0 \) is a measure of the membrane stiffness (See Dowell [3], Marsden et al. [4] and [8]).

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We will summarize this class of equations in the following abstract form. Let us denote by $A$ and $B$ unbounded operators of the Hilbert space $H$, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, satisfying the following hypotheses

H1. $A$ is a positive selfadjoint operator of $H$, such that $D(A^l)$ has compact embedding in $H$ for any $l > 0$, $l \in \mathbb{R}$.

H2. $B$ is a symmetrical operator which comutes with $A$ such that

$$0 \leq (Bw, w) \leq \beta \|Aw\| \|w\| \quad \forall w \in D(A).$$

The corresponding abstract equation is written as follows

$$u_{tt} + M([u(t)])Au + R([u(t)])A^s u + N([u(t)])Bu = 0,$$

$$u(0) = u_0, \quad u_t(0) = u_1,$$

(1.1)

where $M, N,$ and $R$ are nonnegative functions satisfying $M, N, R : \mathbb{R}^5 \to \mathbb{R}$ and by $[u(t)]$ we are denoting the vector

$$[u(t)] = ((u(t), u_t(t)), (Au(t), u_{tt}(t)), \|A^{1/2}u(t)\|^2, \|A^{1/2}u_t(t)\|^2, \|Au(t)\|^2) \in \mathbb{R}^5.$$

For example the panel flutter equation is obtained taking $A$ and $B$ as

$$D(A) = D(B) = \{w \in H^1_0([0, L]) \cap H^2([0, L]) ; w_{xx} \in H^1_0([0, L]) \cap H^2([0, L])\}$$

$$A w = w_{xxxx}, \quad B w = w_{xxxx} + \sqrt{\rho \delta} w,$$

where $\sqrt{\rho \delta}$ is denoting the aerodynamic damping and by $\rho$ the dynamic pressure. The power will be take as $\gamma = \frac{1}{2}$, so we have

$$A^{1/2} = w_{xx}, \quad D(A^{1/2}) = H^1_0([0, L]) \cap H^2([0, L]).$$

and finally

$$M = 1, \quad N = \gamma, \quad R([u(t)]) = I - \rho + K \int_0^L |u_x|^2 \ dx + \sigma \int_0^L u_x u_{tt} \ dx.$$

(See Marsden et al., ibid.).

The nonlinear equation for small amplitude of an elastic string is obtained from (1.1) by taking $N = R = 0$ and $[u] = \|A^{1/2}u\|^2$ which yields

$$u_{tt} + M(\|A^{1/2}u\|^2)Au = 0.$$

(1.2)

This equation has different behaviour with respect to the semilinear wave equation with non local nonlinearity, because due to breaking waves and
the formation of shocks it may not generally has a globally defined smooth solution, no matter how smooth \( u_0 \) and \( u_1 \) are, which is not the case for equation (1.2) that has global solution for analytical initial data as was showed by Arosio \textit{et al.} [1]. The question here is about the existence of global solution when the initial data is taking in the usual Sobolev's spaces (or \( D(A') \times D(A') \) in our framework). To solve this problem for equations relative to (1.2) several authors ([9], [10], [11]) have considered damping terms as \( A^2 u, Au \) or \( A^3 u \), which allows them to secure global estimates.

In this paper we will prove the local existence theorem for equation (1.1) when H1–H2 and

\[ M([u(0)]) > 0; N([u(0)]) \geq 0 \quad \text{and} \quad R([u(0)]) > 0, \]

where by \([u(0)]\) we mean

\[ [u(0)] = ((u_0, u_1), (Au_0, u_1), \|A^{1/2} u_0\|^2, \|A^{1/2} u_1\|^2, \|Au_0\|^2), \]

holds. To get global solution for small data, we will change hypotheses H2 and H3 by

\[ G1. B \text{ is a symmetric operator which commutes with } A \text{ such that } \beta_0 \|w\|^2 \leq (Bw, w) \leq \gamma \|A^{1/2} w\|^2. \]

\[ G2. M \text{ and } N \text{ are } C^1\text{-function such that} \]

\[ M([0]) > 0, \quad N([0]) > 0 \quad \text{and} \quad R([0]) > 0. \]

The main result of this paper is to prove that equation (1.1) has the smoothness effect property, this means that the solution of (1.1) satisfies

\[ u \in C^\infty([0, \infty); D(A^\infty)) \quad \text{where} \quad D(A^\infty) = \bigcap_{k \in \mathbb{N}} D(A^k), \]

no matter the regularity of the initial data \((u_0, u_1)\) has, provided the following hypothesis holds.

\[ S1. B = A; 0 \leq \alpha < 1. \]

Smoothness effect is not expected for any non negative \( C^1 \) function \( M \); in fact if \( u_0 \) is such that \( M([u(0)]) = 0 \) and \( R = 0 \) then the function \( t \mapsto u(t) = u_0 \) is the solution of (1.1) when \( u_1 = 0 \), therefore we have no smoothness effect in this case. As an application of the smoothness effect property we can say that nonlinear oscillations of the panel flutter (to fix ideas) are very smooth in positive time, no matter the regularity of the initial oscillation has.
2. Existence and Regularity

It is easy to verify that for any positive real number $l$ there exist only one solution for the linear equation

$$v_{tt} + M(t)Av + N(t)Bv + R(t)A^Tv = 0$$

$$v(0) = u_0 \in D(A^\gamma); \quad v_1(0) = u_1 \in D(A^{-1/2});$$

satisfying

$$v \in C([0, T]; D(A^\gamma)) \cap C^1([0, T]; D(A^{1-1/2})) \cap C^2([0, T]; D(A^{1-3/2}))$$

provided $M, N$ and $R$ belong to $C(0, T)$ and satisfy

$$M(t) \geq M_0 > 0; \quad N(t), R(t) \geq 0.$$

To prove the existence of local solutions we will use the fixed point Theorem for contractions, for this reason we introduce the space $\mathcal{H}$ given by

$$\mathcal{H} = \{ v \in C([0, T]; D(A^{1/2})); v_1 \in C([0, T]; D(A^{1/2})) \},$$

which is a Banach space with respect to the norm

$$\|v\|_{\mathcal{H}}^2 = \|v\|_{C([0, T]; D(A^{1/2}))}^2 + \|v_1\|_{C([0, T]; D(A^{1/2}))}^2.$$

Let us define the operator $T: \mathcal{H} \to \mathcal{H}$ as

$$w \mapsto Tw = v,$$

where $v$ is the solution of the equation

$$v_{tt} + M([w(t)]) Av + N([w(t)]) Bv + R([w(t)]) A^T v = 0$$

$$v(0) = u_0 \in D(A^\gamma); \quad v_1(0) = u_1 \in D(A^{-1/2});$$

(2.1)

we will show that $T$ has a fixed point when $T$ is small enough. To do this we will use energy estimates. In order to take advantage of the dissipative properties of Eq. (1.1) (case G2 holds), we will introduce the energy function:

$$E(t; v) = \frac{1}{2} \|v\|^2 + \frac{m_0}{2} \|A^{1/2} v(t)\|^2 + \frac{n_0}{2} \|A^{1/2} v(t)\|^2$$

where

$$m_0 := M([u(0)]), \quad n_0 := N([u(0)]) \quad \text{and} \quad r_0 := R([u(0)]).$$
Note that for any $w \in \mathcal{H}$ such that
$$w(0) = u_0, \quad w_t(0) = u_1,$$
we get
$$[w(0)] = [u(0)] = ([u_0, u_1], (Au_0, u_1, \|A^{1/2}u_0\|^2, \|A^{1/2}u_1\|^2, \|Au_0\|^2).$$
So that, if $w$ satisfies (2.2),
$$E(0; w) = \frac{1}{2} \|u_1\|^2 + \frac{m_0}{2} \|A^{1/2}u_0\|^2 + \frac{n_0}{2} \|A^{1/2}u_1\|^2.$$
Let us take $\nu \in \mathcal{H}$ such that $A^{1/2}\nu$ belongs to $\mathcal{H}$ satisfies (2.2). We will denote by $\kappa$ the number
$$\kappa := 3E(0; A^{1/2}v)\nu.$$
It is not difficult to see that there exists a positive constant $c_0$ such that
$$|\nu(t)| \leq c_0 E(t; A^{1/2}v), \quad (2.3)$$
where $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^3$. In order to facilitate our analysis we will introduce the following notations
$$\mu := \max_{|\xi| \leq \infty} \{ |\partial^\alpha M(\xi)|, |\partial^\alpha N(\xi)|, |\partial^\alpha R(\xi)|; |\xi| \leq 1 \},$$
$$F_i(t, w) := M([w(t)]) - M([w(0)]), \quad F_2(t, w) := N([w(t)]) - N([w(0)]),$$
$$F_3(t, w) := R([w(t)]) - R([w(0)]).$$
In these conditions, for a given initial data $(u_0, u_1)$ taken in $D(A^2_0) \times D(A^{1/2})$, we will define the closed convex subset $\mathcal{W}(0, T)$ of $\mathcal{H}$ as the set given by
$$\mathcal{W}(0, T) := \{ w \in \mathcal{H}; A^{1/2}w, A^{1/2}w \in \mathcal{H}, \|A^{1/2}w\| \leq c_1 E(t; A^{1/2}v) \leq c_1 \kappa \},$$
where $c_1$ is such that
$$\|M([w(t)])A\nu + N([w(t)])B\nu + R([w(t)])A\nu\|^2 \leq c_1 \mu E(t; A^{1/2}v). \quad (2.4)$$
So to prove that $\mathcal{F}$ has a fixed point we will show that $\mathcal{W}(0, T)$ is invariant by $\mathcal{F}$ and the restriction of $\mathcal{F}$ to $\mathcal{W}(0, T)$ is a contraction in $\mathcal{W}$. Following remarks are in order.

**Remark 2.1.** With the above notations there exists $c_2 > 0$ satisfying
$$\left| \frac{d}{dt} F_i(t, w) \right| \leq c_2 \kappa, \quad i = 1, 2, 3, \quad \forall w \in \mathcal{W}(0, T).$$
In fact, let us denote by \( \mathcal{L} \) any of the functions \( M, N, \) or \( R \). Applying the chain rule we have

\[
\frac{d}{dt} F_i(t) := \frac{\partial \mathcal{L}}{\partial X_1} \left\{ \| w_i(t) \|^2 + (w(t), w_i(t)) \right\}
+ \frac{\partial \mathcal{L}}{\partial X_2} \left\{ \| A^{1/2} w_i(t) \|^2 + (A w(t), w_i(t)) \right\}
+ \frac{\partial \mathcal{L}}{\partial X_3} \left\{ (A w(t), w_i(t)) \right\}
\frac{\partial \mathcal{L}}{\partial X_4} \left\{ (A w(t), w_i(t)) \right\}
\frac{\partial \mathcal{L}}{\partial X_5} \left\{ (A w(t), w_i(t)) \right\}.
\]

Using the definition of \( \mathcal{W}(0, T) \) and \( \mu \) our assertion follows for some appropriate constant \( c_2 > 0 \). Here we also use the continuous imersion of \( D(A^{1/2}) \subset D(A^{1/2}) \).

Using Remark 2.1 and the Mean Value Theorem (in time) we have

\[
|F_i(t, w)| \leq \mu c_2 k, \quad i = 1, 2, 3, \quad \forall \in \mathcal{W}(0, T).
\]

**Remark 2.2.** Let us take \( \varphi, \vartheta \in \mathcal{W}(0, T) \) then we get

\[
|\varphi(t) - \vartheta(t)| \leq c_3 \sqrt{2k} E(t; \varphi - \vartheta)^{1/2}.
\]

In fact, since

\[
|\varphi - \vartheta| \leq |\varphi - \varphi_i| + |\varphi_i - \vartheta|,
\]

\[
|A \varphi - A \vartheta| \leq |A \varphi - A \varphi_i| + |A \varphi_i - A \vartheta|,
\]

\[
|A^{1/2} \varphi|^2 - |A^{1/2} \vartheta|^2 \leq |(A^{1/2} \varphi + A^{1/2} \vartheta, A^{1/2} \varphi - A^{1/2} \vartheta)|
\]

\[
|A^{1/2} \varphi|^2 - |A^{1/2} \vartheta|^2 \leq |(A^{1/2} \varphi + A^{1/2} \vartheta, A^{1/2} \varphi - A^{1/2} \vartheta)|
\]

\[
|A \varphi|^2 - |A \vartheta|^2 \leq |(A \varphi + A \vartheta, A \varphi - A \vartheta)|
\]

Using the triangle inequality and recalling the definition of \( |\cdot| \) and \( E(t; v) \), our assertion follows.

Let us introduce the function

\[
L_u(t; v) := E(t; v) - \frac{1}{2} F_i(t, w) \| A^{1/2} v(t) \|^2 - \frac{1}{2} F_i(t, w) \| A^{1/2} v(t) \|^2.
\]

**Remark 2.3.** For any \( t < m_0/4k \mu c_2 (1 + \lambda_1^{-1}) := T_i \), we have that

\[
\frac{1}{2} E(t; v) \leq L_u(t; v) \leq \frac{1}{2} E(t; v).
\]
Where $\lambda_1$ is the first eigenvalue of $A$. In fact, using (2.5) and the fact $t \leq T_1$ we have
\[
|F_1(t, w) \|A^{1/2}v\|^2 + F_2(t, w) \|A^{3/2}v\|^2 | \\
\leq t\nu c_\delta (1 + \lambda_1^{t^{-1}}) \|A^{1/2}v\|^2 \leq E(t; v).
\]
Recalling the definition of $L_u(t; v)$, our assertion follows. 

**Theorem 2.1.** Under hypothesis H1–H2 and for any initial datum $(u_0, u_1) \in D(A^2) \times D(A^{3/2})$ there exist $0 < T' < T_1$ and a local strong solution $u$ of (1.1) satisfying
\[
u \in C([0, T']; D(A^2)) \cap C^1([0, T'); D(A^{3/2})) \cap C^2([0, T]; D(A^{1/2})).
\]
Moreover if $(u_0, u_1) \in D(A^\beta) \times D(A^\beta)$ we have
\[
u \in C([0, T']; D(A^\beta)) \cap C^1([0, T']; D(A^{\beta-1/2})) \cap C^2([0, T'; D(A^{\beta-3/2})].
\]

**Proof.** First, using multiplicative techniques, we will prove that $W(0, T_1)$ is invariant by $\mathcal{T}$. The difficulty here appears because we need more regularity in $\nu$ than the obtained in the linear equation for initial data in $D(A^2) \times D(A^{3/2})$. To overcome this problem we will take initial data in $D(A^2) \times D(A^\beta)$ and then we will use density arguments. Let us regard equation (2.1) as
\[
u_{tt} + m_0 Av + N([w(t)]) Bv + r_0 A^\beta v \\
= \{m_0 - M([w(t)])\} Av + \{r_0 - R([w(t)])\} A^\beta v.
\]
Multiplying by $A^\beta v$, we have
\[
\frac{d}{dt} E(t; A^{3/2}v) + N([w(t)])(BA^{3/2}v, A^{3/2}v) \\
\leq\frac{1}{2} F_1(t, w) \frac{d}{dt} \|A^\beta v(t)\|^2 + \frac{1}{2} F_2(t, w) \frac{d}{dt} \|A^{(3+\beta)/2}v(t)\|^2.
\]
From remark (2.1) it follows
\[
\frac{d}{dt} L_u(t; A^{3/2}v) \leq -\frac{1}{2} F_1'(t) \|A^\beta v(t)\|^2 - \frac{1}{2} F_2' \|A^{(3+\beta)/2}v(t)\|^2 \\
\leq \frac{\mu}{2} c_\delta (1 + \lambda_1^{t^{-1}}) \|A^\beta v(t)\|^2.
\]
Integration and applying Remark (2.3) yields
\[ E(t; A^{3/2}v) \leq 3E(0; A^{3/2}v) + \frac{1}{T_1} \int_0^t E(\tau; A^{3/2}v) \, d\tau, \]
From Gronwall inequality we get
\[ E(t; A^{3/2}v) \leq 3E(0; A^{3/2}v) e^{(cT_1)} \leq 3E(0; A^{3/2}v) e := \kappa. \]
Using the density of \( D(A^{1/2}) \times D(A^1) \) in \( D(A^2) \times D(A^{3/2}) \), (2.1) and (2.4), we show that \( v \in \mathcal{H}(0, T_1) \). Finally we will prove that there exists \( T' > 0 \) for which the restriction of \( \mathcal{F} \) to \( \mathcal{H}(0, T') \) is a contraction in \( \mathcal{H} \). To do this let us take \( w^1, w^2 \in \mathcal{H}(0, T') \) and let us denote by
\[ v' = \mathcal{F}w', \quad v = v^1 - v^2, \quad W = w^1 - w^2. \]
In this conditions we have that \( V \) satisfies
\[ V'' + m_0AV + N([w^1(t)]) BV + r_0A^\alpha v \\
- \{ M([w^1(t)]) - M([w^2(t)]) \} AV - \{ N([w^1(t)] - N([w^2(t)]) \} BV^2 \\
- \{ R([w^1(t)] - R([w^2(t)]) \} A^\gamma v^2. \]
Multiplying (2.6) by \( V_t \) and applying Remarks 2.1–2.3 we have
\[ \frac{d}{dt} L_{\gamma'}(t; V) \leq \mu c_5 \kappa (1 + \lambda_1^{2-1}) \| A^2 V(t) \|^2 \\
+ \mu c_3 \sqrt{2\kappa \frac{E(t; W)}{V}} \left\{ \| \gamma(A^2v) \| + \beta \| Av^2 \| + \| A^\gamma v^2 \| \right\}. \]
Note that
\[ w \mapsto \text{Sup}\{ \sqrt{E(t; w)}; 0 \leq t \leq T_1 \} \]
defines a norm in \( \mathcal{H} \) which is equivalent to \( \| \cdot \|_{\mathcal{H}} \). Since \( v^2 \in \mathcal{H}(0, T_1) \) we have \( E(t; A^{3/2}v) \leq \kappa \), hence after a time integration we see that there exists \( c_5 > 0 \) satisfying
\[ E(t, V) \leq t \| W \|_{\mathcal{H}}^2 + c_5 \int_0^t E(\tau, V) \, d\tau. \]
Using Gronwall's inequality we obtain
\[ E(t, V) \leq T' \| W \|_{\mathcal{H}}^2 e^{c_5 T'}, \]
which means that \( \mathcal{F} \) is a contraction on \( \mathcal{H}(0, T') \) small, hence the existence result follows. The regularity follows from the regularity of the linear
equation and since \( M, N, \) and \( R \) are \( C^1 \)-functions, the uniqueness is obtained by standard means. The proof is now complete.

In order to prove the global existence Theorem we will take:
\[
\begin{align*}
m_0 &= M([0]); & F_1(t) &= M([u(t)]) - M([0]); \\
n_0 &= N([0]); & F_2(t) &= N([u(t)]) - N([0]); \\
r_0 &= R([0]); & F_3(t) &= R([u(t)]) - R([0]).
\end{align*}
\]

As in Remark 2.2 we get
\[
|F_i(t)| \leq c_{\delta} E(t; A^{3/2}u) \quad (i = 1, 2, 3). \tag{2.7}
\]

Using the mean value Theorem for several variables we can show without loss of generality that for \( c_\delta \) we also have
\[
|F_i(t)| \leq c_{\delta} E(t; A^{3/2}u) \quad (i = 1, 2, 3). \tag{2.8}
\]

Our next goal is to prove that there exists \( \varepsilon > 0 \) such that for any initial data satisfying
\[
E(0; A^{3/2}u) < \varepsilon \tag{2.9}
\]
equation (1.1) has a global solution.

Recalling Zorn's lemma, Theorem 2.1 says that equation (1.1) possesses only one maximal solution \( u([0, T_m] \to D(A^2), \) so as to prove that \( u \) is globally defined we only need to show that
\[
\lim_{t \to T_m} \frac{\|A^{3/2}u(t)\|^2 + \|A^2u(t)\|^2}{E(t; A^{3/2}u)} \leq C < \infty. \tag{2.10}
\]

Let us note that the continuity of \( u \) together with inequality (2.9) show the existence of an interval \([0, t_0]\) for which we have
\[
E(t; A^{3/2}u) \leq 3\varepsilon, \quad t \in [0, t_0]. \tag{2.11}
\]

Therefore to verify (2.10) it is enough to establish the identity
\[
T_m = \sup_{t > 0} \{ E(t; A^{3/2}u) \leq 3\varepsilon \} := T_s
\]
this will imply that \( T_m = +\infty. \) The existence of global solution for equation (1.1) is shown in this way. This argument will be used in the following Theorem. First note that (2.8) and (2.11) for
\[
\varepsilon < \frac{m_0}{12c_{\delta} \mu (1 + \lambda_1^{3/2})}
\]
imply
\[ \frac{3}{4} E(t; v) \leq L_d(t; v) \leq \frac{5}{4} E(t; v); \quad \forall v \in D(A^{1/2}). \] (2.12)

In these conditions we can establish

**Theorem 2.2.** Under hypotheses H1, G1 and G2 there exists \( \varepsilon > 0 \) such that for any initial datum \( u_0 \in D(A^2) \) and \( u_1 \in D(A^{1/2}) \) satisfying (2.9), equation (1.1) has only one global solution satisfying
\[ u \in C([0, \infty]; D(A^2)) \cap C^1([0, \infty]; D(A^{3/2})) \cap C^2([0, \infty]; D(A^{1/2})). \]

Moreover if \( (u_0, u_1) \in D(A^1) \times D(A^{1/2}) \) we have
\[ u \in C([0, \infty]; D(A')) \cap C^1([0, \infty]; D(A^{1/2})) \cap C^2([0, \infty]; D(A^{1/2})) \]
for \( t \geq 2 \) and
\[ E(t; A^{3/2} u) \leq Ce^{-\varepsilon t}. \]

**Proof.** As discussed lines above we will prove, for \( \varepsilon \) small, that \( T_m = T_\ast \).

Let us suppose the contrary, \( T_m > T_\ast \). From G2, (2.9) and taking \( \varepsilon \) small so that H3 holds, Theorem 2.1 is valid. So we have a local solution satisfying (1.1). Denoting \( v = A' u \) we get
\[ v_{tt} + m_0 A v + n_0 B v_t + r_0 A^2 v = F_1'(t) A v + F_2'(t) B v_t + F_3'(t) A^2 v. \] (2.13)

Multiplying equation (2.13) by \( v_t \) we have
\[ \frac{d}{dt} E(t; v) + n_0 B v_t(t), v_t(t) \]
\[ \leq \frac{1}{2} F_1'(t) \frac{d}{dt} \| A^{1/2} v(t) \|^2 + F_2'(t) (B v_t(t), v_t(t)) + \frac{1}{2} F_3'(t) \frac{d}{dt} \| A^{3/2} v(t) \|^2. \]

Recalling the definition of \( L_d \), the later inequality became
\[ \frac{d}{dt} L_d(t; v) + n_0 (B v_t(t), v_t(t)) \]
\[ \leq \frac{1}{2} F_1'(t) \| A^{1/2} v(t) \|^2 + F_2'(t) (B v_t(t), v_t(t)) \]
\[ - \frac{1}{2} F_3'(t) \| A^{3/2} v(t) \|^2. \]
Using (2.9)–(2.11), with $\varepsilon$ small enough, we obtain
\[
\frac{d}{dt} L(t; v) + \frac{3}{4} m_0 (Bv(t), v(t)) \leq 3 n_0 c_\varepsilon (1 + \lambda_1^{-1}) ||A^{1/2}v(t)||^2. \tag{2.14}
\]

On the other hand, multiplying (2.13) by $v$, using inequalities (2.8) and (2.11) and choosing $\varepsilon$ small enough we get
\[
\frac{d}{dt} (v, v) \leq \|v\|^2 - \frac{3}{4} m_0 \|A^{1/2}v\|^2 + \mu \| (Bv, v) \|
\]

Using G3 and the inequality
\[
\| (Bv, v) \| \leq \| (Bv, v) \|^1 \| (Bv, v) \|^1 \leq \frac{\gamma}{m_0} \| (Bv, v) \| + \frac{m_0}{4\gamma\mu} \| (Bv, v) \|
\]

it follows
\[
\frac{d}{dt} (v, v) \leq \|v\|^2 - \frac{m_0}{2} \|A^{1/2}v\|^2 + \frac{\gamma}{m_0} \| (Bv, v) \|
\]

which together with (2.14) yields
\[
\frac{d}{dt} \{L(t; v) + \sqrt{\varepsilon} (v, v)\} \leq -\left(\frac{3}{4} n_0 - \frac{\gamma \mu^2 \sqrt{\varepsilon}}{m_0} \right) (Bv, v)
\]
\[-\sqrt{\varepsilon} \left( \frac{m_0}{2} - \frac{3}{2} \mu c_\varepsilon (1 + \lambda_1^{-1}) \right) ||A^{1/2}v(t)||^2.
\]

Taking $\varepsilon$ small enough and using G2 we easily see that there exists a positive constant $c_7$ satisfying
\[
\frac{d}{dt} \{L(t; v) + \sqrt{\varepsilon} (v, v)\} \leq -c_7 \sqrt{\varepsilon} E(t; v). \tag{2.15}
\]

Looking inequality (2.12), we obtain for $\varepsilon < \lambda_1 m_0 / 16$, that
\[
\frac{1}{2} E(t; v) \leq L(t; v) + \sqrt{\varepsilon} (v, v) \leq \frac{1}{2} E(t; v). \tag{2.16}
\]

Inequalities (2.15)–(2.16) implies that
\[
L(t; v) + \sqrt{\varepsilon} (v, v) \leq \{ L(0; v) + \sqrt{\varepsilon} (v(0), v(0)) \} e^{-3\varepsilon t}
\]

where $\varepsilon = 2c_7 \sqrt{\varepsilon} / 3$. Taking $v = A^{1/2}u$, from (2.16) and the inequality above we obtain
\[
E(t; A^{1/2}u) \leq 3 E(0; A^{1/2}u) e^{-3\varepsilon t} \leq 3 e e^{-3\varepsilon t} \text{ in } [0, T_\varepsilon] \tag{2.17}
\]
which implies that

\[ E(T_\ast; A^{3/2}u) < 3\epsilon. \]

This inequality is contrary to the definition of \( T_\ast \). Therefore \( T_m = T_\ast \) and the maximal solution must be global. Finally from (2.17) the exponential decay holds while the regularity and the uniqueness are obtained as in Theorem 2.1. The proof is now completed.

In the remaining part of this section we will prove the smoothing effect property of equation (1.1). In order to do this, let us denote \((w_i)_{i \in \mathbb{N}}\) and \((\lambda_i)_{i \in \mathbb{N}}\) the sequence of eigenfunction and eigenvalues of \( A \) such that \( \{w_i; i \in \mathbb{N}\} \) is a complete orthonormal subset of \( X \) and \( \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \to \infty \) as \( i \to \infty \). It is well known that for any \( v \in L^\infty([0, T]; H) \) we have

\[ v \in L^\infty([0, T]; D(A)) \]

\[ \iff \sum_{i=1}^\infty \lambda_i^2 (v, w_i)^2 < \infty \text{ is bounded in } L^\infty([0, T]; D(A)). \]

From now on we will denote by \( g_i(t) = (u(t), w_i) \), so \( g'_i(t) = (u'_t(t), w_i) \), where \( u \) is the solution of (1.1). For simplicity let us denote by

\[ m(t) = M(\[u(t)\]), \ n(t) = N(\[u(t)\]), \ r(t) = R(\[u(t)\]). \]

Hypothesis G2 together with the continuity of \( u \) implies that

\[ m(t) n(t) > 0 \quad \forall t > 0. \]

Therefore we can find \( C_0 \) such that

\[ m(t) n(t) > C_0 n(t) \quad \forall t > 0. \]

Let us define the function \( H_i(t) \) as

\[ H_i(t) := \frac{1}{2} g'_i(t)^2 + C_0 g_i(t) g'_i(t). \]

**Remark 2.4.** For any positive real numbers \( a, b \) and \( c \) we have that

\[ aX^2 + bXY + cY^2 \geq a \left( 1 - \frac{b}{\sqrt{4ac}} \right) X^2 + c \left( 1 - \frac{b}{\sqrt{4ac}} \right) Y^2 \]

as is simple to verify.

In this conditions we have

**Lemma 2.1.** Under hypotheses of Theorem 2.2 and G1–G2, there exist \( I \in \mathbb{N} \) and \( \chi > 0 \) satisfying
(i) \( H_i(t) \leq H_i(0) e^{-\lambda_i \int_{0}^{t} dt} \) for any \( t > 0, \forall i \geq I \),

(ii) \( \frac{1}{2} g^2_i(t) + \chi H_i(t) \leq \frac{1}{2} \{ g^2_i(0) + \chi H_i(0) \} e^{-\lambda_i \int_{0}^{t} dt} \forall t > 0, \forall i \geq I \).

**Proof.** Projecting equation (1.1) on \( R \), we have

\[
g^2_i(t) + \dot{\lambda} m(t) g_i(t) + \lambda n(t) g'_i(t) + \dot{\lambda} \tau(t) g_i(t) = 0. \tag{2.19}
\]

Multiplying equation (2.19) by \( g'_i(t) \) we get

\[
\frac{1}{2} \frac{d}{dt} g^2_i(t) = -\dot{\lambda} m(t) g_i(t) g'_i(t) - \dot{\lambda} n(t) g'_i(t) \tag{2.20}
\]

Denoting by \( \phi_i(t) = g_i(t) g'_i(t) \), equation (2.19) implies

\[
\frac{d}{dt} \phi_i = g_{ii}(t)^2 - \dot{\lambda} m(t) g_i(t)^2 - \dot{\lambda} n(t) \phi_i - \dot{\lambda} \tau(t) g_i(t)^2. \tag{2.21}
\]

Multiplying identity (2.21) by \( C_0 \) and adding the product result to (2.20) yields

\[
\frac{d}{dt} H_i(t) \leq C_0 g'_i(t)^2 - \dot{\lambda} C_0 m(t) \phi_i - \dot{\lambda} C_0 n(t) g'_i(t) \tag{2.22}
\]

Using Remark 2.4, inequality (2.22) can be expressed as

\[
\frac{d}{dt} H_i(t) \leq C_0 g'_i(t)^2 - \dot{\lambda} C_0 m(t) \phi_i - \dot{\lambda} C_0 n(t) g'_i(t) \tag{2.23}
\]

Recalling the definition of \( C_0 \) and \( H_i \), we get

\[
\frac{d}{dt} H_i(t) \leq C_0 g'_i(t)^2 - \dot{\lambda} m(t) H_i(t) - \dot{\lambda} \tau(t) g_i(t) \tag{2.24}
\]

Since \( \alpha < 1 \) and \( \dot{\lambda} \to +\infty \), there exist \( \alpha > 0 \) and \( I \in \mathbb{N} \) such that

\[
\frac{d}{dt} H_i(t) \leq -\alpha \dot{\lambda}_i \{ g^2_i(t) + g_i(t)^2 \} - \dot{\lambda} n(t) H_i(t) \tag{2.25}
\]
In particular
\[ \frac{d}{dt} H_i(t) \leq -\lambda_i n(t) H_i(t) \quad \forall t \geq 1. \]

From where part (i) follows. Differentiating in time equation (2.19) we obtain
\[
\begin{align*}
\dot{g}_i^\gamma(t) + \lambda_i m(t) g_i^\gamma(t) + \lambda_i n(t) g_i^\gamma(t) + \lambda_i^* r(t) g_i^\gamma(t) \\
+ \lambda_i^* r(t) g_i^\gamma(t) + \lambda_i n(t) g_i^\gamma(t) + \lambda_i m(t) g_i(t) = 0.
\end{align*}
\]
(2.24)

Multiplying equation (2.24) by \( g_i^\gamma(t) \) we can show that
\[
\frac{1}{2} \frac{d}{dt} g_i^\gamma(t)^2 = -\lambda_i n(t) g_i^\gamma(t)^2 - R_i(t)
\]
(2.25)

where
\[
R_i(t) = \lambda_i m(t) g_i^\gamma(t) g_i^\gamma(t) + \lambda_i n(t) g_i^\gamma(t) g_i^\gamma(t)
+ \lambda_i^* r(t) g_i^\gamma(t) g_i^\gamma(t) + \lambda_i^* r(t) g_i(t) g_i(t).
\]

Using inequalities of the form \( ab \leq a^2/2c + b^2/2 \) and the continuity of the solution we conclude that there exists a positive constant \( c_9 \) such that
\[
R_i(t) \leq \frac{1}{2} \lambda_i n(t) g_i(t)^2 + c_9 \{ g_i(t)^2 + g_i(t)^2 \}, \quad \forall t > 0.
\]
(2.26)

From part (i) of this lemma, (2.23), (2.25) and (2.26) we obtain
\[
\begin{align*}
\frac{d}{dt} \left\{ \frac{1}{2} g_i^\gamma(t)^2 + \chi H_i(t) \right\} \\
\leq -\lambda_i n(t) g_i^\gamma(t)^2 - \lambda_i n(t) H_i(t) - (\chi c_9 - c_9) \lambda_i \{ g_i(t)^2 + g_i(t)^2 \}.
\end{align*}
\]

Now we choose \( \chi > c_9/c_8 \) to obtain
\[
\frac{d}{dt} \left\{ \frac{1}{2} g_i^\gamma(t)^2 + \chi H_i(t) \right\} \leq -\lambda_i n(t) \left\{ \frac{1}{2} g_i^\gamma(t)^2 + \chi H_i(t) \right\}, \quad \forall t > 0.
\]

Finally multiplying by \( e^{-\frac{\lambda_i n(t)}{2} \int_0^t dt} \) and integrating with respect to time, part (ii) follows.

Let us denote by \( E_i(t; v) \) the functional given by
\[
E_i(t; v) := \| v \|^2 + \frac{m(t)}{2} \| A^{1/2} v \|^2.
\]

Where \( m \) is given as in (2.18). Before to prove the main result of this paper we will show
**Lemma 2.2.** Under hypotheses of Theorem 2.2, if the solution $u$ of (1.1) satisfies

$$u \in L^\infty([\delta, T]; D(A')) \quad 0 < \delta < T, \quad s > 0$$

then we have

$$u_t, u_{tt} \in L^\infty([\delta, T]; D(A'))$$

and

$$\mathcal{X} \|A'u\|^2 + \|A'u_{tt}\|^2 \leq 4C^2_{\mathcal{X}} \|A'u\|^2 + c(t, I) E(0, A'u).$$

Moreover let us denote by $v$ the function

$$v := \sum_{i=1}^n \lambda_i^2 (u(t), w_i) w_i,$$

where $l$ is a positive real number, then we have that there exist $I \in \mathbb{N}$ and a constant $c_0$ independent of $v$ for which we have that

$$E(t, v) \leq c_0 E(s, v) e^{-\lambda(t, I)n(t)} dt, \quad \forall t > s.$$

**Proof.** From inequality (ii) of Lemma 2.1 we get

$$\frac{1}{4} g_i'(t)^2 + \frac{1}{4} \mathcal{X} g_i'(t)^2 \leq \mathcal{X} C^2 \sum_{i=1}^n (g_i'(0)^2 + H_i(0)) \times e^{-\lambda_i n(t)} dt \quad \forall t > 0, \quad \forall i \geq I. \quad (2.27)$$

Multiplying the above inequality by $\lambda_i^2$ and summing up the product result from $I$ to $m$ we get

$$\frac{1}{4} \sum_{i=1}^n \lambda_i^2 g_i'(t)^2 + \frac{1}{4} \mathcal{X} \sum_{i=1}^n \lambda_i^2 g_i'(t)^2$$

$$\leq \mathcal{X} C^2 \sum_{i=1}^n \lambda_i^2 g_i'(t)^2 + \frac{1}{4} \sum_{i=1}^n \{ g_i''(0)^2 + H_i(0) \} \lambda_i^2 e^{-\lambda_i N_0 t}$$

$$\leq \mathcal{X} C^2 \sum_{i=1}^n \lambda_i^2 g_i'(t)^2 + c_0 E(0, A^{3/2}u) \sum_{i=1}^n \lambda_i^2 e^{-\lambda_i N_0 t} \quad \forall t > 0, \forall i \geq I. \quad (2.27)$$

Where $N_0$ is such that $n(t) \geq N_0$. From hypotheses the right hand side of equation (2.27) is finite. Note that there exists a positive constant $c_1$ such that

$$\sum_{i=1}^n \{ \lambda_i^2 g_i'(t)^2 + \lambda_i^2 g_i''(t)^2 \} \leq c_1 E(t, A^{3/2}u)$$

$$\sum_{i=1}^n \{ \lambda_i^2 g_i'(t)^2 + \lambda_i^2 g_i''(t)^2 \}.$$
therefore from Theorem 2.2 we get
\[
\sum_{i=1}^{I} \{ \lambda_i^2 g_i'(t)^2 + \lambda_i^2 g_i(t)^2 \} \leq c_2 E(0, A^{3/2}u) e^{-ct}.
\]
Using equation (2.19) we conclude that
\[
g_i'(t)^2 \leq c_3 \lambda_i^2 \{ g_i'(t)^2 + g_i(t)^2 \}.
\]
So there exists a positive constant such that
\[
\sum_{i=1}^{I} \{ \frac{1}{2} \lambda_i^2 g_i'(t)^2 + \frac{1}{2} \lambda_i^2 g_i(t)^2 \} \leq c_4 \lambda_i^{2+2} E(0, A^2 u) e^{-ct}.
\]
(2.28)
From inequalities (2.27) and (2.28) we get
\[
\zeta \| A' u \|^2 + \| A' u_t \|^2 \leq 4 \zeta C_0 \| A' u \|^2 + c_3 E(0, A^{1/2}u) \sum_{i=1}^{I} \lambda_i^2 e^{-\lambda_i^2 t} + c_4 \lambda_i^{2+2} E(0, A^{3/2}u) e^{-ct}
\]
then we obtain
\[
u , u_t \in L^\infty([\delta, T]; D(A'))
\]
as required. On the other hand it is not difficult to see that \( v \) satisfies the equation:
\[
v_{tt} + m(t) Av + n(t) Av + r(t) A^* v = 0
\]
(2.29)
where \( m, n, \) and \( r \) are given as in (2.18). Multiplying equation (2.29) by \( v \), we get
\[
d \frac{d}{dt} E_1(v; t) = -n(t) \| A^{1/2}v \|^2 + \frac{m(t)}{2} \| A^{1/2}v \|^2 - r(t) (A^{3/2} v, A^{3/2} v).
\]
Multiplying equation (2.29) by \( v \) we get
\[
d \frac{d}{dt} \left\{ \| v(t) \|^2 - m(t) \| A^{1/2}v(t) \|^2 - r(t) \| A^{3/2} v(t) \|^2 + \frac{n(t)}{2} \| A^{1/2}v(t) \|^2 \right\} = \| v(t) \|^2 - m(t) \| A^{1/2}v(t) \|^2 - r(t) \| A^{3/2} v(t) \|^2 + \frac{n(t)}{2} \| A^{1/2}v(t) \|^2.
\]
Since $m, n,$ and $r$ are bounded for any $t > 0$ and $n(t) \geq N_0 > 0$, there exists $C_i > 0$ such that

$$r(t) < C_i n(t). \quad (2.30)$$

So we have that

$$\frac{d}{dt} E_1(t; v) \leq -n(t) \| A^{1/2} v \|^2 + \frac{m'(t)}{2} \| A^{1/2} v \|^2 + \frac{r(t)}{2C_i l_i^{-1}} \| A^{1/2} v \|^2.$$

Using (2.30) we get

$$\frac{d}{dt} E_1(t; v) \leq -n(t) \| A^{1/2} v \|^2 + \frac{m'(t)}{2} \| A^{1/2} v \|^2 + \frac{r(t)}{2C_i l_i^{-1}} \| A^{1/2} v \|^2.$$

Let us denote by

$$K(t, v) := E(t, v) + C \left\{ (v, v) + \frac{n(t)}{2} \| A^{1/2} v \|^2 + \frac{r(t)}{2} \| A^{1/2} v \|^2 \right\},$$

where $C$ is a fixed number given by

$$C = \max \left\{ \frac{C_i l_i^{-1}}{2} , \frac{16 + 8m(t)}{n(t)} \right\}. \quad (2.31)$$

By the definition of $v$ it is not difficulty to see that

$$\| v_i \|^2 \leq \frac{1}{\lambda_i} \| A^{1/2} v_i(t) \|^2.$$

So we have

$$\frac{d}{dt} K(t; v) \leq \left\{ n(t) \lambda_i - C \right\} \| v_i \|^2 + \left\{ \frac{m'(t)}{2} - Cm(t) + \frac{n'(t)}{2} C \right\} \| A^{1/2} v \|^2 - \left\{ C - \frac{C_i l_i^{-1}}{2} \right\} r(t) \| A^{1/2} v \|^2. \quad (2.32)$$
By the definition of $K$ we get easily that
\[
K(t; v) \leq \frac{1}{2} \left( 1 + \frac{C^2}{2\lambda_j^2} \right) \| v_r \|^2 + \frac{m(t) + Cn(t) + 1}{2} \| A^{1/2} v(t) \|^2. \tag{2.33}
\]

From inequalities (2.31)-(2.33) it follows that
\[
\frac{d}{dt} K(t; v) + \frac{8m(t)}{5n(t)} K(t; v) \leq - \left( n(t) \lambda_j - C - \frac{8m(t)}{5n(t)} \right) \| v_r \|^2 - \frac{2m(t) + Cn(t)}{2} \| v_r \|^2.
\]

Note that Theorem (2.2) implies that
\[
|n'(t)| + |v'(t)| + |m'(t)| \leq c_6 E(0, A^{1/2} u) e^{-\alpha t}.
\]

So there exists $t_0$ such that
\[
n'(t) \leq \frac{m(t)}{5}; \quad m'(t) \leq \frac{8m(t)}{5n(t)} \quad \forall t \geq t_0,
\]

therefore we get
\[
\frac{d}{dt} K(t; v) + \frac{8m(t)}{5n(t)} K(t; v) \leq - \left( n(t) \lambda_j - C - \frac{8m(t)}{5n(t)} \right) \| v_r \|^2 - \frac{16 + 8m(t)}{m(t)} \| A^{1/2} v \|^2.
\]

Since $\lambda_j \to \infty$ as $I \to \infty$ we conclude that there exists a natural number $I_0$ for which we get
\[
n(t) \lambda_j - C - \frac{8m(t)}{5n(t)} - \frac{8mC^2}{10\lambda_j n(t)} \geq 0 \quad \text{for} \quad I \geq I_0.
\]

So by our choice of $C$ we get
\[
\frac{d}{dt} K(t; v) + \frac{8m(t)}{5n(t)} K(t; v) \leq 0,
\]

from where it follows
\[
K(t; v) \leq CK(s; v) e^{-8/5 \int_{m(t)}^{m(t)} dr}.
\]
Recalling the definition of $K$ and from our choosing of $I_0 \geq 2C^2/n(t)$ we easily obtain that

$$K(t; v) \geq E_i(t; v) - \frac{1}{4} ||u||^2 - \left\{ C^2 - \frac{n(t)}{2} \lambda_i \right\} ||v||^2 \geq \frac{1}{2} E_i(t; v).$$

Since the solution is bounded for any $t > 0$ our conclusion follows.

**Theorem 2.3.** Under hypotheses of Lemma 2.1, the solution of equation (1.1) satisfies

(i) If $M$, $N$ and $R$ are $C^1$ functions we have that $u \in C^2([0; +\infty[; D(A^\infty))$, and

(ii) If $M$, $R$ and $N$ are $C^\infty$ we have that $u \in C^\infty([0; +\infty[; D(A^\infty)).$

**Proof.** We will prove that for any $\delta > 0$ we will prove that

$$u \in C^2([\delta, T]; D(A^\delta)), \quad \forall \delta \in \mathbb{N}.$$

In fact since $u \in C([0, T]; D(A^2))$, Lemma 2.2 implies that

$$u_j, u_{jj} \in L^\infty([\delta, T]; D(A^2)).$$

Taking $v^* = \sum_{j=1}^n (u_j, w_j) w_j$ as in Lemma 2.2 we get

$$v^* + m(t) A v^* + n(t) A v^*_j + r(t) A^2 v^* = 0.$$

so we have

$$\frac{n(t)}{m(t)} A v^* + A v^*_j = -\frac{1}{n(t)} \{ v^*_j + r(t) A^2 v^* \}.$$ 

Multiplying this by $e^{\int_0^{m(t)\{ m(t) \} dx} A^2 v^*}$ we get

$$\frac{d}{dt} \left\{ e^{\int_0^{m(t)\{ m(t) \} dx} A^2 v^*} \right\} = -\frac{\int_0^{m(t)\{ m(t) \} dx} \{ (v^*_j + r(t) A^2 v^*) \}}{n(t)}.$$ 

Integrating from $t$ to $T$ we get

$$-\int_t^T \frac{\int_0^{m(t)\{ m(t) \} dx} \{ (v^*_j + r(t) A^2 v^*) \}}{n(t)} dt = \int_T^t \frac{\int_0^{m(t)\{ m(t) \} dx} \{ (v^*_j + r(t) A^2 v^*) \}}{n(t)} dt. \quad (2.34)$$
Using similar arguments as in Lemma 2.1 we get
\[ \|A^{j/2}v_n\| \leq 4\mathcal{C}^2_0 + \sum_{i=1}^{N} c_i \lambda_i^j e^{-\lambda_i N\tau} \]
where
\[ s(j,\tau) := \sum_{i=1}^{N} \lambda_i^j e^{-\lambda_i N\tau}. \]

In these conditions we have
\[ (v_n(\tau), A^{j/2}v(\tau)) = (A^{j/2}v_n(\tau), A^{j/2}v(\tau)) \]
\[ \leq 4\mathcal{C}^2_0 + \|A^{j/2}v(\tau)\|^2 + c_0 E(0, A^{1/2}u) s(j, \tau). \]

Using Lemma 2.2 we get
\[ (v_n(\tau), A^{j/2}v(\tau)) \leq c \|A^{j/2}v(\tau)\|^2 e^{-3/5 \| m(t) \| dt} + c_0 E(0, A^{1/2}u) s(j, \tau) \]
for any \( \tau > t. \) Using this inequality and the fact that \( D(A^{(n+j)/2}) \subset D(A^{j/2}) \)
with continuous imbedding, in (2.34) we get
\[
\begin{align*}
& e^{\int_{t}^{T} e^{\| m(t) \| dt} dt} \|A^{(j+1)/2}v(t)\|^2 \\
& \leq e^{\int_{t}^{T} e^{\| m(t) \| dt} dt} \|A^{(j+1)/2}v(T)\|^2 \\
& + \int_{t}^{T} e^{\| m(t) \| dt} \|A^{j/2}v(t)\|^2 \|A^{j/2}v(\tau)\|^2 d\tau \\
& \leq e^{-3/5 \| m(t) \| dt} \|A^{(j+1)/2}v(t)\|^2 \\
& + \int_{t}^{T} \|A^{j/2}v(\tau)\|^2 e^{-3/5 \| m(t) \| dt} d\tau + cE(0, A^{1/2}u) s(j, \tau) d\tau \\
& \leq e^{-3/5 \| m(t) \| dt} \|A^{(j+1)/2}v(t)\|^2 \\
& + \|A^{j+1/2}v(T)\|^2 \left\{ \frac{1}{2} \left( e^{-3/5 \gamma} - e^{-3/5 \gamma T} \right) \right\} \| A^{j+1/2}u \| s(j, \tau) d\tau,
\end{align*}
\]
where
\[ \gamma = \inf_{t>0} \left( \frac{\| m(t) \|}{\| m(t) \|} \right). \]
Letting $T \to \infty$ we get
\[
e^{b_1(t, \cdot)} \left\| A^{(j+1)/2} v(t) \right\|^2 \leq c e^{-\gamma} \left\| A^{(j+\pi)/2} v(t) \right\|^2 + CE(0, A) S(j, t)
\]
for
\[
S(j, t) := \sum_{i=1}^{r} \frac{\lambda_{i}^{j-1}}{N_0} e^{-N_0 \beta_i^j t}.
\]
From where we conclude that there exists a positive constant $C$ such that
\[
\left\| A^{(j+1)/2} v(t) \right\|^2 \leq C \left\| A^{(j+\pi)/2} v(t) \right\|^2 + CE(0, A^{3/2}) S(t, j).
\]
The inequality above is valid for any $j \in \mathbb{R}$ so we have
\[
\left\| A^{(j+1)/2} v(t) \right\|^2 \leq C \left( \left\| A^{(j+\pi)/2} v(t) \right\|^2 + CE(0, A^{3/2}) S(t, j) \right)
\]
\[
\leq \sum_{i=1}^{k-1} C^i E(0, A^{3/2}) S(t, j).
\]
Taking $k \in \mathbb{N}$ such that $k \geq (j+1)/(1-\alpha)$ we get
\[
\left\| A^{(j+1)/2} v(t) \right\|^2 \leq \left\{ \sum_{i=1}^{k-1} C^i + S^i(t, j) \right\} E(0, A^{3/2}) u.
\]
Letting $r \to \infty$ we conclude that
\[
\left\| A^{(j+1)/2} v(t) \right\|^2 \leq C_2(t, j) E(0, A^{3/2}) u,
\]
where
\[
C_2(t, j) := \sum_{i=j}^{k-1} \frac{\lambda_{i}^{j-1}}{N_0} e^{-N_0 \beta_i^j t} + \sum_{i=1}^{k-1} C^i.
\]
Since
\[
\left\| A^{(j+1)/2} u(t) \right\|^2 = \sum_{i=1}^{t-1} \lambda_{i}^{j+1}(u, w_j)^2 + \left\| A^{(j+1)/2} v(t) \right\|^2
\]
we conclude that
\[
\left\| A^{(j+1)/2} u(t) \right\|^2 \leq \left\{ c \lambda_{j}^{j+1} + C_2(j, t) \right\} E(0, A^{3/2}) u
\]
\[\text{NONLINEAR EVOLUTION EQUATION} 123\]
for any \( j \in \mathbb{N} \), where \( C(j, t) \to \infty \) as \( t \to 0 \). Using Lemma 2.2 we conclude that

\[
  u_t, u_{tt} \in L^\infty([0, \infty[; D(A')) \quad \forall k \in \mathbb{N}.
\]

So \( u, u_t \in C([0, \infty[; D(A')) \). Using this relation and equation (1.1) our conclusion follows.

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