Mindlin–Timoshenko systems with Kelvin–Voigt: analyticity and optimal decay rates

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Article history:
Received 17 September 2013
Available online 11 March 2014
Submitted by K. Nishihara
Dedicated to Professor Shuichi Kawashima on the occasion of his 60th Birthday

Keywords:
Mindlin–Timoshenko plate
Kelvin–Voigt
Analyticity
Exponential stability
Polynomial stability
Optimal decay rate

1. Introduction

In this paper we study the asymptotic stability of the following Mindlin–Timoshenko plate model

\[
\rho w_{tt} - KL_1(w, \psi, \varphi) - D_0 \Delta w_t = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.1)
\]

\[
\frac{\rho h^3}{12} \psi_{tt} - DL_2(\psi, \varphi) + K \left( \psi + \frac{\partial w}{\partial x} \right) - D_1 L_2(\psi_t, \varphi_t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.2)
\]

\[
\frac{\rho h^3}{12} \varphi_{tt} - DL_3(\varphi, \psi) + K \left( \varphi + \frac{\partial w}{\partial y} \right) - D_1 L_3(\varphi_t, \psi_t) = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \quad (1.3)
\]

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where \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \) with Lipschitz boundary \( \Gamma = \partial \Omega \), and \( L_1, L_2, L_3 \) are coupling terms defined by

\[
L_1(w, \psi, \varphi) = \frac{\partial}{\partial x} \left( \psi + \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \varphi + \frac{\partial w}{\partial y} \right), \tag{1.4}
\]

\[
L_2(\psi, \varphi) = \frac{\partial^2 \psi}{\partial x^2} + \frac{1 - \mu}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 \varphi}{\partial x \partial y}, \tag{1.5}
\]

\[
L_3(\varphi, \psi) = \frac{\partial^2 \varphi}{\partial y^2} + \frac{1 - \mu}{2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{1 + \mu}{2} \frac{\partial^2 \psi}{\partial x \partial y}. \tag{1.6}
\]

This system models the vibrations of a thin plate with reference configuration \( \Omega \) by taking into account the displacements and rotations caused by the movement. The model was considered in Lagnese [9] and Lagnese and Lions [10] with a comprehensive discussion about its mathematical modeling. Accordingly, the parameters of the model have the following physical meanings. The unknowns \( w \) and \( (\psi, \varphi) \) represent, respectively, the transverse displacement of the reference surface and the rotation angles of the plate filaments. The constants \( \rho, h, K, D \) are positive numbers which represent, respectively, the mass density, plate thickness, shear modulus and flexural rigidity. The constant \( \mu \) is Poisson’s ratio which is taken in \((0, 1/2)\).

The interesting case is when we consider \( D_0 = 0 \) and \( D_1 > 0 \), and so we only have damping on the rotation angles \( \psi \) and \( \varphi \). We notice that the damping terms \( L_2(\psi_1, \varphi_1) \) and \( L_3(\varphi_1, \psi_1) \) correspond to the ones of Kelvin–Voigt type. Indeed, materials with Kelvin–Voigt damping are characterized by having stress proportional to strain and strain rate, that is,

\[
\sigma = a \varepsilon + b \frac{\partial \varepsilon}{\partial t}, \quad a, b > 0. \tag{1.7}
\]

See for instance Bulíček et al. [3]. With respect to Mindlin–Timoshenko models the strain tensor corresponding to rotation equations \( \psi \) and \( \varphi \) is given by

\[
\varepsilon = \left( \begin{array}{cc} \frac{1}{2} (\frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y}) & \frac{1 - \mu}{2} (\frac{\partial \varepsilon}{\partial x} + \mu \frac{\partial \psi}{\partial y}) \\ \frac{1}{2} (\frac{\partial \varphi}{\partial x} + \mu \frac{\partial \varphi}{\partial y}) & \frac{1 - \mu}{2} (\frac{\partial \varepsilon}{\partial y} + \mu \frac{\partial \psi}{\partial x}) \end{array} \right). \tag{1.8}
\]

See for instance van Rensburg et al. [16]. Then, since the balance of the linear momentum is

\[
\rho \frac{\partial^2}{\partial t^2} (\psi, \varphi) = c \nabla \cdot \sigma,
\]

where \( c > 0 \) is a normalizing constant, we get from (1.7)–(1.8),

\[
\left( \begin{array}{c} \rho \psi_{tt} \\ \rho \varphi_{tt} \end{array} \right) = ac \left( \begin{array}{c} \frac{\partial^2 \psi}{\partial x^2} + \frac{1 - \mu}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 \varphi}{\partial x \partial y} \\ \frac{\partial^2 \varphi}{\partial y^2} + \frac{1 - \mu}{2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{1 + \mu}{2} \frac{\partial^2 \psi}{\partial x \partial y} \end{array} \right) + bc \left( \begin{array}{c} \frac{\partial^2 \psi_t}{\partial x^2} + \frac{1 - \mu}{2} \frac{\partial^2 \psi_t}{\partial y^2} + \frac{1 + \mu}{2} \frac{\partial^2 \varphi_t}{\partial x \partial y} \\ \frac{\partial^2 \varphi_t}{\partial y^2} + \frac{1 - \mu}{2} \frac{\partial^2 \varphi_t}{\partial x^2} + \frac{1 + \mu}{2} \frac{\partial^2 \psi_t}{\partial x \partial y} \end{array} \right).
\]

But this corresponds precisely to Eqs. (1.2)–(1.3), without coupling terms involving Eq. (1.1).

To the system (1.1)–(1.3) we add initial conditions

\[
w(x, y, 0) = w_0(x, y), \quad w_t(x, y, 0) = w_1(x, y) \quad \text{in} \quad \Omega,
\]

\[
\psi(x, y, 0) = \psi_0(x, y), \quad \psi_t(x, y, 0) = \psi_1(x, y) \quad \text{in} \quad \Omega,
\]

\[
\varphi(x, y, 0) = \varphi_0(x, y), \quad \varphi_t(x, y, 0) = \varphi_1(x, y) \quad \text{in} \quad \Omega, \tag{1.9}
\]
and two different types of boundary conditions: the Dirichlet condition

$$w = \psi = \varphi = 0 \quad \text{on } \Gamma \times [0, \infty),$$

(1.10)

or a mixed condition

$$w = 0 \quad \text{on } \Gamma \times [0, \infty),$$

$$\psi = 0, \quad \left(1 - \frac{\mu}{2} \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right) \cdot \nu = 0 \quad \text{on } \Gamma_1 \times [0, \infty),$$

(1.11)

$$\varphi = 0, \quad \left(\frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y}, \frac{1}{2} \left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right)\right) \cdot \nu = 0 \quad \text{on } \Gamma_2 \times [0, \infty),$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is a partition of $\Gamma$ given by nonempty disjoint open sets $\Gamma_1, \Gamma_2$, and $\nu = (\nu_1, \nu_2)$ is the unit exterior normal to $\Gamma$.

The conservative Mindlin–Timoshenko system ($D_0 = D_1 = 0$) has been considered by some authors by adding boundary dissipations. Lagnese [9] showed that this problem with a dissipative mixed boundary condition, under suitable geometrical hypotheses, is exponentially stable without restrictions on the system's parameters. A related problem with boundary dissipation of memory type was considered by Muñoz Rivera and Portillo Oquendo [12]. They showed that the associated energy of the system decays to zero exponentially or polynomially depending on the decay rate of the relaxation function.

More recently, Fernández Sare [5] studied the system (1.1)–(1.3) (with $D_0 = D_1 = 0$) with additional internal frictional damping acting only on the equations for the rotation angles (1.2)–(1.3). He proved, for the boundary condition (1.11), that the system is not exponentially stable. In addition, under the boundary condition (1.10), he showed that the system is polynomially stable. The optimality of the decay rates was not considered.

Our work is motivated by the above mentioned papers. We do not consider boundary dissipations. Instead, we consider internal dissipation given by the terms $L_2(\psi_t, \varphi_t)$ and $L_3(\varphi_t, \psi_t)$. Roughly speaking, we show that the semigroup associated to the system (1.1)–(1.3) is analytic if and only if $D_0, D_1$ are positive real numbers. Furthermore, when $D_0 = 0$ but $D_1 \neq 0$, we show that the semigroup is not exponentially stable. The arguments used to discuss the exponential stability is based on spectral properties of semigroup operators, which are in some way similar to the ones used in, for instance, [1,4–6,13].

The main result of this paper is Theorem 5.1. There we show in the case $D_0 = 0$ and $D_1 \neq 0$, that the system stabilizes polynomially to zero with optimal decay rate $t^{-1/2}$, for admissible initial data $(w_0, \phi_0, \varphi_0)$ in $[H^2(\Omega)]^3$. Our approach is different from the classical ones using energy estimates. Instead, we use a recent result by Borichev and Tomilov [2].

Our work is organized as follows. In Section 2 we show the well-posedness of the problem. In Section 3 we state the analyticity of semigroup associated to the Mindlin–Timoshenko system. In Section 4 we show the lack of exponential decay to the partially damped system. Finally in Section 5 we prove the polynomial stability to the partially damped system and the optimality of the rate of decay.

2. Well-posedness

In this section we establish the global existence and uniqueness of Mindlin–Timoshenko model (1.1)–(1.9) under boundary conditions (1.10) or (1.11). We notice that most of the functional setting needed for a semigroup approach to this model is presented in Lagnese [9, Chap. 3]. Let us denote by $U$ the vector-valued function

$$U = (w, W, \psi, \varphi, \Phi)^\ell \quad \text{with } W = w_t, \quad \psi = \psi_t, \quad \varphi = \varphi_t.$$
Then the system (1.1)–(1.9) is equivalent to

$$U_t = AU, \quad U(0) = U_0,$$

(2.1)

where $U_0 = (w_0, w_1, \psi_0, \psi_1, \varphi_0, \varphi_1)^t$ and $A$ is the differential operator given by

$$A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{K}{\rho_1} \Delta & \frac{D_0}{\rho_1} \Delta & \frac{K}{\rho_1} \partial_x & 0 & \frac{K}{\rho_1} \partial_y & 0 \\
0 & 0 & 0 & \frac{1}{\rho_2} (DB_1 - KI) & \frac{D_1}{\rho_2} B_1 & \frac{D}{\rho_2} B_2 \\
0 & -\frac{K}{\rho_2} \partial_x & 0 & \frac{D}{\rho_2} B_2 & \frac{D_1}{\rho_2} B_2 & \frac{1}{\rho_2} (DB_3 - KI) \\
0 & -\frac{K}{\rho_2} \partial_y & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{D}{\rho_2} B_2 & \frac{D_1}{\rho_2} B_2 & \frac{D}{\rho_2} B_2 & \frac{1}{\rho_2} (DB_3 - KI)
\end{pmatrix},$$

(2.2)

where

$$\rho_1 = \rho h \quad \text{and} \quad \rho_2 = \frac{\rho h_3}{12}.$$  

Here we have denoted by $B_i, i = 1, 2, 3,$ the linear differential operators

$$B_1 = \partial_x^2 + \frac{1-\mu}{2} \partial_y^2,$$

$$B_2 = \frac{1+\mu}{2} \partial_{xy}^2,$$

$$B_3 = \partial_y^2 + \frac{1-\mu}{2} \partial_x^2.$$

In order to consider the two sets of boundary conditions (1.10) and (1.11), we define respectively, the spaces

$$H_1 = H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times H^1_0(\Omega) \times L^2(\Omega),$$

and

$$H_2 = H^1_0(\Omega) \times L^2(\Omega) \times H^1_{11}(\Omega) \times L^2(\Omega) \times H^1_{12}(\Omega) \times L^2(\Omega),$$

where

$$H^1_{1i}(\Omega) = \{ u \in H^1(\Omega); \ u|_{\Gamma_i} = 0 \}, \quad i = 1, 2.$$  

We note that for our viscoelastic system the elliptic regularity yields

$$Kw + D_0 W \in H^2(\Omega), \quad D\psi + D_1 \Psi \in H^2(\Omega), \quad D\varphi + D_1 \Phi \in H^2(\Omega).$$

(2.3)

In addition, with respect to the mixed boundary condition, the regular solutions of (2.1) must satisfy

$$\left( \frac{1-\mu}{2} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right), \frac{\partial \varphi}{\partial y} + \mu \frac{\partial \psi}{\partial x} \right) \cdot \nu = 0 \quad \text{on} \, \Gamma_1,$$

(2.4)

$$\left( \frac{\partial \psi}{\partial x} + \mu \frac{\partial \varphi}{\partial y}, \frac{1-\mu}{2} \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right) \cdot \nu = 0 \quad \text{on} \, \Gamma_2.$$

(2.5)
Let us denote by \( A_1 \) the operator \( A \) when the system is considered with Dirichlet boundary condition. Analogously, let us use \( A_2 \) in the case of mixed boundary condition. Then from above remarks we infer that the domains of \( A_1 \) and \( A_2 \) are respectively

\[
D(A_1) = \{ U \in H_1; \ U \text{ satisfies (2.3) and } (W, \Psi, \Phi) \in (H^1_0(\Omega))^3 \} \tag{2.6}
\]

and

\[
D(A_2) = \{ U \in H_2; \ U \text{ satisfies (2.3)–(2.5) and } (W, \Psi, \Phi) \in H^1_0 \times H^1_{F_1} \times H^1_{F_2} \}. \tag{2.7}
\]

Since it is clear when each boundary condition is under consideration, we simply write \( A \) and \( D(A) \) for both cases. Analogously, we write \( H \) instead of \( H_1 \) or \( H_2 \).

To finish the preliminary remarks about problem (2.1) we note that \( H \) is a Hilbert space equipped with the norm

\[
\|U\|_H^2 = \rho_1\|W\|^2_2 + \rho_2\|\Psi\|^2_2 + \rho_2\|\Phi\|^2_2 + K\|\psi + w_x\|^2_2 + K\|\varphi + w_y\|^2_2
\]

\[
+ D(1 - \mu)\left(\|\psi_x\|^2_2 + \|\varphi_y\|^2_2\right) + D\left(\frac{1 - \mu}{2}\right)\|\psi_y + \varphi_x\|^2_2 + D\mu\|\psi_x + \varphi_y\|^2_2,
\]

which is induced by the inner product

\[
(U, \hat{U})_H = \rho_1(W, \hat{W}) + \rho_2(\Psi, \hat{\Psi}) + \rho_2(\Phi, \hat{\Phi}) + K(\psi + w_x, \hat{\psi} + \hat{w}_x)
\]

\[
+ K(\varphi + w_y, \hat{\varphi} + \hat{w}_y) + D(\psi_x, \hat{\psi}_x) + D(\varphi_y, \hat{\varphi}_y)
\]

\[
+ D\left(\frac{1 - \mu}{2}\right)(\psi_y + \varphi_x, \hat{\psi}_y + \hat{\varphi}_x) + D\mu(\psi_x, \hat{\varphi}_y) + D\mu(\varphi_y, \hat{\psi}_x),
\]

where \( U = (w, W, \psi, \varphi, \Phi)^t \), \( \hat{U} = (\hat{w}, \hat{W}, \hat{\psi}, \hat{\varphi}, \hat{\Phi})^t \) and \( (\cdot, \cdot) \) stands for inner product of \( L^2(\Omega) \).

The existence result to the system (1.1)–(1.9) with Dirichlet or mixed boundary conditions is presented through problem (2.1).

**Theorem 2.1.** Given \( U_0 = (w_0, w_1, \psi_0, \psi_1, \varphi_0, \varphi_1)^t \in D(A) \) there exists a unique solution \( U = (w, W, \psi, \varphi, \Phi)^t \) of the problem (2.1) satisfying

\[
U \in C([0, +\infty), D(A)) \cap C^1([0, +\infty), H). \tag{2.9}
\]

In addition, if \( U_0 \in D(A^k), k \in \mathbb{N} \), then the above solution satisfies

\[
U \in \bigcap_{j=0}^{k} C^{k-j}([0, +\infty), D(A^j)). \tag{2.10}
\]

**Proof.** Firstly we note that \( A \) is a dissipative operator in \( H \). Indeed

\[
\text{Re}(AU, U)_H = -D_0\|\nabla W\|^2_2 - D_1(1 - \mu)\left(\|\psi_x\|^2_2 + \|\varphi_y\|^2_2\right)
\]

\[
- D_1\left(\frac{1 - \mu}{2}\right)\|\psi_x + \varphi_y\|^2_2 - D_1\mu\|\psi_x + \varphi_y\|^2_2
\]

\[
\leq 0,
\]

for all \( U \in D(A) \). Using a standard procedure we can show that \( D(A) \) is densely defined in \( H \) and that \( 0 \in \rho(A) \). Therefore from Lummer–Phillips Theorem, \( A \) is the infinitesimal generator of a \( C_0 \)-semigroup of contractions \( S(t) = e^{At} \) over \( H \). Hence our conclusion follows. See for instance Pazy [14].
3. Analyticity and exponential stability

Here we prove that when $D_0, D_1 > 0$, that is, when the viscoelastic damping is effective in the whole Mindlin–Timoshenko system, its associate semigroup is analytic and exponentially stable. The main tool we use is the following characterization of analytic semigroups given by Liu and Zheng [11], namely, a $C_0$-semigroup of contractions $S(t) = e^{At}$ over a Hilbert space $H$ such that $i\mathbb{R} \subseteq \rho(A)$ is analytic if and only if

$$\limsup_{|\beta| \to \infty} \| \beta (i\beta I - A)^{-1} \|_{\mathcal{L}(H)} < \infty, \tag{3.1}$$

where $\rho(A)$ is the resolvent of $A$, $i\mathbb{R} = \{ i\beta; \beta \in \mathbb{R} \}$ and $I$ is the identity operator. To see the exponential stability we use a well-known result by Gearhart [7], Huang [8] and Prüss [15]. It asserts essentially that a $C_0$-semigroup of contractions $S(t) = e^{At}$ is exponentially stable if and only if

$$i\mathbb{R} \subseteq \rho(A) \quad \text{and} \quad \limsup_{|\beta| \to \infty} \| (i\beta I - A)^{-1} \|_{\mathcal{L}(H)} < \infty. \tag{3.2}$$

The main result of this section is the following.

**Theorem 3.1.** Under the conditions of Theorem 2.1, if $D_0, D_1 > 0$, then the corresponding semigroup $S(t) = e^{At}$ of the system is analytic and exponentially stable.

**Proof.** From the proof of Theorem 2.1 we know that $S(t) = e^{At}$ is a $C_0$-semigroup of contractions in $\mathcal{H}$. Then to prove the analyticity of $S(t)$ it suffices to show that $i\mathbb{R} \subset \rho(A)$ and (3.1) holds. Since $A$ is a closed operator and $D(A)$ has compact embedding over the phase space $\mathcal{H}$, then the spectrum $\sigma(A)$ is given only by eigenvalues. Therefore to prove that the imaginary axis is contained in the resolvent set of $A$ it is enough to show that there is no imaginary eigenvalue. To do so, let us suppose that there exist imaginary eigenvalues $i\lambda$, with eigenvector $U$, that is $i\lambda U - AU = 0$. Then from relation (2.11) we get that $W = \Phi = \Psi = 0$, this implies that $U = 0$, which is a contradiction. Therefore $i\mathbb{R} \subset \rho(A)$.

On the other hand, given $F = (f^1, f^2, f^3, f^4, f^5, f^6)^t \in \mathcal{H}$ the resolvent equation of $A$ is given by

$$i\beta U - AU = F \quad \text{in} \; \mathcal{H}, \tag{3.3}$$

which in term of its components is equivalent to

$$i\beta w - W = f^1, \tag{3.4}$$

$$i\beta L_2(\psi, \varphi) + K \frac{\rho_2}{\varphi} \frac{\rho_2}{\psi} \Delta W = f^2, \tag{3.5}$$

$$i\beta \psi - \Psi = f^3, \tag{3.6}$$

$$i\beta L_2(\psi, \varphi) + K \frac{\rho_2}{\varphi} \frac{\rho_2}{\psi} \Delta W = f^4, \tag{3.7}$$

$$i\beta \varphi - \Phi = f^5, \tag{3.8}$$

$$i\beta L_3(\varphi, \psi) + K \frac{\rho_2}{\varphi} \frac{\rho_2}{\psi} (\varphi + w_y) = f^6. \tag{3.9}$$

Taking the inner product of (3.3) with $U$ one has

$$i\beta \| U \|_{\mathcal{H}}^2 = (AU, U)_{\mathcal{H}} = (F, U)_{\mathcal{H}}, \tag{3.10}$$
Then combining the real part of identity (3.10) and condition (2.11) yields
\[
D_0 \| \nabla W \|_2^2 + D_1 (1 - \mu) (\| \Psi_x \|_2^2 + \| \Phi_y \|_2^2)
+ D_1 \mu (\| \Psi_x + \Phi_y \|_2^2 + D_1 \left( \frac{1 - \mu}{2} \right) \| \Psi_x + \Phi_y \|_2^2) \leq \| U \|_H \| F \|_H. \tag{3.11}
\]
To prove that \( i \mathbb{R} \subset \rho(A) \), it is enough to show that
\[
i \beta U - AU = 0 \quad \text{implies} \quad U = 0.
\]
Indeed, from (3.11) with \( F = 0 \),
\[
D_0 \| \nabla W \|_2^2 + D_1 (1 - \mu) (\| \Psi_x \|_2^2 + \| \Phi_y \|_2^2)
+ D_1 \mu (\| \Psi_x + \Phi_y \|_2^2 + D_1 \left( \frac{1 - \mu}{2} \right) \| \Psi_x + \Phi_y \|_2^2) = 0. \tag{3.12}
\]
Then it follows that \( W = \Phi = \Psi = 0 \). On the other hand, from (3.4), (3.6), (3.8) we infer that \( w = \psi = \varphi = 0 \).
Therefore \( U = 0 \), which means that \( A \) has no pure imaginary eigenvalues.
Now we show that condition (3.1) holds. In view of resolvent equation (3.3) it is enough to show that, for some \( C > 0 \) independent of \( \beta \),
\[
| \beta | \| U \|_H \leq C \| F \|_H \quad \text{when} \quad | \beta | \to \infty. \tag{3.13}
\]
Taking the norm of \( H^1(\Omega) \) in (3.4), (3.6), (3.8), we get
\[
| \beta | \| \nabla w \|_2 + | \beta | \| \nabla \psi \|_2 + | \beta | \| \nabla \varphi \|_2 \leq C \| U \|_H \| F \|_H.
\]
Multiplying Eqs. (3.5), (3.7), (3.9), respectively by, \( i \beta \overline{W}, i \beta \overline{\Psi}, i \beta \overline{\Phi} \), we get
\[
| \beta |^2 (\| W \|_2^2 + \| \Psi \|_2^2 + \| \Phi \|_2^2) \leq C | \beta |^2 (\| \nabla w \|_2 + \| \nabla \psi \|_2 + \| \nabla \varphi \|_2) + C \| U \|_H \| F \|_H.
\]
From the last two inequalities we obtain
\[
| \beta | \| U \|_H \leq C \| U \|_H^{1/2} \| F \|_H^{1/2}
\]
and hence (3.13) is satisfied. This proves the analyticity of the semigroup \( S(t) \).
It remains to show that \( S(t) \) is exponentially stable. To this end we note that (3.1) implies (3.2). Indeed, for \( | \beta | \geq 1 \) see that
\[
\| (i \beta I - A)^{-1} \|_{L(H)} \leq \| \beta (i \beta I - A)^{-1} \|_{L(H)}.
\]
Therefore (3.2) holds. \( \square \)

4. Lack of exponential decay

In this section we show that the Mindlin–Timoshenko system, when \( D_0 = 0 \), is not in general exponentially stable (and consequently not analytic). A counterexample is constructed by assuming the mixed boundary condition (1.11).
Theorem 4.1. Under the assumptions of Theorem 2.1, if \( D_0 = 0 \), then the corresponding semigroup of the Mindlin–Timoshenko system is not in general exponentially stable.

Proof. If \( D_0 = D_1 = 0 \), then the system is conservative and therefore there is no exponential stability. Then we assume \( D_0 = 0 \) and \( D_1 > 0 \). In what follows we show that condition (3.2) fails. Let us take \( \Omega = [0, \pi] \times [0, \pi] \) with boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \) where

\[
\Gamma_1 = \{(x, y); \ 0 < x < \pi, \ y = 0, \pi\} \quad \text{and} \quad \Gamma_2 = \{(x, y); \ 0 < y < \pi, \ x = 0, \pi\}. \tag{4.1}
\]

To contradict (3.2) it is enough to show the existence of a sequence of real numbers \((\beta_n)_{n\in\mathbb{N}}\) and a sequence \(F_n\) with \(\|F_n\|_{\mathcal{H}_2} \leq 1\) such that the solution of the resolvent equation

\[
(i\beta_n I_d - A_2)U_n = F_n \tag{4.2}
\]

verifies

\[
\lim_{n \to \infty} \|U_n\|_{\mathcal{H}_2} = \lim_{n \to \infty} \|(i\beta_n I_d - A_2)^{-1}F_n\|_{\mathcal{H}_2} = \infty. \tag{4.3}
\]

For each \(n \in \mathbb{N}\), we choose

\[
F_n = \begin{pmatrix} 0, \alpha_1 \sin(nx) \sin(ny), 0, \alpha_2 \cos(nx) \sin(ny), 0, \alpha_3 \sin(nx) \cos(ny) \end{pmatrix}^t,
\]

where \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\). For simplicity we remove the subindex \(n\). Therefore the resolvent equation (4.2) can be written as

\[
i\beta w - W = 0,
\]

\[
i\beta W - \frac{K}{\rho_1}((\psi + w_x)_x + (\varphi + w_y)_y) = f^2,
\]

\[
i\beta \psi - \psi = 0,
\]

\[
i\beta \psi - \frac{D}{\rho_2} \mathcal{L}_2(\psi, \varphi) + \frac{K}{\rho_2} (\psi + w_x) - \frac{D_1}{\rho_2} \mathcal{L}_2(\psi, \Phi) = f^4,
\]

\[
i\beta \varphi - \varphi = 0,
\]

\[
i\beta \varphi - \frac{D}{\rho_2} \mathcal{L}_3(\varphi, \psi) + \frac{K}{\rho_2} (\varphi + w_y) - \frac{D_1}{\rho_2} \mathcal{L}_3(\Phi, \Psi) = f^6.
\]

Note that

\[
\mathcal{L}_2(\Psi, \Phi) = i\beta \mathcal{L}_2(\psi, \varphi) \quad \text{and} \quad \mathcal{L}_3(\Phi, \Psi) = i\beta \mathcal{L}_3(\varphi, \psi),
\]

and then we obtain a system for \(w, \psi, \varphi\),

\[
-\beta^2 \rho_1 w - K(\psi + w_x)_x - K(\varphi + w_y)_y = \rho_1 f^2,
\]

\[
-\beta^2 \rho_2 \psi - (D + iD_1 \beta) \left( \psi_{xx} + \frac{1 - \mu}{2} \psi_{yy} + \frac{1 + \mu}{2} \varphi_{xy} \right) + K(\psi + w_x) = \rho_2 f^4,
\]

\[
-\beta^2 \rho_2 \varphi - (D + iD_1 \beta) \left( \varphi_{yy} + \frac{1 - \mu}{2} \varphi_{xx} + \frac{1 + \mu}{2} \psi_{xy} \right) + K(\varphi + w_y) = \rho_2 f^6. \tag{4.4}
\]

Because of boundary condition (1.11) we can assume that (4.4) has a solution \((w_n, \psi_n, \varphi_n)\) like
\[ \begin{align*}
\omega_n &= A_n \sin(nx) \sin(ny), \\
\psi_n &= B_n \cos(nx) \sin(ny), \\
\varphi_n &= C_n \sin(nx) \cos(ny),
\end{align*} \tag{4.5} \]

where \( A_n, B_n, C_n \) are constants to be determined. In this context, system (4.4) is equivalent to find \( A_n, B_n, C_n \) as solution of

\[ \begin{align*}
p_1(\beta)A_n + KnB_n + KnC_n &= \rho_1 \alpha_1, \\
KnA_n + p_2(\beta)B_n + p_3(\beta)C_n &= \rho_2 \alpha_2, \\
KnA_n + p_3(\beta)B_n + p_2(\beta)C_n &= \rho_2 \alpha_3, \tag{4.6} \end{align*} \]

where functions \( p_1, p_2, p_3 \) are defined by

\[ \begin{align*}
p_1(\beta) &= -\beta^2 \rho_1 + 2Kn^2, \\
p_2(\beta) &= -\beta^2 \rho_2 + \frac{3-\mu}{2}(D + iD_1 \beta)n^2 + K, \\
p_3(\beta) &= \frac{1+\mu}{2}(D + iD_1 \beta)n^2. \end{align*} \]

Let us choose

\[ \beta_n = \sqrt{\frac{2K}{\rho_1}}n, \quad n \in \mathbb{N}. \]

Then \( p_1(\beta) \equiv 0. \) Rewriting (4.6) under matrix form we have

\[ \begin{pmatrix}
0 & Kn & Kn \\
Kn & p_2(\beta) & p_3(\beta) \\
Kn & p_3(\beta) & p_2(\beta)
\end{pmatrix}
\begin{pmatrix}
A \\
B \\
C
\end{pmatrix} =
\begin{pmatrix}
\rho_1 \alpha_1 \\
\rho_2 \alpha_2 \\
\rho_2 \alpha_3
\end{pmatrix}. \]

Solving the above system one has

\[ A_n = -\frac{\alpha_1 \rho_1}{2Kn^2} + \frac{(\alpha_2 + \alpha_3)\rho_2}{2Kn} + \frac{(\rho_2 K - \rho_1 D)\alpha_1}{K^2} - i\alpha_1 D_1 \sqrt{\frac{2\rho_1}{K^3}}n, \]

\[ B_n = \frac{(\alpha_2 - \alpha_3)\rho_2}{2K + \Theta n^2 + i\Theta_1 n^3} + \frac{\alpha_1 \rho_1}{2Kn}, \]

\[ C_n = \frac{(\alpha_3 - \alpha_2)\rho_2}{2K + \Theta n^2 + i\Theta_1 n^3} + \frac{\alpha_1 \rho_1}{2Kn}, \]

where

\[ \Theta = 2D(1-\mu) - 4K \frac{\rho_2}{\rho_1} \quad \text{and} \quad \Theta_1 = 2D_1(1-\mu) \sqrt{\frac{2K}{\rho_1}}. \]

Now we note \( A_n \approx \sigma_0 n \) and \( \beta_n \approx \sigma_1 n, \) for some \( \sigma_0, \sigma_1 > 0 \) as \( n \to \infty. \) Then using (4.5) we have

\[ W_n = i\beta_n \omega_n = i\beta_n A_n \sin(nx) \sin(ny), \]

from where it follows that
Therefore we have
\[ \lim_{n \to \infty} \|U_n\|_{H^2} = \infty, \]
which shows that (4.3) holds. Therefore the corresponding semigroup is not exponentially stable. \( \square \)

**Remark 4.1.** Theorem 4.1 shows the lack of exponential stability of the Mindlin–Timoshenko system for \( D_0 = 0 \) in the special case of mixed boundary condition and \( \Omega \) is a square. However, since boundary conditions (1.10) and (1.11) are of conservative nature, they should not interfere on the decay rates of the system. Therefore we might expect that, under both boundary conditions, \( S(t) \) is not exponentially stable if \( D_0 = 0 \). In other words, the Mindlin–Timoshenko system with Kelvin–Voigt damping is exponentially stable if and only if \( D_0 > 0 \). \( \square \)

### 5. Optimal polynomial decay rate

In the previous section we have seen that the Mindlin–Timoshenko system with \( D_0 = 0 \) is not exponentially stable. Here we show if however \( D_1 > 0 \) then the system is polynomially stable. In fact we obtain an optimal rate of decay. To this end we apply a recent result by Borichev and Tomilov [2, Thm 2.4]. Accordingly, for a bounded \( C_0 \)-semigroup \( S(t) = e^{At} \) defined on a Hilbert space \( H \) with generator \( A \) satisfying \( i\mathbb{R} \subset \rho(A) \), one has
\[ \|T(t)A^{-1}\|_{\mathcal{L}(H)} = O(t^{-1/\alpha}) \text{ if and only if } \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} = O(|\beta|^\alpha), \]  
(5.1)
for \( t \to \infty, \beta \to \infty \).

**Theorem 5.1.** Let us assume the conditions of Theorem 2.1 with \( D_0 = 0 \) and \( D_1 > 0 \). Then for \( U_0 \in D(A) \), there exists a constant \( C > 0 \) such that
\[ \|e^{At}U_0\|_H \leq Ct^{-1/2}\|U_0\|_{D(A)} \text{ when } t \to \infty. \]  
(5.2)
In the case \( U_0 \in D(A^k), k \in \mathbb{N} \), then there exists a constant \( C_k > 0 \) such that
\[ \|e^{At}U_0\|_H \leq C_k t^{-k/2}\|U_0\|_{D(A^k)} \text{ when } t \to \infty. \]  
(5.3)
In addition, the above rates of decay are optimal.

**Proof.** We shall prove that second condition in (5.1) holds. We consider the resolvent equation used in the proof of Theorem 3.1 and also the relations (3.3)–(3.10) with \( D_0 = 0 \). In what follows \( C > 0 \) will denote several different constants.

As in Theorem 3.1 it is easy to see that \( i\mathbb{R} \subset \rho(A) \), because relation (2.11) implies that \( \Phi = \Psi = 0 \), but using the spectral system we get that \( w = 0 \). Therefore \( U = 0 \), and this implies that there exists no imaginary eigenvalue.

Now we claim that for some \( C > 0 \),
\[ \|U\|^2_H \leq C|\beta|^4\|F\|^2_H, \quad \forall|\beta| \geq 1. \]  
(5.4)
Since the proof of the (5.4) is long and technical it will be postponed to Lemma 5.5 below.
Let us assume by now that (5.4) holds. Then there exists a constant \( C > 0 \) such that
\[
\| (i \beta I - A)^{-1} \|_{L(H)} \leq C|\beta|^2,
\]
for \(|\beta|\) large enough. Using Borichev–Tomilov condition (5.1) we obtain
\[
\| e^{At}A^{-1} \|_{L(H)} = O(t^{-1/2}), \quad t \to \infty,
\]
which implies (5.2). Indeed, given \( U_0 \in D(A) \) there exists a unique \( F \in \mathcal{H} \) such that \( U_0 = A^{-1}F \) and \( \| F \|_{\mathcal{H}} = \| U_0 \|_{D(A)} \). Then the above identity implies that
\[
\| e^{At}U_0 \|_{\mathcal{H}} = \| e^{At}A^{-1}F \|_{\mathcal{H}} \leq Ct^{-1/2}\| U_0 \|_{D(A)}, \quad t \to \infty.
\]
The decay rate (5.3) follows by induction over \( k \).

Finally we show the optimality of the decay rate (5.2). Using similar reasoning as in Theorem 4.1 we shall construct an example which contradicts the second condition of (5.1) if \( \alpha < 2 \). To this end we suppose \( \Omega = [0, \pi] \times [0, \pi] \) with boundary \( \Gamma = \Gamma_1 \cup \Gamma_2 \) and verifying (4.1).

Assume that the rate of polynomial decay can be improved to \( t^{-1/(2-\delta)} \) with \( 0 < \delta < 2 \), that is, there exists a constant \( C > 0 \) such that
\[
|\beta|^\delta - 2 \| (i \beta I - A)^{-1} \|_{L(H)} \leq C \quad \text{when } \beta \to \infty.
\]
Thus for \( F \in \mathcal{H} \) and noting that \( i\mathbb{R} \subset \rho(A) \) we obtain from the resolvent equation
\[
|\beta|^\delta - 2 \| U \|_{\mathcal{H}} \leq C \| F \|_{\mathcal{H}},
\]
for some constant \( C > 0 \) and \( \beta \) large enough. On the other hand, proceeding as in the proof of Theorem 4.1, for each \( n \in \mathbb{N} \) we define
\[
F_n = (0, \alpha_1 \sin(nx) \sin(ny), 0, \alpha_2 \cos(nx) \sin(ny), 0, \alpha_3 \sin(nx) \cos(ny))^t,
\]
where \( \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \). As in (4.7) we have
\[
\| U_n \|_{H_2}^2 \geq \rho_1 \| W_n \|_{2}^2 = \frac{\rho_1 \pi^2}{4} |\beta_n A_n|^2,
\]
which implies that
\[
|\beta_n|^{\delta - 2} \| U_n \|_{H_2} \geq \frac{\rho_1^{1/2}}{2} |\beta_n^{\delta - 1} A_n| \geq Cn^\delta.
\]
So we have
\[
\lim_{n \to \infty} |\beta_n|^{\delta - 2} \| U_n \|_{H_2} = \infty,
\]
which contradicts (5.5). Therefore the decay rate \( t^{-1/2} \) cannot be improved. \( \square \)
Remark 5.1. The optimality result in Theorem 5.1 is shown for the case of mixed boundary condition and \( \Omega \) is a square. As discussed in Remarks 4.1, because of the boundary conditions are of conservative type, we might expect that optimal polynomial decay rate holds for both boundary conditions. □

We end this paper with the proof of inequality (5.4) which is divided into several lemmas. In what follows we are in the context of the proof of Theorem 3.1 with \( D_0 = 0 \). We denote by \( C > 0 \) different constants in different places.

Lemma 5.1. There exists \( C > 0 \) independent of \( \beta \) such that

\[
\|\Psi\|_{H^1(\Omega)}^2 + \|\Phi\|_{H^1(\Omega)}^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}, \tag{5.7}
\]

and

\[
|\beta|^2 \|\nabla \psi\|_2^2 + |\beta|^2 \|\nabla \varphi\|_2^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + C \|\Phi\|_{H^1(\Omega)}^2. \tag{5.8}
\]

Proof. From Eqs. (3.7) and (3.9) with \( D_0 = 0 \),

\[
D_1(1 - \mu) [\|\Psi_x\|_2^2 + \|\Phi_y\|_2^2] + D_1 \mu \|\Psi_x + \Phi_y\|_2^2 + D_1 \left( \frac{1 - \mu}{2} \right) \|\Psi_x + \Phi_y\|_2^2 \leq \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}. \tag{5.9}
\]

Then (5.7) holds. Inequality (5.8) follows by using (3.6) and (3.8). □

Lemma 5.2. There exists a constant \( C > 0 \) independent of \( \beta \) such that

\[
\min \{ \|\Psi\|_2^2, \|\Phi\|_2^2 \} \leq \frac{C}{|\beta|} \|w\|_2 \|U\|_{H^1(\Omega)} \|F\|_{H^1(\Omega)}^{1/2} \frac{C}{|\beta|} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{C}{|\beta|} \|F\|_{H^1(\Omega)}^2, \tag{5.10}
\]

for \( |\beta| \geq 1 \).

Proof. From Eqs. (3.7) and (3.9) with \( D_0 = 0 \) we get

\[
|\beta| \|\Psi\|_{H^{-1}(\Omega)} \leq C \left\{ \|\nabla \psi\|_2 + \|\nabla \varphi\|_2 + \|w\|_2 + \|\nabla \Phi\|_2 + \|\nabla \Psi\|_2 + \|F\|_{\mathcal{H}} \right\}, \tag{5.11}
\]

\[
|\beta| \|\Phi\|_{H^{-1}(\Omega)} \leq C \left\{ \|\nabla \psi\|_2 + \|\nabla \varphi\|_2 + \|w\|_2 + \|\nabla \Psi\|_2 + \|\nabla \Phi\|_2 + \|F\|_{\mathcal{H}} \right\}, \tag{5.12}
\]

for some constant \( C > 0 \), where as usual \( H^{-1}(\Omega) \) means the dual space of \( H^1_0(\Omega) \). By interpolation, we conclude that there exists \( C_0 > 0 \) such that

\[
\|\Psi\|_2^2 \leq C_0 \|\Psi\|_{H^{-1}(\Omega)} \|\Psi\|_{H^1(\Omega)}.
\]

From this and using estimate (5.11), Young inequality and estimates (5.7)–(5.8), we get

\[
\|\Psi\|_2^2 \leq \frac{C}{|\beta|} \left\{ \|\nabla \psi\|_2 + \|\nabla \varphi\|_2 + \|w\|_2 + \|\nabla \Phi\|_2 + \|\nabla \Psi\|_2 + \|F\|_{\mathcal{H}} \right\} \|\Psi\|_{H^1(\Omega)}
\]

\[
\leq \frac{C}{|\beta|} \|w\|_2 \|\Psi\|_{H^1(\Omega)} + \frac{C}{|\beta|} \left\{ \|\nabla \psi\|_2 + \|\nabla \varphi\|_2 \right\} \|\Psi\|_{H^1(\Omega)}
\]

\[
+ \frac{C}{|\beta|} \left\{ \|\nabla \Phi\|_2 + \|\nabla \Psi\|_2 \right\} \|\Psi\|_{H^1(\Omega)} + \frac{C}{|\beta|} \|F\|_{\mathcal{H}} \|\Psi\|_{H^1(\Omega)}
\]

\[
\leq \frac{C}{|\beta|} \|w\|_2 \|\Psi\|_{H^1(\Omega)} + C \|\Phi\|_{H^1(\Omega)} + \frac{C}{|\beta|} \|F\|_{\mathcal{H}} \|\Psi\|_{H^1(\Omega)}
\]
Lemma 5.3. There exists a constant $C > 0$ independent of $\beta$ such that

$$
\|w\|_2 \leq C|\beta| \left\{ \|\Psi\|_{H^{-1}(\Omega)} + \|\Phi\|_{H^{-1}(\Omega)} \right\} + C\|U\|^\frac{1}{2}_H \|F\|^\frac{1}{2}_H + C\|F\|_H,
$$

for $|\beta| \geq 1$.\hfill \Box

Proof. Differentiating Eqs. (3.7) and (3.9) with respect to $x$ and $y$ respectively, we get

$$
-\Delta w = F,
$$

where

$$
F = i\frac{\rho_2}{K} \beta \psi_x - \frac{D}{K} \mathcal{L}_2(\psi, \varphi)_x + \psi_x - \frac{D_1}{K} \mathcal{L}_2(\Phi, \varphi)_x - \frac{\rho_2}{K} f_x^4 + i\frac{\rho_2}{K} \beta \Phi_y - \frac{D}{K} \mathcal{L}_3(\varphi, \psi)_y + \varphi_y - \frac{D_1}{K} \mathcal{L}_3(\Phi, \psi)_y - \frac{\rho_2}{K} f_y^6.
$$

Using the elliptic regularity we obtain

$$
\|w\|_2 \leq C\|F\|_{H^{-2}(\Omega)}.
$$

Note that

$$
\|F\|_{H^{-2}(\Omega)} \leq C|\beta| \left\{ \|\Psi\|_{H^{-1}(\Omega)} + \|\Phi\|_{H^{-1}(\Omega)} \right\} + C\{\|\nabla \psi\|_2 + \|\nabla \Phi\|_2 + \|\nabla \varphi\|_2 + \|\nabla \varphi\|_2 + \|F\|_H \},
$$

for some constant $C > 0$. Then using Lemma 5.1 and the above inequalities (5.14)–(5.15) our conclusion follows.\hfill \Box

Lemma 5.4. There exists a constant $C > 0$ independent of $\beta$ such that

$$
\|\nabla w\|_2^2 \leq C|\beta|^2 \|U\|_H \|F\|_H + C|\beta|^3 \|F\|_H^2,
$$

and

$$
\|W\|_2^2 \leq C|\beta|^2 \|U\|_H \|F\|_H + C|\beta|^3 \|F\|_H^2,
$$

for $|\beta| \geq 1$.\hfill \Box
On the other hand, multiplying (3.5) by \(\epsilon > 0\), we obtain
\[
|\beta|\|\Psi\|_{H^{-2}(\Omega)} \leq C\left\{\|w\|_{H^{-1}(\Omega)} + \|\Phi\|_2 + \|F\|_\mathcal{H}\right\},
\]
(5.18)
\[
|\beta|\|\Phi\|_{H^{-2}(\Omega)} \leq C\left\{\|w\|_{H^{-1}(\Omega)} + \|\Psi\|_2 + \|F\|_\mathcal{H}\right\}.
\]
Using interpolation inequality, (5.18) and (3.5), it follows that
\[
\|\Psi\|_{H^{-1}(\Omega)} \leq C\|\Psi\|_{H^{-2}(\Omega)}^{1/2}\|\Psi\|_2^{1/2}
\leq C\left\{|\beta|^{1/2}\left\{\|w\|_{H^{-1}(\Omega)} + \|\Psi\|_2^{1/2} + \|\Phi\|_2^{1/2} + \|F\|^{1/2}_{\mathcal{H}}\right\}\|\Psi\|_2^{1/2}\right\}
\leq C\left\{|\beta|^{1/2}\left\{\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\|\Phi\|_2^{1/2} + \|\Psi\|_2 + \|\Phi\|_2 + \|F\|^{1/2}_{\mathcal{H}}\|\Psi\|_2^{1/2}\right\}\right\}
\leq C\left\{|\beta|^{1/2}\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\|\Phi\|_2^{1/2} + \frac{C}{|\beta|^{1/2}}\left\{|\|\Psi\||_2 + \|\Phi\||_2 + \|F\|^{1/2}_{\mathcal{H}}\|\Psi\|_2^{1/2}\right\}\right\},
\]
for some constant \(C > 0\) independent of \(\beta\), that is,
\[
\|\Psi\|_{H^{-1}(\Omega)} \leq \frac{C}{|\beta|^{1/2}}\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\|\Psi\|_2^{1/2} + \frac{C}{|\beta|^{1/2}}\left\{|\|\Psi\||_2 + \|\Phi\||_2 + \|F\|^{1/2}_{\mathcal{H}}\|\Psi\|_2^{1/2}\right\}. \tag{5.19}
\]
Similarly we have
\[
\|\Phi\|_{H^{-1}(\Omega)} \leq \frac{C}{|\beta|^{1/2}}\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\|\Phi\|_2^{1/2} + \frac{C}{|\beta|^{1/2}}\left\{|\|\Psi\||_2 + \|\Phi\||_2 + \|F\|^{1/2}_{\mathcal{H}}\|\Phi\|_2^{1/2}\right\}. \tag{5.20}
\]
Combining Lemma 5.3 with estimates (5.19)–(5.20) and applying Young inequality yields
\[
\|w\|_2 \leq \frac{C}{|\beta|^{1/2}}\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\left\{|\|\Psi\||_2^{1/2} + \|\Phi\||_2^{1/2}\right\} + C|\beta|^{1/2}\{\|\Psi\||_2 + \|\Phi\||_2\}
+ C\|U\|^{1/2}_{\mathcal{H}}\|F\|_{\mathcal{H}}^{1/2} + C|\beta|^{1/2}\|F\|_{\mathcal{H}},
\]
where \(C > 0\) is independent of \(\beta\) and \(|\beta| \geq 1\). Using Lemma 5.2 it follows
\[
\|w\|_2 \leq \frac{C}{|\beta|^{1/2}}\|\nabla w\|_{H^{-1}(\Omega)}^{1/2}\left\{|\|\Psi\||_2^{1/2} + \|\Phi\||_2^{1/2}\right\} + C\|U\|^{1/2}_{\mathcal{H}}\|F\|_{\mathcal{H}}^{1/2} + C|\beta|^{1/2}\|F\|_{\mathcal{H}}.
\]
Using again Young inequality with \(\epsilon > 0\) and Lemma 5.1, we obtain
\[
\|w\|_2 \leq \frac{\epsilon}{|\beta|}\|\nabla w\|_2 + C_\epsilon\|U\|^{1/2}_{\mathcal{H}}\|F\|_{\mathcal{H}}^{1/2} + C|\beta|^{1/2}\|F\|_{\mathcal{H}}, \tag{5.21}
\]
where \(C, C_\epsilon > 0\) are independent of \(\beta\) and \(|\beta| \geq 1\). This implies that
\[
\|W\|_2 \leq \epsilon\|\nabla w\|_2 + C_\epsilon|\beta|\|U\|^{1/2}_{\mathcal{H}}\|F\|_{\mathcal{H}}^{1/2} + C|\beta|^{3/2}\|F\|_{\mathcal{H}}. \tag{5.22}
\]
On the other hand, multiplying (3.5) by \(\Phi\) and integrating by parts over \(\Omega\), we have
\[
\|\nabla w\|^2_2 = -\frac{i\rho_1}{K}\beta(W, w) + (\psi_x, w) + (\varphi_y, w) + \frac{\rho_1}{K}(f^2, w).
\]
Using Poincaré’s inequality and Eq. (3.4) it follows that
\[
\|\nabla w\|_2^2 \leq \frac{p_1}{k} |\beta| \|w\|_2 \|W\|_2 + \|\psi_x\|_2 \|w\|_2 + \|\varphi_y\|_2 \|w\|_2 + \frac{p_1}{k} \|f \|^2 \|w\|_2
\]
\[
\leq C \left\{ \|W\|_2^2 + C \|F\|_H^2 \right\} + C \left\{ \|\nabla \psi\|_2 + \|\nabla \varphi\|_2 + \|F\|_H \right\} \|\nabla w\|_2,
\]
for some constant \(C > 0\). Taking \(\epsilon > 0\) small enough and using Lemma 5.1 we obtain
\[
\|\nabla w\|_2^2 \leq C \|W\|_2^2 + C \|U\|_H \|F\|_H + C \|F\|_H^2,
\]
(5.23)

and also
\[
\|\nabla w\|_2 \leq C \|W\|_2 + C \|U\|^{1/2}_H \|F\|^{1/2}_H + C \|F\|_H.
\]
(5.24)

Substitution of (5.24) into (5.22) for \(\epsilon > 0\) small enough, we get
\[
\|W\|_2^2 \leq C |\beta|^2 \|U\|_H \|F\|_H + C |\beta|^3 \|F\|_H^2,
\]
for \(|\beta| \geq 1\). This proves inequality (5.17). Finally, inserting this last estimate into (5.23) we obtain (5.16).

\section*{Acknowledgment}

The authors thank Professor Luci H. Fatori for her useful remarks and comments. The first two authors were supported by FAPESP, grants 2008/00123-7 and 2012/19274-0. The third author was supported by CNPq, grant 306615/2010-0.

\section*{References}