General decay to the full von Kármán system with memory

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The full von Kármán system accounting for in plane acceleration and general memory effects is considered. We establish a general decay of the solution as time goes to infinity. Our work allows certain kernel functions which are not necessarily of exponential or polynomial decay and, therefore, generalizes and improves earlier results in the literature.

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\textbf{1. Introduction}

For the past several decades, various types of equations have been employed some mathematical models describing physical, chemical, biological and engineering systems. Among them, the mathematical models of vibrating, flexible structures have been considerably stimulated in recent years by an increasing number of questions of practical concern. Research on the stabilization of distributed parameter systems has largely focused on the stabilization of dynamic models of individual structural members such as strings, membranes and beams.

In this paper we consider the full dynamic von Kármán system of equations for viscoelastic thin plates under the presence of a long-range memory. The main purpose of this paper is to present a general decay of the solution of a full von Kármán system with memory effects. For this, let $\Omega$ be a bounded domain of $\mathbb{R}^2$, with smooth boundary $\partial \Omega = \Gamma$. We consider the deflections of a thin plate occupying the domain $\Omega$. For $U = (u, v)$ and $w$ we are denoting the plane and vertical displacements of the plate respectively, both of them depending on the spatial variables $(x, y)$ and $t > 0$.

We assume that the boundary is divided into two parts, such that

$$\Gamma = \Gamma_0 \cup \Gamma_1 \quad \text{with} \quad \Gamma_0 \cap \Gamma_1 = \emptyset, \quad \text{and} \quad \Gamma_0 \neq \emptyset. \quad (1.1)$$

Let us denote by $\nu = (\nu_1, \nu_2)$ the unit vector normal to $\Gamma$ and by $\eta = (-\nu_2, \nu_1)$ the unitary tangent vector positively oriented on $\Gamma$.

Let us denote by $\mathcal{C}$ a continuous map from the space of $2 \times 2$ symmetric matrices into itself. We consider for example

$$\mathcal{C}(A) = \alpha [\mu (\text{tr}(A))] I + (1 - \mu) A, \quad (1.2)$$

for any $2 \times 2$ matrix $A$. In (1.2) $\text{tr}(A)$ denotes the trace of $A$, $0 < \mu < \frac{1}{2}$ is the Poisson ratio, $\alpha$ is a positive constant (which depends on the density of the plate, the Young modulus and $\mu$) and $"I"$ denotes the identity operator. The constitutive relation we will use in this paper is the following:

$$\delta = \mathcal{C}(\varepsilon[U] + f(\nabla w)) - g_1 \ast \mathcal{C}(\varepsilon[U] + f(\nabla w)).$$

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Here we denote by \( g * G \) the convolution product, defined as follows
\[
g * G = \int_0^t g(t - \tau)G(\tau) \, d\tau.
\]
and \( \varepsilon[U] \) is defined as
\[
\varepsilon[U] = \frac{1}{2} (\nabla U + \nabla U^t).
\]
The nonlinear function \( f \) is defined as \( f(\xi) = \frac{1}{2} \xi \otimes \xi \) where \( \otimes \) denotes the tensor product, in our case \( f(\nabla w) \) is given by
\[
f(\nabla w) = \frac{1}{2} \begin{pmatrix} w_x^2 & w_x w_y & w_y^2 \end{pmatrix}.
\]
To simplify the notation we will denote by \( \sigma_0 \) the following expression
\[
\sigma_0 = \varepsilon[U] + f(\nabla w).
\]
Then, the corresponding motion equations are given by
\[
\begin{align*}
U_\varepsilon - \text{Div } \delta &= 0 \quad \text{in } \Omega \times [0, \infty[, & (1.3) \\
w_\varepsilon - \gamma \Delta w + \Delta^2 w - g_2 * \Delta^2 w - \text{div} \{ \delta \nabla w \} &= 0 \quad \text{in } \Omega \times [0, \infty[. & (1.4)
\end{align*}
\]
We consider the following initial conditions
\[
U(x, y, 0) = U_0(x, y), \quad U_t(x, y, 0) = U_t(x, y), \quad w(x, y, 0) = w_0(x, y), \quad w_t(x, y, 0) = w_t(x, y),
\]
and system (1.3)–(1.4) is subjected to the boundary conditions
\[
\begin{align*}
U &= 0, \quad w = \frac{\partial w}{\partial y} = 0 \quad \text{on } \Gamma_0 \times [0, \infty[, & (1.6) \\
\delta v &= 0 \quad \text{on } \Gamma_1 \times [0, \infty[, & (1.7) \\
B_1 w - B_1 \left\{ \int_0^t g_2(t - \tau)w(\tau) \, d\tau \right\} &= 0 \quad \text{on } \Gamma_1 \times [0, \infty[, & (1.8) \\
B_2 w - \gamma \frac{\partial w}{\partial y} - B_2 \left\{ \int_0^t g_2(t - \tau)w(\tau) \, d\tau \right\} - \delta v \cdot \nabla w &= 0 \quad \text{on } \Gamma_1 \times [0, \infty[. & (1.9)
\end{align*}
\]
In (1.3), \( \text{Div } \delta \) is given by
\[
\text{Div } \delta = \text{Div } \left( \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \right) = \left( \begin{pmatrix} \frac{\partial f_{11}}{\partial x} + \frac{\partial f_{12}}{\partial y} \\ \frac{\partial f_{21}}{\partial x} + \frac{\partial f_{22}}{\partial y} \end{pmatrix} \right).
\]
Here the operators \( B_1 \) and \( B_2 \) are defined as
\[
\begin{align*}
B_1 w &= \Delta w + (1 - \mu)B_1 w, \\
B_2 w &= \frac{\partial \Delta w}{\partial y} + (1 - \mu) \frac{\partial B_2 w}{\partial y},
\end{align*}
\]
where \( B_1 \) and \( B_2 \) are given by
\[
\begin{align*}
B_1 w &= 2v_1 v_2 \frac{\partial^2 w}{\partial x \partial y} - v_1^2 \frac{\partial^2 w}{\partial y^2} - v_2^2 \frac{\partial^2 w}{\partial x^2} \\
B_2 w &= (v_1^2 - v_2^2) \frac{\partial^2 w}{\partial x \partial y} + v_1 v_2 \left\{ \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right\}.
\end{align*}
\]

**Remark 1.1.** In (1.3)–(1.4) we are assuming that the history of \( U \) and \( w \) vanishes identically when \( t < 0 \), that is \( U(x, y, t) = 0 \), \( w(x, y, t) = 0 \), for \( t < 0 \). Note that we do not assume that
\[
\begin{align*}
U(x, y, 0^+) &= U(x, y, 0^-), & U_t(x, y, 0^+) &= U_t(x, y, 0^-), \\
w(x, y, 0^+) &= w(x, y, 0^-), & w_t(x, y, 0^+) &= w_t(x, y, 0^-).
\end{align*}
\]
When the history of $U$ and $w$ is not zero, their contribution to the memory can be incorporated into the body forces. Precise derivation and physical justification of the model is given in [1].

The dissipation in system (1.3)–(1.4) is due to the terms $-g_1 \ast \sigma_0$ and $-g_2 \ast \Delta^2 w$.

We assume that the kernel $g_i \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ satisfies

$$g_i, g_i', g_i'' \in L^1(0, \infty), \quad g_i(t) > 0, \quad g_i(t) < 0, \quad \alpha_i = 1 - \int_0^{\infty} g_i(\tau) \, d\tau > 0, \quad i = 1, 2. \quad (1.10)$$

For our goal we will suppose that there exist two differential functions $\xi_i$ satisfying

$$g_i'(t) \leq -\xi_i(t) g_i(t), \quad \xi_i(t) > 0, \quad \xi_i(t) < 0, \quad t \geq 0, \quad i = 1, 2 \quad (1.11)$$

$$\frac{|\xi_i'(t)|}{\xi_i(t)} \leq k \quad (1.12)$$

where $k$ is a positive constant that will be chosen later.

There are many functions satisfying (1.11) and (1.12), for example, for $a > 0$ and $b > 0$ to be chosen properly,

- if $g_i(t) = ae^{-b(t+1)^p}$, $0 < p < 1$ we have $\xi_i(t) = pb(t+1)^{p-1}$
- if $g_i(t) = (a(t+1)^4, q < -1$ we have $\xi_i = q(t+1)^{-1}$.

**Remark 1.2.** Since $\xi_i$ are nonincreasing functions then $\xi_i(t) \leq \xi_i(0) = M_i$, $i = 1, 2$.

We can mention some results associated to viscoelastic wave, von Kármán and full von Kármán equations. In [2] the nonlinear viscoelastic wave equation with boundary dissipation has been considered. Then with suitable conditions on the initial data and the relaxation function, the authors proved the existence and uniqueness of its global solution by means of the Galerkin method and showed the uniform decay rate of the energy. Chueshov and Lasiecka [3] studied the asymptotic behavior of solutions to von Karman thermoelastic plate equations. A distinct feature of the work is that the model considered had not other dissipation added. The thermal buckling of nanocolumns with nonlocal effect and shear deformation has been investigated in [4] based on the nonlocal elasticity theory and the Timoshenko beam theory. In [5] the existence of solutions is proved for a full system of dynamic von Kármán equations expressing vibrations of geometrically nonlinear viscoelastic plate, the viscosity of which has the character of a short memory.

Now, let us mention briefly some previous related work to the problem we address. In [1] Lagnese derived system (1.3)–(1.4) explaining the meaning of the boundary conditions (1.6)–(1.9) in the case when $g_i = 0$, $i = 1, 2$. Lasiecka [6] studied the uniqueness of a global weak solution of system (1.3)–(1.4) with $g_i = 0$ and nonlinear boundary dissipation; to obtain the result the author used a method proposed by Sedenko in [7]. In [8] Lasiecka studied the full von Kármán system with the nonlinear velocity feedback acting on a part of the edge of the plate, and proved the uniform energy decay rates. In [9] Lasiecka considered system (1.3)–(1.4) with $\gamma = g_i = 0$ coupled with two heat equations and nonlinear dissipation at the boundary, and proved uniform rates of decay of the solutions. Muñoz Rivera and Perla Menzala [10] studied system (1.3)–(1.4) with the following hypotheses on the relaxation functions $g_i \in C^2(\mathbb{R}_+, \mathbb{R}_+)$:

$$g_i, g_i', g_i'' \in L^1(0, \infty), \quad \alpha_i := 1 - \int_0^{\infty} g_i(\tau) \, d\tau > 0, \quad (1.13)$$

$$g_i(t) \geq 0, \quad g_i'(t) \leq 0, \quad i = 1, 2 \quad (1.14)$$

$$-c_0 g_i(t) \leq g_i'(t) \leq -c_1 g_i(t); \quad 0 \leq g_i''(t) \leq c_2 g_i(t); \quad g_i(0) > 0, \quad (1.15)$$

and

$$-c_0 g_i^{1+p}(t) \leq g_i'(t) \leq -c_1 g_i^{1+p}(t); \quad 0 \leq g_i''(t) \leq c_2 g_i^{1+p}(t); \quad p > 2; \quad g_i(0) > 0, \quad i = 1, 2 \quad (1.16)$$

$$\beta_i := \int_0^{\infty} g_i^{1-p}(\tau) \, d\tau < \infty, \quad p > 2; \quad g_i(0) > 0, \quad i = 1, 2 \quad (1.17)$$

where $c_0$, $c_1$ and $c_2$ are positive constants. They showed that system (1.3)–(1.4) is exponentially and polynomially stable with rates that depend on the relaxation functions. Recently, Li [11] has considered the limit behavior of the solution to the nonlinear viscoelastic Marguerre von Kármán system; he proved that the limiting system is the von Kármán model with memory for thin plates.

In the present work we generalize the earlier decay result to solutions of (1.3)–(1.4). More precisely, we show that the energy decays with a similar rate of decay of the kernel functions. Our result allows a larger class of kernel functions (see Section 4 for details). It is important to observe that all the existing results in the literature about the Full von Kármán system with memory considered the case $p > 2$, which is a particular case of our hypotheses on $g_i$ (see Section 4).
The modified von Kármán system for viscoelastic plates with general memory was studied by Raposo and Santos [12]. Thus, this work can be considered as an extension of the results given in [12].

The notations we use in this paper are standard and can be found in Lion’s book [13,14]. In the sequel we denote by $c$ (sometime $c_0$, $c_1$,...) various positive constants which do not depend on $t$ or on the initial data.

This work is divided into four sections. In Section 2 we establish the existence, regularity and uniqueness of strong solutions for the studied problem (1.3)–(1.9). Section 3 is devoted to the uniform stability of the solutions of (1.3)–(1.9). We use the multiplier techniques introduced by Lions [13] along with some technical lemmas and some technical ideas as in [12,15] with the necessary modifications required by the nature of our problem. Finally in Section 4 some examples of functions $g_i$ and $\xi_i$ are given.

2. Existence of global solutions

To study the existence of solutions of system (1.3)–(1.4) we introduce the following spaces:

\[ V := \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_0 \}, \]
\[ W := \{w \in H^2(\Omega); w = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma_0 \}. \]

Let $0 < \mu < \frac{1}{2}$, we define the bilinear form $a(\cdot, \cdot)$ as follows,

\[ a(u, v) = \int_\Omega \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + \mu \left( \frac{\partial^2 u}{\partial x^2 \partial y^2} + \frac{\partial^2 u}{\partial y^2 \partial x^2} \right) \right) \cdot \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + \mu \left( \frac{\partial^2 u}{\partial x^2 \partial y^2} + \frac{\partial^2 u}{\partial y^2 \partial x^2} \right) \right) \, dA. \]

We will assume that the relaxation functions $g_i$ satisfy the hypotheses (1.10).

**Lemma 2.1.** Let $u$ and $v$ be functions in $H^4(\Omega) \cap W$. Then, we have

\[ \int_\Omega (\Delta^2 u) v \, dA = a(u, v) + \int_{\Gamma_1} (B_2 u) v - (B_1 u) \frac{\partial v}{\partial n} \, d\Gamma_1. \]  

(2.1)

**Proof.** From Green’s formula we get

\[ \int_\Omega (\Delta^2 u) v \, dA = \int_{\Gamma_1} \left( \frac{\partial \Delta u}{\partial n} \right) v \, d\Gamma_1 - \int_\Omega \Delta u \frac{\partial v}{\partial n} \, d\Gamma_1 + \int_\Omega \Delta u \Delta v \, dA \]

\[ = \int_{\Gamma_1} \left( \frac{\partial \Delta u}{\partial n} \right) v \, d\Gamma_1 - \int_\Omega \Delta u \frac{\partial v}{\partial n} \, d\Gamma_1 + \int_\Omega \Delta u \Delta v \, dA \]

\[ + (1 - \mu) \int_\Omega \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \, dA - 2(1 - \mu) \int_\Omega \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \, dA. \]

Recalling the definition of $B_1$ and $B_2$ and using the expression

\[ \int_\Omega \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \, dA - 2 \int_\Omega \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \, dA = \int_{\Gamma_1} \left( \frac{\partial B_2 u}{\partial n} \right) v - (B_1 u) \frac{\partial v}{\partial n} \, d\Gamma_1 \]

our result follows. \[ \square \]

To simplify our calculations we will use the same notations as in [10] given by

\[ g_1 \square \mathcal{O}[\sigma_0] = \int_0^t g_1(t - \tau) \mathcal{O}(\sigma_0(\cdot, t) - \sigma_0(\cdot, \tau)) \cdot [\sigma_0(\cdot, t) - \sigma_0(\cdot, \tau)] \, d\tau, \]
\[ g_2 \square \partial^2 w = \int_0^t g_2(t - \tau) a(w(\cdot, t) - w(\cdot, \tau), w(\cdot, t) - w(\cdot, \tau)) \, d\tau \]

where $\mathcal{O}$ and $\sigma_0$ are given in Section 1 and the dot “.” means the inner product of the $2 \times 2$ matrices. The following properties are useful.

**Lemma 2.2.** Under the above notations the following formulas

\[ g_1 \ast [\sigma_0]_t = \frac{1}{2} \frac{d}{dt} \int_0^t g_1 \, d\tau \cdot \sigma_0 - \frac{1}{2} \frac{g_1(t)}{2} \cdot [\sigma_0(t)], \]
\[ g_2 \ast \varphi \psi_1 = \frac{1}{2} \frac{d}{dt} \int_0^t g_2 \, d\tau |\varphi|^2 - \frac{1}{2} g_2 \square \varphi + \frac{1}{2} g_2 \square \psi - \frac{1}{2} g_2(t) |\psi|^2 \]

hold.
Proof. We will prove the first identity. Using the symmetry of the operator \( C \) we get
\[
\frac{d}{dt} \int_0^t g_1(t - \tau) C(\sigma_0(\cdot, t) - \sigma_0(\cdot, \tau)) \cdot (\sigma_0(\cdot, t) - \sigma_0(\cdot, \tau)) \, d\tau
\]
\[
= g'_1 \Box C[\sigma_0] + 2 \int_0^t g_1(t - \tau) C(\sigma_0(\cdot, t) - \sigma_0(\cdot, \tau)) \cdot [\sigma_0]_h \, d\tau
\]
\[
= g'_1 \Box C[\sigma_0] - 2 \int_0^t g_1(t - \tau) C(\sigma_0) d\tau \sigma_0]_h + 2 \int_0^t g_1(t - \tau) C(\sigma_0)[\sigma_0]_h \, d\tau
\]
which implies
\[
\frac{d}{dt} g_1 \Box C[\sigma_0] = g'_1 \Box C[\sigma_0] - 2 g_1 \ast C(\sigma_0)[\sigma_0] + \frac{d}{dt} \left( \int_0^t g_1(t \, d\tau \cdot \sigma_0) \cdot \sigma_0 \right) - g_1(t) C(\sigma_0) \cdot \sigma_0(t),
\]
from where the first identity follows. The second identity can be obtained using the same procedure as in the first identity. \( \Box \)

To get the first order estimate let us introduce the following functional
\[
E(t) = \frac{1}{2} \int_\Omega \left( |U_t|^2 + \left( 1 - \int_0^t g_1 \, d\tau \right) C(\sigma_0) \sigma_0 + g'_1 \Box C[\sigma_0] \right) \, dA
\]
\[
+ \frac{1}{2} \int_\Omega \left( |w_t|^2 + \gamma |\nabla w_t|^2 \right) \, dA + \left( 1 - \int_0^t g_2 \, d\tau \right) a(w, w) + \frac{1}{2} g_2 \Box \partial^2 w.
\]
(2.2)

Remark 2.1. Let us denote by \( A \) a \( 2 \times 2 \) symmetric matrix and let us denote by \( B \) and \( C \) matrices \( 1 \times 2 \). Then, we have that
\[
A \times B \cdot C = A \cdot B \otimes C.
\]

Using the above remark, one can prove that system (1.3)–(1.4) is a dissipative system.

Lemma 2.3. Under the above notations we have that
\[
\frac{d}{dt} E(t) = \frac{1}{2} \int_\Omega \left( g_1 \, dA - \frac{1}{2} g_1(t) \int_\Omega \sigma_0 \, dA + \frac{1}{2} - \frac{1}{2} g_2(t) a(w, w). \right)
\]

Proof. Let us take the inner product of (1.3) with \( U_t \), we obtain
\[
\int_\Omega U_t \cdot U_t \, dA = \int_{\Gamma_1} \delta v \cdot \nu U_t \, d\Gamma_1 - \int_\Omega \delta \cdot \nabla U_t \, dA.
\]
Next we multiply Eq. (1.3) by \( U_t^\ast \) and integrating over \( \Omega \) we get
\[
\int_\Omega U_t \cdot U_t^\ast \, dA = \int_{\Gamma_1} \delta v \cdot \nu U_t^\ast \, d\Gamma_1 - \int_\Omega \delta \cdot \nabla U_t^\ast \, dA.
\]
Summing up the above two identities we arrive at
\[
\frac{d}{dt} \int_\Omega |U_t|^2 \, dA = -2 \int_\Omega \delta \cdot \varepsilon[U_t] \, dA.
\]
Now, multiplying Eq. (1.4) by \( w_t \) and integrating over \( \Omega \) we get
\[
\frac{d}{dt} \left\{ \int_\Omega \left( |w_t|^2 + \gamma |\nabla w_t|^2 \right) \, dA + a(w, w) \right\}
\]
\[
= a(g_2 \ast \Delta w, \Delta w_t) - \int_\Omega \delta \times \nabla w \cdot \nabla w_t \, dA
\]
\[
\times - \int_{\Gamma_1} \left\{ \frac{\partial w_t}{\partial v} + B_2 w + B_2 g_2 \ast w + S \times \nu \cdot \nabla w \right\} w_t \, d\Gamma - \int_{\Gamma_1} \left[ B_2 g_2 \ast w - B_2 w \right] \frac{\partial w_t}{\partial v} \, d\Gamma.
\]
\[\approx 0\]
Using Remark 2.1, we have
\[
\frac{d}{dt} \int_{\Omega} |U_t|^2 \, dA = - \int_{\Omega} C(\sigma_0) \sigma \cdot \varepsilon[U_t] \, dA + \int_{\Omega} g_1 \ast C(\sigma_0) \cdot \varepsilon[U_t] \, dA + \frac{d}{dt} \int_{\Omega} \left( |w_t|^2 + \gamma |\nabla w_t|^2 \right) \, dA + a(w, w)
\]
\[
= a(g_2 \ast w, w) - \int_{\Omega} C(\sigma_0) \cdot \nabla w \otimes \nabla w_t \, dA + \int_{\Omega} g_1 \ast C(\sigma_0) w \otimes \nabla w_t \, dA.
\]

Summing up the above identities we get
\[
\frac{d}{dt} \left\{ \int_{\Omega} (|U_t|^2 + |w_t|^2 + \gamma |\nabla w_t|^2 + |\Delta w|^2) \, dA + a(w, w) \right\}
\]
\[
= - \int_{\Omega} C(\sigma_0) \cdot [\sigma_0]_t \, dA + \int_{\Omega} g_1 \ast C(\sigma_0) \cdot [\sigma_0]_t \, dA + a(g_2 \ast w, w_t).
\]

Using Lemma (1.6) and the symmetry of the operator $C$, our conclusion follows. \hfill \Box

The existence result is given by the following theorem.

**Theorem 2.1.** Let us suppose that the initial data satisfy
\[
(U_0, U_1) \in [H^2(\Omega)]^2 \times [H^1(\Omega)]^2, \quad (w_0, w_1) \in H^3(\Omega) \times H^2(\Omega),
\]

together with the following compatibility conditions
\[
U_0 = 0, \quad w_0 = \frac{\partial w_0}{\partial v} \quad \text{on } \Gamma_0 \times ]0, \infty[, \quad C[U_0 + f(\nabla w_0)]v = 0, \quad \text{on } \Gamma_1 \times ]0, \infty[, \quad B_1 w_0 = 0, \quad \text{on } \Gamma_1 \times ]0, \infty[.
\]

Suppose that $g_i(t), \ i = 1, 2,$ satisfy the conditions (1.10). Then, for any $T > 0$, there exists a unique solution of (1.3)–(1.9) satisfying
\[
(U, w) \in L^\infty(0, T; [H^2(\Omega)]^2 \times H^3(\Omega)), \quad (U_1, w_1) \in L^\infty(0, T; [H^1(\Omega)]^2 \times H^2(\Omega)).
\]

**Proof.** The proof is obtained using the Galerkin method. Then thanks to Lemma 2.3, we can follow step by step the method proposed by Lasiecka [8] (see also [16] section 4) to show existence, uniqueness and continuous dependence as well as the regularity of the global solution. \hfill \Box

**Remark 2.2.** Taking the initial data satisfying
\[
(U_0, U_1) \in [H^3(\Omega)]^2 \times [H^2(\Omega)]^2, \quad (w_0, w_1) \in H^4(\Omega) \times H^3(\Omega),
\]

with the following compatibility conditions
\[
U_0 = 0, \quad w_0 = \frac{\partial w_0}{\partial v}, \quad \text{on } \Gamma_0 \times ]0, \infty[, \quad C[U_0 + f(\nabla w_0)]v = 0, \quad \text{on } \Gamma_1 \times ]0, \infty[, \quad B_1 w_0 = 0, \quad \text{on } \Gamma_1 \times ]0, \infty[, \quad B_1 w_0 - \gamma \frac{\partial w_2}{\partial v} = 0, \quad \text{on } \Gamma_1 \times ]0, \infty[.
\]

where $w_2 = w_{|t=0}$, we can show that the solution of system (1.3)–(1.9) satisfies
\[
(U, w) \in L^\infty(0, T; [H^3(\Omega)]^2 \times H^4(\Omega)), \quad (U_1, w_1) \in L^\infty(0, T; [H^2(\Omega)]^2 \times H^3(\Omega)).
\]

**Proof.** The proof is obtained using the Galerkin method. Then thanks to Lemma 2.3, we can follow step by step the method proposed by Lasiecka [6,8] (see also [16] section 4) to show existence, uniqueness and continuous dependence as well as the regularity of the global solution. \hfill \Box

3. **General decay**

In this section we consider a wider class of kernel functions, and we establish a general decay result, which contains the usual exponential and polynomial decay rates as special cases. To get our result, let us prove first the following lemmas.
Lemma 3.1. Assume that \( g \) satisfy assumptions (1.10)-(1.11). Then, there exists a positive constant \( c \) such that

\[
\begin{align*}
|U|^2 &\leq c \cdot \sigma_0 + cE(0)a(w, w), \\
g \Box (\nabla w) &\leq cE(0).
\end{align*}
\]

Proof. Due to the definition of \( C \) it follows that there exist positive constants \( \alpha_0 \) and \( c_0 \) such that

\[
\alpha_0 |s|^2 \leq C(s) \cdot s \leq c_0 |s|^2.
\]

Using the Cauchy–Schwarz inequality and the last inequality we get

\[
\int_\Omega C(\sigma_0) \cdot \sigma_0 \, dA \geq \alpha_0 \int_\Omega |\sigma_0|^2 \, dA
\]

\[
= \alpha_0 \int_\Omega |\varepsilon[U] + f(\nabla w)|^2 \, dA
\]

\[
= \alpha_0 \int_\Omega |\varepsilon[U]|^2 \, dA + 2\alpha_0 \int_\Omega \varepsilon[U] \cdot f(\nabla w) \, dA + \alpha_0 \int_\Omega |f(\nabla w)|^2 \, dA
\]

\[
\geq \frac{\alpha_0}{2} \int_\Omega |\varepsilon[U]|^2 \, dA - 3\alpha_0 \int_\Omega |f(\nabla w)|^2 \, dA.
\]

(3.1)

Since

\[
||\nabla w||_4 \leq \varepsilon a(w, w)^{1/2},
\]

it follows from (3.1) that

\[
\frac{\alpha_0}{2} \int_\Gamma |\varepsilon[U]|^2 \, dA \leq \frac{2}{\alpha_0} \int_\Omega C(\sigma_0) \cdot \sigma_0 \, dA + cE(0)a(w, w),
\]

which proves the first part of the lemma.

Now by using the following inequality

\[
\int_\Omega |\nabla w|^4 \, dA \leq c[a(w, w)]^2,
\]

we obtain

\[
\int_0^t g_2(t - \tau) \int_\Omega |f(\nabla w)(\cdot, t) - f(\nabla w)(\cdot, \tau)|^2 \, dA \, d\tau
\]

\[
= \frac{1}{4} \int_0^t g_2(t - \tau) \int_\Omega |\nabla w \otimes \nabla w(\cdot, t) - \nabla w \otimes \nabla w(\cdot, \tau)|^2 \, dA \, d\tau
\]

\[
= \frac{1}{4} \int_0^t g_2(t - \tau) \int_\Omega ([\nabla w(\cdot, t) + \nabla w(\cdot, \tau)] \otimes [\nabla w(\cdot, t) - \nabla w(\cdot, \tau)])^2 \, dA
\]

\[
\leq \frac{1}{4} \int_0^t g_2(t - \tau) \int_\Omega |\nabla w(\cdot, t) + \nabla w(\cdot, \tau)|^2 |\nabla w(\cdot, t) - \nabla w(\cdot, \tau)|^2 \, dA
\]

\[
\leq \frac{1}{4} \left( \int_0^t g_2(t - \tau) \int_\Omega |
abla w(\cdot, t) + \nabla w(\cdot, \tau)|^4 \, dx \right)^{1/2} \left( \int_0^t g_2(t - \tau) \int_\Omega |
abla w(\cdot, t) - \nabla w(\cdot, \tau)|^4 \, dA \right)^{1/2}
\]

\[
\leq cE(0) \left( \int_0^t g_2(t - \tau) a(w(\cdot, t) - w(\cdot, \tau), w(\cdot, t) - w(\cdot, \tau)) \right)
\]

\[
\leq cE(0) g_2 \Box \Delta^2 w,
\]

which conclude the proof of Lemma 3.1. \( \square \)

Let us introduce the functional

\[
J_1(t) = -\int_\Omega U_t \cdot [g_1(0)U + g_1^* U] \, dA - \int_\Omega [w_t - \gamma \Delta w_t][g_2(0)w + g_2^* w] \, dA
\]

\[
- \int_{\Gamma_1} \frac{\partial w_t}{\partial v} [g_2(0)w + g_2^* w] \, d\Gamma + \frac{1}{2} a(g_2^* w, g_2^* w).
\]

The following two lemmas play a crucial role in our discussion.
Lemma 3.2. Let $U$ and $w$ be the unique solution of system (1.3)-(1.9) and let us suppose that $g_i$ satisfies the conditions (1.10)-(1.11). Then for any $\delta > 0$ there exists $C_\delta > 0$ such that

$$\frac{d}{dt} J_1(t) \leq -\frac{g_1(0)}{2} |U|^2 - \frac{g_2(0)}{2} |w|^2 + \gamma |\nabla w|^2 + C_\delta g_1(t) \cdot \sigma_0 + C_\delta E(0) g_2(t) a(w, w) + C_\delta + C_\delta E(0) + C_\delta g_2(t) a(w, w) + \delta \cdot \sigma_0.$$  

Proof. Multiplying (1.3) by $(g_1 \ast U)$, we have

$$\int_{\Omega} U_t \cdot [g_1(0) U + g_1'(U) \ast U] \, dA + \int_{\Omega} \delta \cdot \varepsilon [g_1(0) U + g_1'(U) \ast U] \, dA = 0,$$

which implies

$$\frac{d}{dt} \int_{\Omega} U_t \cdot [g_1(0) U + g_1'(U) \ast U] \, dA = g_1(0) \int_{\Omega} |U_t|^2 \, dA + \int_{\Omega} U_t \cdot [g_1'(U) \ast U] \, dA - \int_{\Omega} \varepsilon [g_1(0) \cdot \varepsilon[U]] \, dA + g_1'(U) \ast [g_1(0) \cdot \varepsilon[U]] \, dA,$$

from where it follows that

$$\frac{d}{dt} \int_{\Omega} U_t \cdot [g_1(0) U + g_1'(U) \ast U] \, dA = g_1(0) \int_{\Omega} |U_t|^2 \, dA + g_1'(U) \ast [g_1(0) \cdot \varepsilon[U]] \, dA - \int_{\Omega} \varepsilon [g_1(0) \cdot \varepsilon[U]] \, dA + g_1'(U) \ast [g_1(0) \cdot \varepsilon[U]] \, dA.$$

Since

$$\|C(A)\| \leq \alpha \|A\|,$$

then using Lemma 3.1 we conclude that

$$I_1 \leq \alpha \int_{\Omega} (|g_1(t - \tau)| \|\varepsilon[u](\cdot, \tau) - \varepsilon[u](\cdot, \tau)\| \, dA \leq \alpha \left\{ \int_{\Omega} |g_1(t - \tau)| \|\varepsilon[u](\cdot, \tau) - \varepsilon[u](\cdot, \tau)\|^2 \, dA \right\}^{1/2} \leq C \left\{ \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA \right\}^{1/2} \leq C_1 \left( g_1 \ast C[\sigma_0] + E(0) g_2 \ast \sigma_0 \right)^{1/2}.$$  

from where it follows that for any $\delta > 0$ we can find $C_\delta$ such that

$$I_1 \leq \delta \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_\delta (g_1 \ast C[\sigma_0] + E(0) g_2 \ast \sigma_0).$$

Let us estimate $I_2$. 

$$I_2 = \int_{\Omega} g_1 \ast C(\sigma_0) \cdot \varepsilon \left[ g_1(t) U + \int_{0}^{t} g_1'(t - \tau) [U(\cdot, \tau) - U(\cdot, \tau)] \, d\tau \right] \, dA$$

$$= \int_{\Omega} \int_{0}^{t} g_1(t - \tau) \, d\tau \left( \varepsilon \left[ g_1(t) U + \int_{0}^{t} g_1'(t - \tau) [U(\cdot, \tau) - U(\cdot, \tau)] \, d\tau \right] \right) \, dA$$

$$+ \int_{0}^{t} g_1(t - \tau) \, d\tau \int_{\Omega} \varepsilon [g_1(t) U + \int_{0}^{t} g_1'(t - \tau) [U(\cdot, \tau) - U(\cdot, \tau)] \, d\tau] \, dA$$

from where it follows that

$$I_2 \leq \delta \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_\delta \left( g_1 \ast C[\sigma_0] + g_2 \ast \sigma_0 \ast \sigma_0 \right).$$
Using the above inequalities we get

\[- \frac{d}{dt} \int_{\Omega} U_1 \cdot [g_1(0)U + g'_1 \ast U] \, dA \leq - \frac{g_1(0)}{2} \int_{\Omega} |U_1|^2 \, dA + \frac{2}{5} \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_3 \left\{ g_1 \, \square C[\sigma_0] + g_2 \, \square \Delta^2 w + g_1(t) \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + g_2(t)a(w, w) \right\} \tag{3.3} \]

On the other hand, multiplying Eq. (1.4) by \((g_2 \ast w)_t\), we obtain

\[
\frac{d}{dt} \int_{\Omega} \left[ w_1 - \gamma \Delta w_1 \right] [g_2(0)w + g'_2 \ast w] \, dA \\
= \int_{\Omega} \left[ w_1 - \gamma \Delta w_1 \right] [g_2(0)w + g'_2 \ast w] \, dA + \int_{\Omega} \left[ w_1 - \gamma \Delta w_1 \right] [g_2(0)w_1 + g'_2(0)w + g''_2 \ast w] \, dA \\
= - \int_{\Omega} \Delta^2 w [g_2(0)w + g'_2 \ast w] \, dA + \int_{\Omega} g_2 \ast \Delta^2 w [g_2(0)w + g'_2 \ast w] \, dA \\
+ \int_{\Omega} \left[ \delta \cdot \nabla w \right] [g_2(0)w + g'_2 \ast w] \, d\Gamma_1 - \int_{\Omega} \left[ \delta \cdot \nabla w \right] [g_2(0)\nabla w + g'_2 \ast \nabla w] \, d\Gamma \\
+ g_2(0) \int_{\Omega} (|w_1|^2 + \gamma |\nabla w|^2) \, dA - \gamma \int_{\Gamma_1} \frac{\partial w_1}{\partial \nu} [g_2(0)w_1 + g'_2(0)w + g''_2 \ast w] \, d\Gamma \\
= I_0 \]

\[
+ \int_{\Omega} w_1 [g'_2(0)w + g''_2 \ast w] \, dA + \int_{\Omega} \nabla w_1 \cdot [g'_2(0)\nabla w + g''_2 \ast \nabla w] \, dA.
\]

Note that

\[
I_0 = \int_{\Gamma_1} \frac{\partial w_1}{\partial \nu} [g_2(0)w + g'_2 \ast w] \, d\Gamma \\
= \frac{d}{dt} \int_{\Gamma_1} \frac{\partial w_1}{\partial \nu} [g_2(0)w + g'_2 \ast w] \, d\Gamma - \int_{\Gamma_1} \frac{\partial w_{1t}}{\partial \nu} [g_2(0)w + g'_2 \ast w] \, d\Gamma,
\]

from which it follows that

\[
\frac{d}{dt} \left\{ \int_{\Omega} \left[ w_1 - \gamma \Delta w_1 \right] [g_2(0)w + g'_2 \ast w] \, dA + \gamma \int_{\Gamma_1} \frac{\partial w_1}{\partial \nu} [g_2(0)w + g'_2 \ast w] \, d\Gamma \right\} \\
= -g_2(t)a(w, w) - a(w, \int_0^t g_2(t - \tau) (w(\cdot, t) - w(\cdot, \tau))) \\
\quad = I_3 \\
+ \frac{1}{2} \frac{d}{dt} \left( a(g_2 \ast w, g_2 \ast w) - \int_{\Omega} C(\sigma_0) \cdot \nabla w [g_2(0)\nabla w + g'_2 \ast \nabla w] \, dA \right) \\
= I_4 \\
+ \int_{\Omega} g_1 \ast C(\sigma_0) \cdot \nabla w [g_2(0)\nabla w + g'_2 \ast \nabla w] \, dA + g_2(0) \int_{\Omega} (|w_1|^2 + \gamma |\nabla w|^2) \, dA \\
= I_5 \\
+ \int_{\Gamma_1} \left[ \delta \cdot \nabla w \right] [g_2(0)w + g'_2 \ast w] \, d\Gamma_1 - \int_{\Gamma_1} \left[ \delta \cdot \nabla w \right] [g_2(0)w + g'_2 \ast \nabla w] \, d\Gamma_1 \\
+ g_2'(0) \int_{\Omega} w_1 \, dA + \int_{\Omega} w_1 \int_0^t g''_2(t - \tau) (w(\cdot, t) - w(\cdot, \tau)) \, dA \\
+ \gamma g_2'(t) \int_{\Omega} \nabla w \cdot \nabla w_1 \, dA + \gamma \int_{\Omega} \nabla w_1 \cdot \int_0^t g''_2(t - \tau) (\nabla w(\cdot, t) - \nabla w(\cdot, \tau)) \, dA.
\]
Using essentially the same arguments as we did at the beginning of the proof of this lemma, one can estimate the functionals $I_3$, $I_4$ and $I_5$ as follows

\[
I_3 \leq \frac{\delta}{5} a(w, w) + C_4 g_2 \nabla^2 w, \\
I_4 \leq \frac{\delta}{5} \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_4 \{g_2(t) a(w, w) + g_2 \nabla^2 w\}, \\
I_5 \leq \frac{\delta}{5} \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_4 \{g_1 \nabla C(\sigma_0) + g_2(t) a(w, w) + g_2 \nabla^2 w\}.
\]

From the above inequalities we have

\[
- \frac{d}{dt} \left\{ \int_{\Omega} [w_t - \gamma \Delta w_t][g_2(0)w + g_2^* w] \, dA + \int_{\Gamma_1} \frac{\partial w_t}{\partial n}[g_2(0)w + g_2^* w] \, d\Gamma \right\} \\
\leq - \frac{g_2(0)}{2} \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA + \delta a(w, w) + \frac{3}{5} \delta \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA \\
+ C_4 \{g_1 \nabla C(\sigma_0) + g_2(t) a(w, w) + g_2 \nabla^2 w\}.
\]

Summing inequalities (3.3)–(3.4) our conclusion follows. □

**Lemma 3.3.** Let us assume that $U$, $w$ and $g_i$ satisfy the conditions of Lemma 3.2. Then for any $\delta > 0$ we can find $C_5 > 0$ such that

\[
\frac{d}{dt} J_2(t) \leq \int_{\Omega} |U_t|^2 \, dA + \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA - \left(1 - \delta - \int_0^1 g_2 \, dt\right) a(w, w) \\
- \left(1 - \delta - \int_0^1 g_1 \, dt\right) \int_{\Omega} C(\sigma_0) \cdot \sigma_0 \, dA + C_5 \int_{\Omega} (g_1 \nabla C(\sigma_0) + g_2 \nabla^2 w) \, dA.
\]

**Proof.** Let us multiply Eq. (1.3) by $U$ and Eq. (1.4) by $w$, we have

\[
\frac{d}{dt} \int_{\Omega} U \cdot U_t \, dA = \int_{\Omega} |U_t|^2 \, dA + \int_{\Omega} U \cdot U_{tt} \, dA \\
= \int_{\Omega} |U_t|^2 \, dA - \int_{\Omega} \varepsilon[U] \, dA \\
= \int_{\Omega} |U_t|^2 \, dA - \int_{\Omega} C(\sigma_0) \cdot \varepsilon[U] \, dA + \int_{\Omega} g_1 \cdot C(\sigma_0) \cdot \varepsilon[U] \, dA
\]

and

\[
\frac{d}{dt} \int_{\Omega} (w w_t + \gamma \nabla w \nabla w_t) \, dA = \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA + \int_{\Omega} (w w_{tt} + \gamma \nabla w \nabla w_{tt}) \, dA \\
= \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA + \int_{\Omega} w[w_{tt} - \gamma \Delta w_t] \, dA + \gamma \int_{\Gamma_1} w \frac{\partial w_t}{\partial n} \, d\Gamma_1 \\
= \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA - \int_{\Omega} w[\Delta^2 w - g_2 \Delta^2 w - \text{div} \, \delta \nabla w] \, dA \\
+ \gamma \int_{\Gamma_1} w \frac{\partial w_t}{\partial n} \, d\Gamma_1 \\
= \int_{\Omega} (|w_t|^2 + \gamma |\nabla w_t|^2) \, dA - a(w, w) + a(g_2 \cdot w, w) - \int_{\Omega} \delta \nabla w \otimes \nabla w \, dA \\
+ \gamma \int_{\Gamma_1} \frac{\partial w_t}{\partial n} \, d\Gamma_1 - \int_{\Gamma_1} w B_2 \{w - g_1 \cdot w\} \, d\Gamma_1 \\
+ \int_{\Gamma_1} (\delta v \nabla w) w \, d\Gamma_1 + \int_{\Gamma_1} B_1 \{w - g_2 \cdot w\} \frac{\partial w}{\partial n} \, d\Gamma_1 = 0
\]
from where it follows that
\[
\frac{d}{dt} \int_\Omega (w u_t + \gamma w \nabla w \cdot \nabla w_t) \, dA = \int_\Omega (|u_t|^2 + \gamma |\nabla u_t|^2) \, dA - a(w, w) + a(g_2 * w, w) - \int_\Omega (\delta \nabla w) \cdot \nabla w \, dA
\]
\[
= \int_\Omega (|u_t|^2 + \gamma |\nabla u_t|^2) \, dA - \left(1 - \int_0^t g_2 \, d\tau\right) a(w, w)
+ a\left( w, \int_0^t g_2(t - \tau) (w(\cdot, t) - w(\cdot, \tau)) \right)
- \int_\Omega \mathcal{C}(\sigma_0) \cdot f(\nabla w) \, dA
+ \int_\Omega g_1 * \mathcal{C} \cdot f(\nabla w) \, dA,
\]

Using the relations (3.5)–(3.6) we conclude that the functional \( J_2 \) satisfies
\[
\frac{d}{dt} J_2(t) = \int_\Omega |U|^2 \, dA - \int_\Omega \mathcal{C}(\sigma_0) \cdot \varepsilon[U] \, dA + \int_\Omega g_1 * \mathcal{C}(\sigma_0) \cdot \varepsilon[U] \, dA + \int_\Omega (|u_t|^2 + \gamma |\nabla u_t|^2) \, dA
- \left(1 - \int_0^t g_2 \, d\tau\right) a(w, w)
+ \int_\Omega \mathcal{C}(\sigma_0) \cdot \sigma_0 a \, dA + C_3 \int_\Omega g_1 \mathcal{C} \mathcal{C}[\sigma_0] \, dA + C_3 g_2 \mathcal{D} \delta^2 w,
\]

which proves the lemma. \( \square \)

Now we will show the main result of this section.

**Theorem 3.1.** Let \((U, w)\) be the global solution of system (1.3)–(1.9) with the initial data satisfying the compatibility conditions as in Remark 2.2. Assume that \(g_i\) satisfies the conditions (1.10)–(1.11) and that \(\xi_1\) satisfies the hypothesis (1.12) and (3.8). Then, for each \(t_0 > 0\), there exist two small strictly positive constants \(\lambda\) and \(\beta\) such that
\[
E(t) \leq \lambda E(0)e^{-\beta \int_{t_0}^t \xi(t) \, d\tau}, \quad \forall t \geq t_0
\]  

where \(\xi(t) = \min(\xi_1(t), \xi_2(t))\).

**Proof.** Let \(\mathcal{L}\) be the following functional
\[
\mathcal{L}(t) := E(t) + \varepsilon_1 \xi_1(t) J_1(t) + \varepsilon_2 \xi_2(t) J_2(t).
\]

Then differentiating with respect to time the functional \(\mathcal{L}\) and using the Lemmas 2.3, 3.2 and 3.3 we obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq \frac{1}{2} \int_\Omega g_1 \mathcal{D} \mathcal{C}[\sigma_0] \, dA + \frac{1}{2} g_2 \mathcal{D} \delta^2 w - \frac{1}{2} g_2(t) \int_\Omega \mathcal{C}(\sigma_0) \cdot \sigma_0 a \, dA - \frac{1}{2} g_2(t) a(w, w)
+ \varepsilon_1 \xi_1(t) \left[ - \int_\Omega U_t \cdot [g_1(0) U + g_1' * U] \, dA - \int_\Omega [w_t - \gamma \Delta w_t] [g_2(0) w + g_2' * w] \, dA \right.
- \int_\Omega \frac{\partial w_t}{\partial v} [g_2(0) w + g_2' * w] \, d\Gamma + \frac{1}{2} a(g_2 * w, g_2 * w)]
\]  

Then differentiating with respect to time the functional \(\mathcal{L}\) and using the Lemmas 2.3, 3.2 and 3.3 we obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq \frac{1}{2} \int_\Omega g_1 \mathcal{D} \mathcal{C}[\sigma_0] \, dA + \frac{1}{2} g_2 \mathcal{D} \delta^2 w - \frac{1}{2} g_2(t) \int_\Omega \mathcal{C}(\sigma_0) \cdot \sigma_0 a \, dA - \frac{1}{2} g_2(t) a(w, w)
+ \varepsilon_1 \xi_1(t) \left[ - \int_\Omega U_t \cdot [g_1(0) U + g_1' * U] \, dA - \int_\Omega [w_t - \gamma \Delta w_t] [g_2(0) w + g_2' * w] \, dA \right.
- \int_\Omega \frac{\partial w_t}{\partial v} [g_2(0) w + g_2' * w] \, d\Gamma + \frac{1}{2} a(g_2 * w, g_2 * w)]
\]
which implies,

\[ \frac{d}{dt} L(t) \leq \frac{1}{2} \int_{\Omega} g_1^2 \Box C[\sigma_0] dA + \frac{1}{2} g_2^2 \partial^2 w - \frac{1}{2} g_2(t) a(w, w) \]

\[
+ \xi_1(t) \left( - \frac{g_1(0)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left| \nabla \sigma_0 \right| dA \\
+ \xi_1(t) \left( - \frac{g_2(0)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left| \nabla \omega_1 \right|^2 dA \\
+ \xi_1(t) \left( - \frac{g_1(t)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left( \left| \nabla \sigma_0 \right| + \gamma |\nabla \omega_1| \right) dA \\
\]

Choosing \( \delta > 0 \) small enough and \( \epsilon_1 > 0 \) \textbf{small enough} such that

\[ \epsilon_1 \max(2C_3, (C_4E(0) + C_3)) \leq \frac{1}{2} \]

we have

\[ \frac{d}{dt} L(t) \leq \frac{1}{2} \int_{\Omega} g_1^2 \Box C[\sigma_0] dA + \frac{1}{2} g_2^2 \partial^2 w \]

\[
+ \xi_1(t) \left( - \frac{g_1(0)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left| \nabla \sigma_0 \right| dA \\
+ \xi_1(t) \left( - \frac{g_2(0)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left| \nabla \omega_1 \right|^2 dA \\
+ \xi_1(t) \left( - \frac{g_1(t)}{2} + (\epsilon_1 + \epsilon_2) \right) \int_{\Omega} \left( \left| \nabla \sigma_0 \right| + \gamma |\nabla \omega_1| \right) dA \\
\]
where and using the hypothesis (1.12) 

Estimate (3.12) is also true for 

This completes the proof.

Now, choosing \( \varepsilon_2 \) small enough with respect to \( \varepsilon_1 \) (for example \( \varepsilon_2 < \varepsilon_1 \min\left(\frac{\varepsilon_1(0)}{4}, \frac{\varepsilon_1(0)}{4}\right) \)) and \( k > 0 \) such that

\[
 k < \frac{27\varepsilon_1}{(\varepsilon_1 + \varepsilon_2)} \min\left(\frac{g_1(0)}{4}, \frac{g_2(0)}{4}, \frac{1 - \delta - \int_0^t g_1(\tau) \, d\tau}{1 - \int_0^t g_1(\tau) \, d\tau}, \frac{1 - \delta - \int_0^t g_2(\tau) \, d\tau}{1 - \int_0^t g_2(\tau) \, d\tau}\right),
\]

and using the hypothesis (1.12) for \( \xi_1 \), we obtain

\[
 \frac{d}{dt} \mathcal{L}(t) \leq -\frac{1}{2} \int_0^t \xi_1(t - \tau) g_1(t - \tau) \mathcal{E}(\sigma_0(., t) - \mathcal{E}(\sigma_0(., \tau)), \mathcal{E}(\sigma_0(., t) - \mathcal{E}(\sigma_0(., \tau))) \, d\tau \, dA
 + \xi_1(t) \left( (\varepsilon_1 + \varepsilon_2) C_4 + \varepsilon_1 + \varepsilon_2 \right) \int_\Omega g_1 \mathcal{E}[\sigma_0] \, dA
 - \frac{1}{2} \int_0^t \xi_2(t - \tau) g_2(t - \tau) a(w(., t) - w(., \tau), w(., t) - w(., \tau)) \, d\tau
 + \xi_1(t) \left( (\varepsilon_1 + \varepsilon_2) \frac{k c_1}{4} + \varepsilon_1 C_4(0) + \varepsilon_2 C_3 \right) g_2 \mathcal{E} \, dA
 - \xi(t) c_2 \left[ \int_\Omega |U_1|^2 \, dA + \int_\Omega (|w_1|^2 + \gamma |\nabla w_1|^2) \, dA + a(w, w) + \int_\Omega \mathcal{E}(\sigma_0) \cdot \sigma_0 \, dA \right].
\]

Since \( \xi_1 \) and \( \xi_2 \) are nonincreasing functions we have

\[
 \frac{d}{dt} \mathcal{L}(t) \leq - \left( \frac{\xi_1(t)}{2} - \xi_1(t) ((\varepsilon_1 + \varepsilon_2) C_4 + \varepsilon_1 + \varepsilon_2) \frac{k c_1}{2} \right) \int_\Omega g_1 \mathcal{E}[\sigma_0] \, dA
 - \left( \frac{\xi_2(t)}{2} - \xi_1(t) (\varepsilon_1 + \varepsilon_2) \frac{k c_1}{4} + \varepsilon_1 C_4(0) + \varepsilon_2 C_3 \right) g_2 \mathcal{E} \, dA
 - \xi_1(t) c_2 \left[ \int_\Omega |U_1|^2 \, dA + \int_\Omega (|w_1|^2 + \gamma |\nabla w_1|^2) \, dA + a(w, w) + \int_\Omega \mathcal{E}(\sigma_0) \cdot \sigma_0 \, dA \right]
 \leq -c_3 \min(\xi_1(t), \xi_2(t)) \mathcal{E}(t), \quad \forall t \geq t_0.
\]

It is easy to prove that there are two positive constants \( c_4 \) and \( c_5 \) such that

\[
 c_4 \mathcal{E}(t) \leq \mathcal{L}(t) \leq c_5 \mathcal{E}(t), \quad \forall t \geq t_0.
\]

Then from the inequalities (3.9) and (3.10) we obtain

\[
 \frac{d}{dt} \mathcal{L}(t) \leq - \frac{c_3}{c_5} \xi(t) \mathcal{L}(t), \quad \forall t \geq t_0.
\]

Using the fact that \( \mathcal{E}(t) \) is equivalent to \( \mathcal{L}(t) \), we conclude that there exists a positive constant \( \beta \) such that

\[
 \frac{d}{dt} \mathcal{E}(t) \leq -\beta \xi(t) \mathcal{E}(t), \quad \forall t \geq t_0.
\]

The simple integration of (3.11) leads to

\[
 \mathcal{E}(t) \leq \mathcal{E}(0) e^{-\beta \int_0^t \xi(s) \, ds}.
\]

This completes the proof. \( \square \)

**Remark 3.1.** Estimate (3.12) is also true for \( t \in [0, t_0] \) by virtue of continuity and boundedness of \( \mathcal{E}(t) \) and \( \xi(t) \), that is,

\[
 \mathcal{E}(t) \leq \beta_1 e^{-\beta_2 \int_0^t \xi(s) \, ds}
\]

where \( \beta_1 \) and \( \beta_2 \) are positive constants.
4. Examples

In this section, we give some examples to illustrate our results. In Example 1, and from the general assumption (1.11), we obtain several decay rates in which the exponential and polynomial rates are only particular cases.

Example 1. Here we consider some examples of the function $g_i$.

- Let
  \[ g_i(t) = ae^{-b(1+t)\nu} \]
  then it is clear that (1.11) holds for $\xi_i(t) = bv(1+t)^{\min(0,\nu-1)}$. Consequently, applying the first inequality in (3.13), we obtain the following exponential decay
  \[ E(t) \leq \tilde{c}_1 e^{-\tilde{c}_2 b(1+t)^{\min(1,\nu)}} \]
  where $\tilde{c}_1$ and $\tilde{c}_2$ are positive constants.

- If
  \[ g_i(t) = ae^{-b[\ln(1 + t)]^{\nu}}, \]
  then for
  \[ \xi_i(t) = b\frac{v(\ln(1+t))^{\nu-1}}{1 + t}, \]
  the inequality (3.13) gives
  \[ E(t) \leq \tilde{c}_1 e^{-\tilde{c}_2 b(\ln(1+t))^{\nu}}. \]

- If
  \[ g_i(t) = \frac{a}{(2 + t)^{\nu}(\ln(2 + t))^{b}}, \]
  where
  \[ a > 0 \quad \text{and} \quad \nu > 1 \quad \text{and} \quad b \in \mathbb{R} \quad \text{or} \quad \nu = 1 \quad \text{and} \quad b > 1. \]
  Then for
  \[ \xi_i(t) = \frac{\nu(\ln(2 + t)) + b}{(2 + t)(\ln(2 + t))^{b}}, \]
  we obtain from the inequality (3.13)
  \[ E(t) \leq \beta_2 \left[ (\ln(2 + t))^{b} \right]^{\beta_1}. \]

Remark 4.1. Note that the exponential and polynomial decay estimates given in the literature [10], are only particular cases of (3.13). More precisely, we obtain exponential decay for $\xi_i(t) = a$ and polynomial decay for $\xi_i(t) = a(1 + t)^{-1}$, where $a > 0$ is a constant.

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References


