Asymptotic behavior to a von Kármán plate with boundary memory conditions

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Abstract

In this paper, we study the stability of solutions to a von Kármán system for Kirchhoff plate equations with a memory condition working at the boundary. We show that such dissipation is strong enough to produce exponential decay of the solution provided the relaxation functions also decay exponentially. When the relaxation functions decay polynomially, we show that the solution decays polynomially.

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1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions to a von Kármán system of the plate equation with a boundary condition of memory type. To introduce this model we need some notation. Let $\Omega$ be an open-bounded set of $\mathbb{R}^2$ with regular boundary $\Gamma$. Let us denote by $v = (v_1, v_2)$ the external unit normal vector on $\Gamma$ and

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\end{footnotesize}
by \( \mathbf{r} = (-v_2, v_1) \) the corresponding unit tangent vector. Finally, by the brackets \([\cdot, \cdot]\) we denote the binary differential operator given by
\[
[u, v] := \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} - 2 \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2}.
\]
Taking into account these notations, we consider the following initial boundary value problem:
\[
\begin{align*}
    u_{tt} + \Delta^2 u &= [u, v] \quad \text{in } \Omega \times (0, \infty), \\
    \Delta^2 v &= -[u, u] \quad \text{in } \Omega \times (0, \infty), \\
    v &= \frac{\partial v}{\partial v} = 0 \quad \text{on } \Gamma \times (0, \infty), \\
    \frac{\partial u}{\partial v} + \int_0^t g_1(t - s) \left( \mathfrak{B}_1 u(s) + \rho_1 \frac{\partial u}{\partial v}(s) \right) \, ds &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
    u - \int_0^t g_2(t - s)(\mathfrak{B}_2 u(s) - \rho_2 u(s)) \, ds &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
    u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x) \quad \text{in } \Omega.
\end{align*}
\]
Here, by \( \mathfrak{B}_1, \mathfrak{B}_2 \) we are denoting the following differential operators:
\[
\begin{align*}
    \mathfrak{B}_1 u &= \Delta u + (1 - \mu) B_1 u, \\
    \mathfrak{B}_2 u &= \frac{\partial \Delta u}{\partial v} + (1 - \mu) \frac{\partial B_2 u}{\partial \tau},
\end{align*}
\]
where \( \mu \in ]0, \frac{1}{2}[ \) is Poisson’s ratio and
\[
\begin{align*}
    B_1 u &= 2v_1v_2 \frac{\partial^2 u}{\partial x \partial y} - v_1^2 \frac{\partial^2 u}{\partial y^2} - v_2^2 \frac{\partial^2 u}{\partial x^2}, \\
    B_2 u &= (v_1^2 - v_2^2) \frac{\partial^2 u}{\partial x \partial y} + v_1v_2 \left( \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} \right).
\end{align*}
\]
In (1.1) and (1.2), \( u \) denotes the transversal displacement and \( v \) the Airy’s stress function of the vibrating plate. In (1.4) and (1.5) the relaxation functions \( g_1, g_2 \in C^1(0, \infty) \) are positive and nondecreasing, with \( g_1(0) > 0, \ g_2(0) > 0, \) and \( \rho_1, \ \rho_2 \) are small positive constants. Also, let us assume that there exists \( x_0 \in \mathbb{R}^2 \) such that
\[
\Gamma = \{ x \in \Gamma; \ v(x) \cdot (x - x_0) > 0 \}.
\]
Denoting by \( m(x) := x - x_0 \), the compactness of \( \Gamma \) implies that there exists \( \delta_0 > 0 \) such that
\[
m(x) \cdot v(x) \geq \delta_0 > 0, \quad \forall x \in \Gamma.
\]
The problem of stability of the solutions to a von Kármán system with dissipative effects has been studied by several authors. For example, in [4,5] the authors study the von Kármán
equations in the presence of thermal effects. They proved that the thermal dissipation is strong enough to produce a uniform rate of decay for the solution. In [1,2,6,9–12,17] the authors consider the von Kármán system with frictional dissipations effective in the whole plate, in a part of the plate or at the boundary. It is shown in these works that these dissipations produce uniform rate of decay of the solution when $t$ goes to infinity. It is observed that [1,2] are papers very much important in relation to the present paper with free boundary conditions. In [8] Favini and co-workers show global existence, uniqueness of weak solutions as well as the regularity of solutions with smooth data for the von Kármán system with nonlinear boundary dissipation.

The problem of stability of the solutions to a von Kármán system for viscoelastic plates with memory was studied by Rivera and Menzala in [16]. They proved that the energy decays uniformly exponentially or algebraically with the same rate of decay as the relaxation function, that is, when the relaxation function decays exponentially, the corresponding solution also decays exponentially. On the other hand, when the relaxation function decays polynomially, the solution decays polynomially at the same rate.

A distinctive feature of our paper is to deal with memory effect acting only in the boundary. That is, we show that the energy of system (1.1)–(1.6) decays uniformly in time, with rates depending on the rate of decay of the relaxation functions. More precisely, we show that the energy decays exponentially to zero provided $g_1, g_2$ decay exponentially to zero. When $g_1, g_2$ decay polynomially, we show that the corresponding energy of system (1.1)–(1.6) also decays polynomially to zero with the same rate of decay. This means that the memory effect produces strong dissipation capable of making a uniform rate of decay for the energy.

The method used is based on the construction of a suitable functional $\mathcal{L}$ satisfying

$$\frac{d}{dt} \mathcal{L}(t) \leq -c_1 \mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt} \mathcal{L}(t) \leq -c_1 \mathcal{L}(t)^{1+\alpha} + \frac{c_2}{(1+t)^{2+1}},$$

with positive constants $c_1, c_2, \gamma$ and $\alpha$.

Let us define the following bilinear form:

$$a(u, v) = \int_{\Omega} \left\{ \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} + \mu \left( \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial x^2} \right) \right\} + 2(1-\mu) \frac{\partial^2 u}{\partial x \partial y} \frac{\partial^2 v}{\partial x \partial y} \int dxdy.$$  

The following Green’s formula will be often used.

**Lemma 1.1.** For any $u \in H^4(\Omega)$ and $v \in H^2(\Omega)$, we have

$$\int_{\Omega} (\Delta^2 u) v \, dx = a(u, v) + \int_{\Gamma} \left\{ (\mathfrak{B}_2 u) v - (\mathfrak{B}_1 u) \frac{\partial v}{\partial n} \right\} \, d\Gamma. \quad (1.7)$$

**Proof.** See [12,13]. □

The notations we use in this paper are standard and can be found in Lion’s book [14]. In the sequel by $c$ (sometime $c_1, c_2, \ldots$) we denote various positive constants which do not
depend on \( t \) or on the initial data. The organization of this paper is as follows. In Section 2 we establish the existence, regularity and uniqueness of strong solutions to the von Kármán system. In Section 3 we show the uniform rate of exponential decay. Finally, in Section 4 we prove the polynomial rate of decay.

2. Existence, uniqueness and regularity

In this section we shall establish existence and uniqueness of solutions for system \((1.1)\)–\((1.6)\). Moreover we shall prove that the solution is regular provided the initial data are also regular. To facilitate our analysis, we introduce the following binary operators:

\[
(g \Box \varphi)(t) = \int_0^t g(t - s) |\varphi(t) - \varphi(s)|^2 \, ds,
\]

\[
(g \ast \varphi)(t) = \int_0^t g(t - s) \varphi(s) \, ds,
\]

where \( \ast \) is the convolution product. An important relation between these two binary operators is given by the following lemma.

**Lemma 2.1.** For \( g, \varphi \in C^1([0, \infty[; \mathbb{R}) \) we have

\[
(g \ast \varphi) \varphi_t = \frac{1}{2} g' \Box \varphi - \frac{1}{2} g(t) |\varphi|^2 - \frac{1}{2} \frac{d}{dt} \left[ g \Box \varphi - \left( \int_0^t g(s) \, ds \right) |\varphi|^2 \right].
\]

**Proof.** Differentiating the term \( g \Box \varphi \) we arrive at the above inequality. \( \square \)

Let us use boundary conditions \((1.4)\) and \((1.5)\) to find an appropriate expression to \( \mathcal{B}_1 u, \mathcal{B}_2 u \) on the boundary. Differentiating Eqs. \((1.4)\) and \((1.5)\) we arrive at the following Volterra equations:

\[
\mathcal{B}_1 u + \rho_1 \frac{\partial u}{\partial v} + \frac{1}{g_1(0)} g_1' \ast \left( \mathcal{B}_1 u + \rho_1 \frac{\partial u}{\partial v} \right) = -\frac{1}{g_1(0)} \frac{\partial u}{\partial v} t,
\]

\[
\mathcal{B}_2 u - \rho_2 u + \frac{1}{g_2(0)} g_2' \ast (\mathcal{B}_2 u - \rho u) = \frac{1}{g_2(0)} u_t.
\]

Inverting Volterra’s integral operator we get

\[
\mathcal{B}_1 u + \rho_1 \frac{\partial u}{\partial v} = -\frac{1}{g_1(0)} \left\{ \frac{\partial u}{\partial v} \frac{1}{k_1 \ast \frac{\partial u}{\partial v}} \right\} + \frac{1}{g_2(0)} \left\{ u_t + k_2 \ast u_t \right\},
\]

\[
\mathcal{B}_2 u - \rho_2 u = \frac{1}{g_2(0)} \left\{ u_t + k_2 \ast u_t \right\},
\]

where the resolvent kernels \( k_i, \ i = 1, 2 \), are given by the solutions of

\[
k_i + \frac{1}{g_i(0)} g_i' \ast k_i = -\frac{1}{g_i(0)} g_i'.
\]
Denoting by $\eta_1 = 1/g_1(0)$ and $\eta_2 = 1/g_2(0)$ we can rewrite $\mathcal{B}_1u$, $\mathcal{B}_2u$ as

$$\mathcal{B}_1u = -\rho_1 \frac{\partial u}{\partial v} - \eta_1 \left( \frac{\partial u_t}{\partial v} + k_1(0) \frac{\partial u}{\partial v} - k_1(t) \frac{\partial u_0}{\partial v} + k_1' \frac{\partial u}{\partial v} \right), \quad (2.1)$$

$$\mathcal{B}_2u = \rho_2u + \eta_2 [u_t + k_2(0)u - k_2(t)u_0 + k_2' * u]. \quad (2.2)$$

Since we are interested in relaxation functions of exponential or polynomial type and the boundary conditions (2.1) and (2.2) involve the resolvent kernels $k_i$, we want to know whether $k_i$ has the same decay properties or not. The following lemma answers this question.

Let $h$ be a relaxation function and $k$ its resolvent kernel, that is,

$$k(t) - k * h(t) = h(t). \quad (2.3)$$

**Lemma 2.2.** If $h$ is a positive continuous function, then $k$ is also a positive continuous function. Moreover,

1. If there exist positive constants $c_0$ and $\gamma$ with $c_0 < \gamma$ such that

$$h(t) \leq c_0 e^{-\gamma t},$$

then function $k$ satisfies

$$k(t) \leq \frac{c_0(\gamma - \varepsilon)}{\gamma - \varepsilon - c_0} e^{-at},$$

for all $0 < \varepsilon < \gamma - c_0$.

2. Given $p > 1$, let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1 + s)^{-p} (1 + t - s)^{-p} (1 + s)^{-p} \, ds$. If there exists a positive constant $c_0$ with $c_0 c_p < 1$ such that

$$h(t) \leq c_0 (1 + t)^{-p},$$

then function $k$ satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1 + t)^{-p}.$$

**Proof.** Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf \{ t \in \mathbb{R}^+ : k(t) = 0 \}$, so $k(t) > 0$ for all $t \in [0, t_0]$. If $t_0 \in \mathbb{R}^+$, from Eq. (2.3) we get that $-k * h(t_0) = h(t_0)$, but this is a contradiction. Therefore, $k(t) > 0$ for all $t \in \mathbb{R}^+_0$. Now, let us fix $\varepsilon$, such that $0 < \varepsilon < \gamma - c_0$, and denote by

$$k_\varepsilon(t) := e^{at} k(t), \quad h_\varepsilon(t) := e^{at} h(t).$$

Multiplying Eq. (2.3) by $e^{at}$ we get $k_\varepsilon(t) = h_\varepsilon(t) + k_\varepsilon * h_\varepsilon(t)$; hence

$$\sup_{s \in [0, t]} k_\varepsilon(s) \leq \sup_{s \in [0, t]} h_\varepsilon(s) + \left( \int_0^\infty c_0 e^{(\varepsilon - \gamma)s} \, ds \right) \sup_{s \in [0, t]} k_\varepsilon(s) \leq c_0 + \frac{c_0}{\gamma - \varepsilon} \sup_{s \in [0, t]} k_\varepsilon(s).$$
Therefore,
\[
k_p(t) \leq c_0 (\gamma - \varepsilon),
\]
which implies our first assertion. To show the second part let us introduce the following notations:
\[
k_p(t) := (1 + t)^p k(t), \quad h_p(t) := (1 + t)^p h(t).
\]
Multiplying Eq. (2.3) by \((1 + t)^p\) we get
\[
k_p(t) = h_p(t) + \int_0^t k_p(t - s)(1 + t - s)^{-p}(1 + t)^p h(s) \, ds;
\]
hence
\[
\sup_{s \in [0, t]} k_p(s) \leq \sup_{s \in [0, t]} h_p(s) + c_0 c_p \sup_{s \in [0, t]} k_p(s) \leq c_0 + c_0 c_p \sup_{s \in [0, t]} k_p(s).
\]
Therefore,
\[
k_p(t) \leq \frac{c_0}{1 - c_0 c_p},
\]
which proves our second assertion. \(\square\)

**Remark.** The finiteness of the constant \(c_p\) can be found in [18, Lemma 7.4].

Due to the above lemma, in the remainder of this paper we shall use conditions (2.1) and (2.2) instead of (1.4) and (1.5). Some important properties for the brackets’ binary operator are given in the following lemmas.

**Lemma 2.3.** Let \(u, v\) and \(w\) be functions in \(H^2(\Omega)\), such that \(v \in H^3_0(\Omega)\), where \(\Omega\) is an open-bounded and connected subset of \(\mathbb{R}^2\) with smooth boundary. Then we have
\[
\int_\Omega w[v, u] \, dx = \int_\Omega v[w, u] \, dx.
\]

**Proof.** See [12]. \(\square\)

Let \(G\) be the inverse of the biharmonic operator with Dirichlet boundary condition, that is, \(G(f) = w\), where \(w\) is the solution of
\[
\Delta^2 w = f \quad \text{in } \Omega, \quad w = \partial w / \partial n = 0 \quad \text{on } \Gamma.
\]

**Lemma 2.4.** (i) The map \((u, v) \to [u, v]\) is bounded from \(H^2(\Omega) \times H^3(\Omega) \to L^2(\Omega)\).

(ii) The map \((u, w) \to G([-u, w])\) is bounded from \(H^2(\Omega) \times H^2(\Omega) \to H^3(\Omega) \cap W^{2, \infty}(\Omega) \cap W^{4, 1}(\Omega)\), for \(\hat{\Omega} \subset \Omega\).

**Proof.** See addendum to [8]. \(\square\)

**Remark.** Lemma 2.4, is a critical result to the proof of uniqueness in [8].
Definition 2.1. We say that the couple \((u, v)\) is a weak solution of Eqs. (1.1)–(1.6) when
\[
u \in C^0([0, T] : H^2(\Omega)) \cap C^1([0, T] : L^2(\Omega)),
\]
and satisfies
\[
\begin{align*}
- \int_0^T \int_\Omega u_t \theta_t \, dx \, dt + \int_0^T a(u, \theta) \, dt &= \int_0^T \int_\Omega [u, \theta] v \, dx \, dt + \int_\Omega u_1 \theta(\cdot, 0) \, dx \\
- \eta_1 \int_0^T \int_\Gamma \left\{ \frac{\partial u_t}{\partial v} + \left( \frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u}{\partial v} - k_1(t) \frac{\partial u_0}{\partial v} + k'_1 \frac{\partial u}{\partial v} \right\} \frac{\partial \theta}{\partial v} \, d\Gamma \, dt, \\
- \eta_2 \int_0^T \int_\Gamma \left\{ u_t + \left( \frac{\rho_2}{\eta_2} + k_2(0) \right) u - k_2(t) u_0 + k'_2 u \right\} \theta \, d\Gamma \, dt, \\
\int_0^T \int_\Omega \Delta v \Delta \phi \, dx \, dt &= - \int_0^T \int_\Omega [u, \phi] u \, dx \, dt,
\end{align*}
\]
for any functions \(\theta \in C^0([0, T] : H^2(\Omega)) \cap C^1([0, T] : L^2(\Omega))\) such that \(\theta(., T) = 0, \theta_t(., T) = 0\) and \(\phi \in C^0([0, T] : H^2_0(\Omega))\).

Note that the bilinear form given by
\[
(u, w) \mapsto a(u, w) + \int_\Gamma \left( \rho_1 \frac{\partial u}{\partial v} \frac{\partial w}{\partial v} + \rho_2 u w \right) d\Gamma
\]
is strictly coercive on \(H^2(\Omega)\). Let us introduce the energy functional
\[
E(t, u, v) := \frac{1}{2} \int_\Omega |u_t|^2 \, dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_\Gamma \left( \rho_1 \left| \frac{\partial u}{\partial v} \right|^2 + \rho_2 |u|^2 \right) d\Gamma + \frac{1}{4} \int_\Omega |\Delta v|^2 \, dx
\]
\[
+ \frac{\eta_1}{2} \int_\Gamma \left( k_1(t) \left| \frac{\partial u}{\partial v} \right|^2 - k'_1 \frac{\partial u}{\partial v} \frac{\partial u}{\partial v} \right) d\Gamma + \frac{\eta_2}{2} \int_\Gamma (k_2(t) |u|^2 - k'_2 \Box u) d\Gamma.
\]
Now, we are able to prove the existence of weak solutions to von Kármán system (1.1)–(1.6).

Theorem 2.1. Let \(k_i \in C^2(0, \infty)\) be such that \(k_i, -k'_i, k''_i \geq 0\) for \(i = 1, 2\). If the initial data \((u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)\), then there exists a unique weak solution to system (1.1)–(1.6).

Proof. The main idea is to use the Galerkin method. To do this let us take a basis \(\{w_j\}_{j \in \mathbb{N}}\) to \(H^2(\Omega)\) which is orthonormal in \(L^2(\Omega)\) and we represent by \(V_m\) the subspace of \(H^2(\Omega)\).
generated by the first $m$ vectors. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$u^m(\cdot, t) = \sum_{i=1}^{m} h_{i,m}(t)w_i(\cdot), \quad v^m(\cdot, t) = G([-u^m, u^m]),$$

of the approximate system

$$\int_{\Omega} u^m_{i,j} w_j \, dx + a(u^m, w_j) - \int_{\Omega} [u^m, w_j]v^m \, dx$$

$$= -\eta_1 \int_{\Gamma} \left\{ \frac{\partial u^m_i}{\partial v} + \left( \frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m_i}{\partial v} - k_1(t) \frac{\partial u^m_{0,m}}{\partial v} + k_1^* \frac{\partial u^m_{i,m}}{\partial v} \right\} \frac{\partial w_j}{\partial v} \, d\Gamma$$

$$- \eta_2 \int_{\Gamma} \left\{ u^m_t + \left( \frac{\rho_2}{\eta_2} + k_2(0) \right) u^m - k_2(t)u^m_{0,m} + k_2^* u^m \right\} w_j \, d\Gamma, \quad (2.6)$$

for $j = 1, \ldots, m$ with initial data

$$u^m(\cdot, 0) = u_{0,m} \rightarrow u_0 \quad \text{in} \ H^2(\Omega),$$

$$u^m_t(\cdot, 0) = u_{1,m} \rightarrow u_1 \quad \text{in} \ L^2(\Omega).$$

The extension of these solutions to the whole interval $[0, T]$, $0 < T < \infty$, is a consequence of the estimate which we are going to prove below.

A priori estimate: Multiplying Eq. (2.6) by $h'_{j,m}$ and summing up the product result in $j = 1, 2, \ldots, m$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^m_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} a(u^m, u^m) - \int_{\Omega} [u^m, u^m_t]v^m \, dx$$

$$= -\eta_1 \int_{\Gamma} \left\{ \frac{\partial u^m_i}{\partial v} + \left( \frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m_i}{\partial v} - k_1(t) \frac{\partial u^m_{0,m}}{\partial v} + k_1^* \frac{\partial u^m_{i,m}}{\partial v} \right\} \frac{\partial u^m_t}{\partial v} \, d\Gamma$$

$$+ k_1^* \frac{\partial u^m_t}{\partial v} \, d\Gamma - \eta_2 \int_{\Gamma} \left\{ |u^m_t|^2 + \left( \frac{\rho_2}{\eta_2} + k_2(0) \right) u^m u^m_t \right\} \, d\Gamma.$$

Using Lemma 2.1 we have

$$\int_{\Gamma} k_1^* \frac{\partial u^m_i}{\partial v} \frac{\partial u^m_{i,m}}{\partial v} \, d\Gamma = - \frac{1}{2} k_1'(t) \int_{\Gamma} \left| \frac{\partial u^m_i}{\partial v} \right|^2 \, d\Gamma + \frac{1}{2} \int_{\Gamma} k_1'' \partial u^m_i \partial v \, d\Gamma$$

$$- \frac{1}{2} \frac{d}{dt} \int_{\Gamma} \left[ k_1^* \frac{\partial u^m_i}{\partial v} - \left( \int_{0}^{t} k_1'(s) \, ds \right) \left| \frac{\partial u^m_i}{\partial v} \right|^2 \right] \, d\Gamma,$$
Noting that
\[ \int_\Omega [u^m, u^m_t] v^m \, dx = \frac{1}{2} \int_\Omega \frac{d}{dt} (\|u^m\|^2) v^m \, dx = -\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta v^m|^2 \, dx \]
and taking into account the last equalities, we obtain
\[ \frac{d}{dt} E(t, u^m, v^m) \leq c E(0, u^m, v^m). \]
Integrating it over \([0, t]\) and taking into account the definition of the initial data of \(u^m\) we conclude that
\[ E(t, u^m, v^m) \leq c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}. \]
From the above estimate it follows
\[ u^m \to u \quad \text{weak-star in} \quad L^\infty(0, T; H^2(\Omega)), \]
\[ u^m_t \to u_t \quad \text{weak-star in} \quad L^\infty(0, T; L^2(\Omega)), \]
\[ v^m \to v \quad \text{weak-star in} \quad L^\infty(0, T; H^0_0(\Omega)). \]
Multiplying Eq. (2.6) by \(\beta \in C^2([0, T]),\) such that \(\beta(T) = \beta'(T) = 0,\) and integrating over \([0, T]\) we have
\[ \int_0^T \int_\Omega u^m w_j \beta_t \, dx \, dt + \int_0^T a(u^m, w_j) \beta \, dt \]
\[ = \int_0^T \int_\Omega [u^m, w_j] v^m \beta \, dx \, dt \]
\[ - \eta_1 \int_0^T \int_\Gamma \left\{ \frac{\partial u^m}{\partial v} + \left( \frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m}{\partial v} - k_1(t) \frac{\partial u_{0,m}}{\partial v} \right\} \partial w_j \beta \, d\Gamma \, dt \]
\[ - \eta_2 \int_0^T \int_\Gamma \left\{ u^m_t + \left( \frac{\rho_2}{\eta_2} + k_2(0) \right) u^m - k_2(t) u_{0,m} + k'_2 u^m \right\} w_j \beta \, d\Gamma \, dt. \]
Since
\[ [u^m, w_j] \to [u, w_j] \quad \text{weakly in} \quad L^2(0, T; L^2(\Omega)) \]
and
\[ v^m \to v \quad \text{strongly in} \quad L^2(0, T; L^2(\Omega)), \]
it follows letting \(m \to \infty\) and using the density of the set \(\{w_j \beta : j \in \mathbb{N}, \beta \in C^2([0, T])\}\) in \(C^0([0, T] : H^2(\Omega)) \cap C^1([0, T] : L^2(\Omega))\) that \((u, v)\) is a weak solution of (1.1)–(1.6). This completes the proof of the existence.

**Uniqueness:** Let \((u^1, v^1)\) and \((u^2, v^2)\) be two solutions of (1.1)–(1.5) with the same initial data and take \((u, v) := (u^1 - u^2, v^1 - v^2)\). In this condition \((u, v)\) has null initial data and
satisfies
\[
\frac{d}{dt} E_0(t, u) = \int_{\Omega} \left( |u|^1 u_t + |u|^2 v |u_t| \right) dx - \frac{\eta_1}{2} \int_\Gamma \left( \frac{\partial u}{\partial v} \right)^2 + k''(t) \frac{\partial u}{\partial v} - k'(t) \left( \frac{\partial u}{\partial v} \right) \right) d\Gamma \\
- \frac{\eta_2}{2} \int_\Gamma (2|u|^2 + k''_2 u - k'_2(t)|u|^2) d\Gamma,
\]
where
\[
E_0(t, u) := \frac{1}{2} \int_{\Omega} |u_t|^2 dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_\Gamma \left( \rho_1 \left( \frac{\partial u}{\partial v} \right)^2 + \rho_2 |u|^2 \right) d\Gamma \\
+ \frac{\eta_1}{2} \int_\Gamma \left( k_1(t) \frac{\partial u}{\partial v} - k'_1(t) \frac{\partial u}{\partial v} \right) d\Gamma + \frac{\eta_2}{2} \int_\Gamma (k_2(t)|u|^2 - k''_2 u) d\Gamma.
\]

Now, we will estimate the first term of the right-hand side of (2.8). Following Remark 0.2 in addendum to [8] and the second part of Lemma 2.4 we get
\[
\int_{\Omega} \left( |u|^1 u_t + |u|^2 v |u_t| \right) dx \leq \||u, v^1]\|_L^2 ||u_t||_L^2 + ||u^2, v||_L^2 ||u_t||_L^2 \\
\leq c(||u||_{H^2} ||v||_{W^{2,\infty}} ||u_t||_L^2 + ||u^2||_{H^2} ||v||_L ||u_t||_L^2) \\
\leq c(||u||_{H^2} ||v||_{H^2} ||u_t||_L^2 + ||u^2||_{H^2} ||v||_{H^2} ||u_t||_L^2).
\]

Since $\Delta^2 v = -|u, u^1 + u^2|$ and using the same argument as above, we obtain
\[
\int_{\Omega} \left( |u|^1 u_t + |u|^2 v |u_t| \right) dx \\
\leq c(||u||_{H^2} ||v||_{H^2} ||u_t||_L^2 + ||u^2||_{H^2} ||u||_{H^2} ||u^1 + u^2||_{H^2} ||u_t||_L^2) \\
\leq c||u||_{H^2} ||u_t||_L^2.
\]

Substitution of the last inequalities into (2.8) implies
\[
\frac{d}{dt} E_0(t, u) \leq c \left\{ \int_{\Omega} |u_t|^2 dx + a(u, u) \right\} - \frac{\eta_1}{2} \int_\Gamma \left( \frac{\partial u}{\partial v} \right)^2 + k''(t) \frac{\partial u}{\partial v} - k'(t) \left( \frac{\partial u}{\partial v} \right) \right) d\Gamma \\
- \frac{\eta_2}{2} \int_\Gamma (2|u|^2 + k''_2 u - k'_2(t)|u|^2) d\Gamma.
\]

Integrating this inequality over $[0, t]$, taking into account that the initial data are null and applying Gronwall’s inequality our conclusion follows. \hfill \square

To show the regularity result we will use the following lemma.
Lemma 2.5. Suppose that \( f \in L^2(\Omega) \), \( g \in H^{1/2}(\Gamma) \) and \( h \in H^{3/2}(\Gamma) \); then any solution of
\[
a(v, w) = \int_\Omega f w \, dx + \int_\Gamma g w \, d\Gamma + \int_\Gamma h \frac{\partial w}{\partial n} \, d\Gamma \quad \forall w \in H^2(\Omega)
\]
satisfies
\[
v \in H^3(\Omega).
\]

Proof. See [15]. \( \Box \)

The regularity of the solution is established in the next theorem.

Theorem 2.2. Let \( k_i \in C^2(0, \infty) \) be such that \( k_i, -k_i', k_i'' \geq 0 \) for \( i = 1, 2 \). If the initial data \( (u_0, u_1) \) belong to \( H^4(\Omega) \times H^2(\Omega) \) and satisfy the compatibility conditions
\[
\mathcal{B}_1 u_0 = -\rho_1 \frac{\partial u_0}{\partial v} - \eta_1 \frac{\partial u_1}{\partial v}, \quad \mathcal{B}_2 u_0 = \rho_2 u_0 + \eta_2 u_1 \quad \text{on} \ \Gamma,
\]
then the solution of (1.1)–(1.6) has the following regularity:
\[
u \in C^1([0, T] : H^2(\Omega)) \cap C^0([0, T] : H^4(\Omega)),
\]
\[
v, \quad \xi \in C^0([0, T] : H^4(\Omega) \cap H^2(\Omega)).
\]

Proof. Differentiating Eq. (2.6) with respect to time, we get
\[
\int_\Omega u_{tt}^m w_j \, dx + a(u_t^m, w_j)
\]
\[
= \int_\Omega [u_t^m, w_j] v_m \, dx + \int_\Omega [u_t^m, w_j] v_m \, dx
\]
\[
- \eta_1 \int_\Gamma \left\{ \frac{\partial u_t^m}{\partial v} + \left( \frac{\rho_1}{\eta_1} + k_1(0) \right) \frac{\partial u^m}{\partial v} + k_1' \frac{\partial u^m}{\partial v} \right\} \frac{\partial w_j}{\partial v} \, d\Gamma
\]
\[
- \eta_2 \int_\Gamma \left\{ u_{tt}^m + \left( \frac{\rho_2}{\eta_2} + k_2(0) \right) u_t^m + k_2' u_t^m \right\} w_j \, d\Gamma. \tag{2.10}
\]

Multiplying Eq. (2.10) by \( h_j^{''m} \), summing up the product result in \( j = 1, 2, \ldots, m \) and using Lemma 2.1, we obtain
\[
\frac{d}{dt} \left\{ E(t, u_t^m, u_t^m) - \frac{1}{2} \int_\Omega [u_t^m, u_t^m] v_m \, dx \right\}
\]
\[
= -\frac{3}{2} \int_\Omega [u_t^m, u_t^m] v_m \, dx - \frac{\eta_1}{2} \int_\Gamma \left\{ \left| \frac{\partial u_t^m}{\partial v} \right|^2 - k_1'(t) \left| \frac{\partial u_t^m}{\partial v} \right|^2 + k_1'' \frac{\partial u_t^m}{\partial v} \right\} \, d\Gamma
\]
\[
- \frac{\eta_2}{2} \int_\Gamma \left| 2u_t^m \right|^2 - k_2'(t) \left| u_t^m \right|^2 + k_2'' \frac{\partial u_t^m}{\partial v} \right\} \, d\Gamma.
\]
Integrating this identity over $[0, t]$, we obtain

$$E(t, u_t^m, v_t^m) \leq cE(0, u_t^m, v_t^m) + \frac{1}{2} \int_{\Omega} [u_t^m, v_t^m] u_t^m \, dx - \frac{3}{2} \int_{0}^{t} \int_{\Omega} [u_t^m, v_t^m] u_t^m \, dx \, dt.$$  

(2.11)

Using Lemma 2.4 we have the following estimates:

$$\int_{\Omega} [u_t^m, v_t^m] u_t^m \, dx \leq \|[u_t^m, v_t^m]\|_{L^2} \|u_t^m\|_{L^2}$$

$$\leq c\|u_t^m\|_{H^2} \|v_t^m\|_{W^{2, \infty}} \|u_t^m\|_{L^2}$$

$$\leq c\|u_t^m\|_{H^2} \|u_t^m\|_{H^2} \|u_t^m\|_{L^2}$$

$$\leq \epsilon\|u_t^m\|_{H^2}^2 + C\epsilon\|u_t^m\|_{H^2}^4 \|u_t^m\|_{L^2}^2$$

$$\leq \epsilon\|u_t^m\|_{H^2}^2 + C\epsilon,$$

$$- \int_{0}^{t} \int_{\Omega} [u_t^m, v_t^m] u_t^m \, dx \, dt \leq \int_{0}^{t} \|[u_t^m, v_t^m]\|_{L^2} \|u_t^m\|_{L^2} \, ds$$

$$\leq C \int_{0}^{t} \|u_t^m\|_{H^2} \|v_t^m\|_{W^{2, \infty}} \|u_t^m\|_{L^2} \, ds$$

$$\leq C \int_{0}^{t} \|u_t^m\|_{H^2}^2 \|u_t^m\|_{H^2} \|u_t^m\|_{L^2} \, ds$$

$$\leq C \int_{0}^{t} \|u_t^m\|_{H^2}^2 \, ds,$$

where $C\epsilon$ and $C$ are positive constants that depend on $u_0$ and $u_1$. By substitution of these two estimates into (2.11) and using Gronwall’s inequality we conclude that

$$u_t^m$$

is bounded in $L^\infty(0, T; L^2(\Omega))$,

$$u_t^m$$

is bounded in $L^\infty(0, T; H^2(\Omega))$,

$$v_t^m$$

is bounded in $L^\infty(0, T; H^0(\Omega))$.

Integrating by parts Eq. (2.4) with respect to time we get

$$a(u, w) = - \int_{\Omega} \{u_t - [u, v]\} w \, dx - \eta_2 \int_{\Gamma} \{u_t + k_2(0)u - k_2(t)u_0 + k_2^* u\} w \, d\Gamma$$

$$+ \eta_1 \int_{\Gamma} \left\{ - \frac{\partial u_t}{\partial v} - k_1(0) \frac{\partial u}{\partial v} + k_1(t) \frac{\partial u_0}{\partial v} - k_1^* \frac{\partial u}{\partial v} \right\} \frac{\partial w}{\partial v} \, d\Gamma,$$

for any $w \in H^2(\Omega)$. From Lemma 2.5 we get

$$u \in L^\infty(0, T; H^4(\Omega)).$$

The proof is now complete. \(\Box\)
3. Exponential decay

In this section we show that the solution of system (1.1)–(1.6) decays exponentially.

provided the resolvent kernels satisfy

\[ k_i(0) > 0, \quad k_i'(t) \leq -c_1 k_i(t), \quad k_i''(t) \geq -c_2 k_i'(t), \quad i = 1, 2, \tag{3.1} \]

for any \( t \geq 0 \) and some positive constants \( c_1, c_2 \).

Let us denote by \( E(t) := E(t, u, v) \). It is easy to verify that any strong solution of system (1.1)–(1.6) has the following dissipative property:

\[
\frac{d}{dt} E(t) = -\frac{\eta_1}{2} \int_{\Gamma} \left( \left( \frac{\partial u_t}{\partial v} \right)^2 + k_1'' \frac{\partial u}{\partial v} - k_1'(t) \left| \frac{\partial u}{\partial v} \right|^2 - 2k_1(t) \left( \frac{\partial u_0}{\partial v} \frac{\partial u_t}{\partial v} \right) \right) d\Gamma \\
- \frac{\eta_2}{2} \int_{\Gamma} (|u_t|^2 + k_2'' u - k_2'(t)|u|^2 - 2k_2(t)u_0u_t) d\Gamma,
\]

from where it follows, by using Young’s inequality, that

\[
\frac{d}{dt} E(t) \leq -\frac{\eta_1}{2} \int_{\Gamma} \left( \left( \frac{\partial u_t}{\partial v} \right)^2 + k_1'' \frac{\partial u}{\partial v} - k_1'(t) \left| \frac{\partial u}{\partial v} \right|^2 - k_1^2(t) \left| \frac{\partial u_0}{\partial v} \right|^2 \right) d\Gamma \\
- \frac{\eta_2}{2} \int_{\Gamma} (|u_t|^2 + k_2'' u - k_2'(t)|u|^2 - k_2^2(t)|u_0|^2) d\Gamma. \tag{3.2}
\]

The following identity will be used later.

**Lemma 3.1.** For every \( \varphi \in H^4(\Omega) \) we have

\[
\int_{\Omega} (m \cdot \nabla \varphi) \Delta^2 \varphi \, dx \\
= a(\varphi, \varphi) + \int_{\Gamma} \left[ (B_2 \varphi)m \cdot \nabla \varphi - (B_1 \varphi) \frac{\partial}{\partial v}(m \cdot \nabla \varphi) \right] d\Gamma \\
+ \frac{1}{2} \int_{\Gamma} m \cdot v \left[ \frac{\partial^2 \varphi}{\partial x^2}^2 + \frac{\partial^2 \varphi}{\partial y^2}^2 \right] + 2\mu \left[ \frac{\partial^2 \varphi}{\partial x^2} \frac{\partial^2 \varphi}{\partial y^2} \right] + 2(1-\mu) \left[ \left( \frac{\partial^2 \varphi}{\partial x \partial y} \right)^2 \right] d\Gamma.
\]

**Proof.** See [12]. \( \square \)

Let us denote by

\[
N(t) := \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx + \frac{1}{2} a(u, u) + \frac{1}{2} \int_{\Gamma} \left( \rho_1 \left| \frac{\partial u}{\partial v} \right|^2 + \rho_2 |u|^2 \right) d\Gamma + \frac{1}{4} \int_{\Omega} |\Delta v|^2 \, dx.
\]

Let us introduce the functional

\[
\psi(t) = \int_{\Omega} \left( m \cdot \nabla u + \frac{1}{2} u \right) u_t \, dx.
\]

The following lemma plays an important role in the construction of the desired functional.
Lemma 3.2. Any strong solution of (1.1)–(1.6) satisfies

$$\frac{d}{dt} \psi(t) \leq - \lambda_0 N(t) + c \int_{\Gamma} \left\{ \left| \frac{\partial u_t}{\partial v} \right|^2 + k_1^2(t) \left| \frac{\partial u}{\partial v} \right|^2 + k_1' \frac{\partial u}{\partial v} + k_1^2(t) \left| \frac{\partial u}{\partial v} \right|^2 \right\} d\Gamma$$

$$+ c \int_{\Gamma} \left\{ |u_t|^2 + k_2^2(t)|u|^2 + k_2' u + k_2^2(t)|u_0|^2 \right\} d\Gamma,$$

for some positive constants $\lambda_0$, $\epsilon$. Here, the binary operator $\diamond$ is given by

$$(k \diamond h)(t) := \int_0^t k(t-s)(h(t)-h(s)) \, ds.$$

**Proof.** Differentiating $\psi$, using Eq. (1.1) and taking $\phi = u$ in Lemma 3.1, we get

$$\frac{d}{dt} \psi(t) = \int_{\Omega} \left( m \cdot \nabla u_t + \frac{1}{2} u_t \right) u_t \, dx + \int_{\Omega} \left( m \cdot \nabla u + \frac{1}{2} u \right) u_{tt} \, dx$$

$$= \frac{1}{2} \int_{\Gamma} m \cdot |u_t|^2 \, d\Gamma - \frac{1}{2} \int_{\Omega} |u_t|^2 \, dx - \frac{3}{2} a(u, u) - \frac{1}{2} \int_{\Omega} |\Delta v|^2 \, dx$$

$$- \frac{1}{2} \int_{\Gamma} m \cdot \left[ \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + \left( \frac{\partial^2 u}{\partial y^2} \right)^2 + 2 \mu \frac{\partial^2 u}{\partial x \partial y} \right] d\Gamma$$

$$- \int_{\Gamma} \left\{ \mathcal{B}_2 u \left( m \cdot \nabla u + \frac{1}{2} u \right) - \mathcal{B}_1 u \frac{\partial}{\partial y} \left( m \cdot \nabla u + \frac{1}{2} u \right) \right\} d\Gamma$$

$$+ \int_{\Omega} (m \cdot \nabla u)[u, v] \, dx. \quad (3.3)$$

Now, let us calculate the term $\int_{\Omega} (m \cdot \nabla u)[u, v] \, dx$. According to Lemma 2.3 we have

$$\int_{\Omega} [u, v] m \cdot \nabla u \, dx = \int_{\Omega} [m \cdot \nabla u, u] v \, dx$$

$$= \frac{1}{2} \int_{\Omega} \text{div}([u, u]m) + [u, u]v \, dx$$

$$= - \frac{1}{2} \int_{\Omega} [u, u] m \cdot \nabla v \, dx + \int_{\Omega} [u, u] v \, dx$$

$$= \frac{1}{2} \int_{\Omega} \Delta^2 v (m \cdot \nabla) \, dx - \int_{\Omega} \Delta^2 v v \, dx$$

$$= - \frac{1}{2} \int_{\Omega} |\Delta v|^2 \, dx - \frac{1}{2} \int_{\Gamma} m \cdot v |\Delta v|^2 \, d\Gamma. \quad (3.4)$$
Next, we analyze the boundary terms in (3.3). Applying Young’s inequality we have, for \( \varepsilon > 0 \),

\[
\mathfrak{B}_1 u \frac{\partial}{\partial v} \left( m \cdot \nabla u + \frac{1}{2} u \right)
\]

\[
\leq \varepsilon \left| \frac{\partial}{\partial v} (m \cdot \nabla u) \right|^2 + C_\varepsilon |\mathfrak{B}_1 u|^2 + \frac{1}{2} \mathfrak{B}_1 u \frac{\partial u}{\partial v}
\]

\[
\leq \varepsilon \left| \frac{\partial}{\partial v} (m \cdot \nabla u) \right|^2 + C_\varepsilon |\mathfrak{B}_1 u|^2 - \frac{\rho_1}{2} \left| \frac{\partial u}{\partial v} \right|^2 + \frac{1}{2} \left( \mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial v} \right) \frac{\partial u}{\partial v}
\]

\[
\leq \varepsilon \left| \frac{\partial}{\partial v} (m \cdot \nabla u) \right|^2 - \frac{\rho_1}{4} - C_\varepsilon \rho_1^2 \left| \frac{\partial u}{\partial v} \right|^2 + C_\varepsilon |\mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial v}|^2.
\] (3.5)

Similarly, we obtain

\[
-\mathfrak{B}_2 u \left( m \cdot \nabla u + \frac{1}{2} u \right) \leq \varepsilon |m \cdot \nabla u - \left( \frac{\rho_2}{4} - C_\varepsilon \rho_2^2 \right) |u|^2 + C_\varepsilon |\mathfrak{B}_2 u - \rho_2 u|^2.
\] (3.6)

On the other hand, since the bilinear form \( a(u, w) + \int_\Gamma (\rho_1 (\partial u/\partial v)(\partial w/\partial v) + \rho_2 uw) \, d\Gamma \) is strictly coercive on \( H^2(\Omega) \), there exists a constant \( c > 0 \) such that

\[
\int_\Gamma \left\{ \left| \frac{\partial}{\partial v} (m \cdot \nabla u) \right|^2 + |m \cdot \nabla u|^2 \right\} \, d\Gamma
\]

\[
\leq c \left\{ a(u, u) + \int_\Gamma \left( \rho_1 \left| \frac{\partial u}{\partial v} \right|^2 + \rho_2 |u|^2 \right) \, d\Gamma \right\}
\]

\[
+ c \int_\Gamma \frac{m \cdot \nabla u}{\delta_0 \left[ \left( \frac{\partial u}{\delta x} \right)^2 + \left( \frac{\partial u}{\delta y} \right)^2 \right] + \frac{\partial^2 u}{\delta x^2} \frac{\partial^2 u}{\delta y^2} + 2(1 - \mu) \left( \frac{\partial^2 u}{\delta x \delta y} \right)^2 \right] \, d\Gamma.
\] (3.7)

Using (3.4)–(3.7), fixing \( \varepsilon \) small enough and taking into account that \( \rho_1, \rho_2 \) are small, inequality (3.3) becomes

\[
\frac{d}{dt} \psi(t) \leq \frac{1}{2} \int_\Gamma m \cdot \nabla |u|^2 \, d\Gamma - \frac{1}{2} \int_\Omega |u_t|^2 \, dx - \frac{3}{2} a(u, u) - \int_\Omega |\Delta v|^2 \, dx
\]

\[
- \lambda_0 \int_\Gamma \left( \rho_1 \left| \frac{\partial u}{\partial v} \right|^2 + \rho_2 |u|^2 \right) \, d\Gamma
\]

\[
+ c \int_\Gamma \left\{ \left| \mathfrak{B}_1 u + \rho_1 \frac{\partial u}{\partial v} \right|^2 + |\mathfrak{B}_2 u - \rho_2 u|^2 \right\} \, d\Gamma,
\]
where $\lambda_0$ is a small positive constant. Since boundary conditions (2.1) and (2.2) can be written as
\[
\mathcal{B}_1 u = -\rho_1 \frac{\partial u}{\partial v} - \eta_1 \left\{ \frac{\partial u_t}{\partial v} + k_1(t) \frac{\partial u}{\partial v} - k'_1 \frac{\partial u}{\partial v} - k_1(t) \frac{\partial u}{\partial v} \right\},
\]
\[
\mathcal{B}_2 u = \rho_2 u + \eta_2 \{ u_t + k_2(t)u - k'_2 u - k_2(t)u_0 \},
\]
our conclusion follows. □

**Lemma 3.3.** Let $f$ be a real positive function of class $C^1$. If there exist positive constants $\gamma_0, \gamma_1$ and $c_0$ such that
\[
f'(t) \leq -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},
\]
then there exist positive constants $\gamma$ and $c$ such that
\[
f(t) \leq (f(0) + c) e^{-\gamma t}.
\]
**Proof.** First, let us suppose that $\gamma_0 < \gamma_1$. Define $F(t)$ by
\[
F(t) := f(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.
\]
Then,
\[
F'(t) = f'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 F(t).
\]
Integrating from 0 to $t$ we arrive at
\[
F(t) \leq F(0) e^{-\gamma_0 t} \Rightarrow f(t) \leq \left( f(0) + \frac{c_0}{\gamma_1 - \gamma_0} \right) e^{-\gamma_0 t}.
\]
Now, we shall assume that $\gamma_0 \geq \gamma_1$. In this condition we get
\[
f'(t) \leq -\gamma_1 f(t) + c_0 e^{-\gamma_1 t} \Rightarrow [e^{\gamma_1 t} f(t)]' \leq c_0.
\]
Integrating from 0 to $t$ we obtain
\[
f(t) \leq (f(0) + c_0 t) e^{-\gamma_1 t}.
\]
Since $t \leq (\gamma_1 - \varepsilon) e^{(\gamma_1 - \varepsilon)t}$ for any $0 < \varepsilon < \gamma_1$, we conclude that
\[
f(t) \leq [f(0) + c_0 (\gamma_1 - \varepsilon)] e^{-\varepsilon t}.
\]
This completes the proof. □

Let us introduce the functional
\[
\mathcal{L}(t) = NE(t) + \psi(t),
\]
with $N > 0$. It is not difficult to see that $\mathcal{L}(t)$ verifies
\begin{equation}
q_0 E(t) \leq \mathcal{L}(t) \leq q_1 E(t),
\end{equation}
for some positive constants $q_0, q_1$. Now we are in a position to show the main result of this paper.

**Theorem 3.1.** Let us suppose that the initial data $(u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)$. If the resolvent kernels $k_1, k_2$ satisfy condition (3.1) and $\rho_1, \rho_2$ are small positive constants, then there exist positive constants $c, \gamma$ such that
\begin{equation}
E(t) \leq c e^{-\gamma t} E(0),
\end{equation}
for all $t \geq 0$.

**Proof.** We shall prove this theorem for strong solutions, that is, for solutions with initial data $(u_0, u_1) \in H^4(\Omega) \times H^2(\Omega)$ satisfying compatibility conditions (2.9). Our conclusion follows by using standard density arguments. Using hypothesis (3.1) in inequality (3.2) we have
\begin{equation}
\frac{d}{dt} E(t) \leq -\eta_1 c \int_G \left( \left| \frac{\partial u_t}{\partial y} \right|^2 - k_1'( \square u ) + k_1(t) \left| \frac{\partial u}{\partial y} \right|^2 - k_1^2(t) \left| \frac{\partial u_0}{\partial y} \right|^2 \right) d\Gamma
\end{equation}
\begin{equation}
- \eta_2 c \int_G \left( |u_t|^2 - k_2 \square u + k_2(t)|u|^2 - k_2^2(t)|u_0|^2 \right) d\Gamma.
\end{equation}
On the other hand, from Lemma 3.2 we obtain
\begin{equation}
\frac{d}{dt} \psi(t) \leq - \lambda_0 N \psi(t) + c \int_G \left( \left| \frac{\partial u_t}{\partial y} \right|^2 + k_1(t) \left| \frac{\partial u}{\partial y} \right|^2 - k_1'( \square u ) + k_1^2(t) \left| \frac{\partial u_0}{\partial y} \right|^2 \right) d\Gamma
\end{equation}
\begin{equation}
+ c \int_G \left( |u_t|^2 + k_2(t)|u|^2 - k_2 \square u + k_2^2(t)|u_0|^2 \right) d\Gamma,
\end{equation}
from where it follows that, for large $N$,
\begin{equation}
\frac{d}{dt} \mathcal{L}(t) \leq - \lambda_0 E(t) + c N R^2(t) E(0),
\end{equation}
where $R(t) := k_2(t) + k_1(t)$. In view of inequality (3.8) we conclude that
\begin{equation}
\frac{d}{dt} \mathcal{L}(t) \leq - \frac{\lambda_0}{q_1} \mathcal{L}(t) + c N R^2(t) E(0),
\end{equation}
Applying Lemma 3.3 we arrive at
\begin{equation}
\mathcal{L}(t) \leq \{ \mathcal{L}(0) + c \} e^{-\gamma t},
\end{equation}
for some $c, \gamma > 0$. From (3.8) our conclusion follows. \qed
4. Polynomial rate of decay

Here our attention will be focused on the uniform rate of decay when \( k_1 \) and \( k_2 \) decay polynomially like \((1 + t)^{-p}\). In this case we will show that the solution also decays polynomially at the same rate. Let us consider the following hypothesis, for \( i = 1, 2 \):

\[
0 < k(0), \quad k'_i(t) \leq -b_1 k_i^{1+1/p}(t), \quad k''_i(t) \geq b_2 [-k'_i(t)]^{1+1/(p+1)}, \tag{4.1}
\]

where \( p > 1 \) and \( b_1, b_2 \) are positive constants. The following lemmas will play an important role in the sequel.

Lemma 4.1. Let \( m \) and \( h \) be integrable functions, \( 0 \leq r < 1 \) and \( q > 0 \). Then, for \( t \geq 0 \)

\[
\int_0^t |m(t-s)h(s)| \, ds \leq \left( \int_0^t |m(t-s)|^{1+(1-r)/q} |h(s)| \, ds \right)^{q/(q+1)}
\]

\[
\times \left( \int_0^t |m(t-s)|^r |h(s)| \, ds \right)^{1/(q+1)}.
\]

Proof. In fact, let us take

\[
v(s) := |m(t-s)|^{1-r/(q+1)} |h(s)|^{q/(q+1)}, \quad w(s) := |m(t-s)|^{r/(q+1)} |h(s)|^{1/(q+1)}.
\]

Applying Hölder’s inequality to \(|m(s)h(s)| = v(s)w(s)\) with exponents \( \delta = q/(q+1) \) for \( v \) and \( \delta^* = q + 1 \) for \( w \) our conclusion follows.

Lemma 4.2. Let us denote by \((\phi_1, \phi_2) = (\nabla u / \nabla v, u)\) where \((u, v)\) is a solution of (1.1)–(1.5). Then, for \( p > 1, 0 \leq r < 1 \) and \( t \geq 0 \), we have

\[
\left( \int_I |k'_i| \Box \phi_i \, d\Gamma \right)^{1+1/(1-r)(p+1)} \leq 2 \left( \int_0^t |k'_i(s)|^r \, ds \| \phi_i \|_{L^\infty(0; L^2(I))} \right)^{1/(1-r)(p+1)} \int_I |k'_i|^{1+1/(p+1)} \Box \phi_i \, d\Gamma,
\]

while for \( r = 0 \) we get

\[
\left( \int_{I_1} |k'_i| \Box \phi_i \, d\Gamma \right)^{(p+2)/(p+1)} \leq 2 \left( \int_0^t \| \phi_i(s) \|_{L^2(I)}^2 \, ds + t \| \phi_i(s) \|_{L^2(I)}^2 \right)^{p+1} \int_I |k'_i|^{1+1/(p+1)} \Box \phi_i \, d\Gamma,
\]

for \( i = 1, 2 \).

Proof. The above inequalities are a consequence of Lemma 4.1 for

\[
m(s) := |k'_i(s)|, \quad h(s) := \int_I |\phi_i(t, x) - \phi_i(s, x)|^2 \, d\Gamma, \quad q := (1-r)(p+1).
\]

This concludes our assertion.
Lemma 4.3. Let \( f \geq 0 \) be a differentiable function satisfying

\[
f'(t) \leq -\frac{c_1}{f(0)^{1/2}} f(t)^{1+1/\alpha} + \frac{c_2}{(1 + t)^\beta} f(0) \quad \text{for } t \geq 0,
\]

for some positive constants \( c_1, c_2, \alpha, \) and \( \beta \) such that

\[
\beta \geq \alpha + 1.
\]

Then there exists a constant \( c > 0 \) such that

\[
f(t) \leq \frac{c}{(1 + t)^{1/2}} f(0) \quad \text{for } t \geq 0.
\]

Proof. Let us denote by

\[
F(t) = f(t) + \frac{2c_2}{\alpha} (1 + t)^{-\alpha} f(0).
\]

Differentiating this function we have

\[
F'(t) = f'(t) - 2c_2 (1 + t)^{-(\alpha+1)} f(0)
\]

\[
\leq -\frac{c_1}{f(0)^{1/2}} f(t)^{1+1/\alpha} - c_2 (1 + t)^{-(\alpha+1)} f(0)
\]

\[
\leq -\frac{c}{f(0)^{1/2}} \left[ f(t)^{1+1/\alpha} + \left( \frac{f(0)}{(1 + t)^{\alpha}} \right)^{1+1/\alpha} \right]
\]

\[
\leq -\frac{c}{F(0)^{1/2}} F(t)^{1+1/\alpha}.
\]

From this it follows that

\[
F(t) \leq \frac{F(0)}{(1 + ct)^{1/2}} \leq \frac{c}{(1 + t)^{1/2}} f(0).
\]

Therefore,

\[
f(t) \leq \frac{c}{(1 + t)^{1/2}} f(0).
\]

This completes the proof. \( \square \)

Theorem 4.1. Let us suppose that the initial data \((u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)\). If the resolvent kernels \( k_1, k_2 \) satisfy condition \((4.1)\) and \( \rho_1, \rho_2 \) are small positive constants, then there is a positive constant \( c \) such that

\[
E(t) \leq \frac{c}{(1 + t)^{p+1}} E(0).
\]

Proof. We shall prove this theorem for strong solutions, that is, for solutions with initial data \((u^0, u^1) \in H^4(\Omega) \times H^2(\Omega)\) satisfying compatibility conditions \((2.9)\). Our conclusion
follows by using standard density arguments. Using hypothesis (4.1), inequality (3.2) can be written as

\[
\frac{d}{dt} E(t) \leq -\frac{\eta_1 c}{2} \int_I \left( \frac{\partial u}{\partial v} \right)^2 + | -k_1' |^{1+1/(p+1)} \frac{\partial u}{\partial v} + k_1^{1+1/p} (t) \left| \frac{\partial u}{\partial v} \right|^2 \left| \frac{\partial u_0}{\partial v} \right|^2 \right) \, d\Gamma \\
- \frac{\eta_2 c}{2} \int_I (|u|^2 + | -k_2' |^{1+1/(p+1)} \frac{\partial u}{\partial v} + k_2^{1+1/p} (t) |u|^2 - k_2^1 (t) |u_0|^2 ) \, d\Gamma,
\]

(4.2)

for some positive constant \( c \). Using (4.1) again, there exists another positive constant \( c > 0 \) such that

\[
\left| k_1' \frac{\partial u}{\partial v} \right| \leq c | -k_1' |^{1+1/(p+1)} \frac{\partial u}{\partial v}, \quad \left| k_2' \frac{\partial u}{\partial v} \right| \leq c | -k_2' |^{1+1/(p+1)} \frac{\partial u}{\partial v}.
\]

Using this estimate in Lemma 3.2, we obtain

\[
\frac{d}{dt} \psi(t) \leq -\lambda_0 N (t) + c \int_I \left( |u|^2 + k_2^{1+1/p} (t) |u|^2 + | -k_2' |^{1+1/(p+1)} \frac{\partial u}{\partial v} + k_2^1 (t) |u_0|^2 \right) \, d\Gamma \\
+ c \int_I \left( \left| \frac{\partial u}{\partial v} \right|^2 + k_1^{1+1/p} (t) \left| \frac{\partial u}{\partial v} \right|^2 + | -k_1' |^{1+1/(p+1)} \frac{\partial u}{\partial v} + k_1^1 (t) \left| \frac{\partial u_0}{\partial v} \right|^2 \right) \, d\Gamma.
\]

(4.3)

Taking large \( N \), from (4.2)–(4.3) we arrive at

\[
\frac{d}{dt} \mathcal{L}(t) \leq -\lambda_0 N (t) - \lambda_1 \int_I \left( | -k_1' |^{1+1/(p+1)} \frac{\partial u}{\partial v} + | -k_2' |^{1+1/(p+1)} \frac{\partial u}{\partial v} \right) \, d\Gamma \\
+ c N R^2 (t) E(0),
\]

(4.4)

for some \( \lambda_1 > 0 \). Let us fix \( 0 < r < 1 \) such that \( 1/(p+1) < r < p/(p+1) \). In this condition from hypothesis (4.1) we have

\[
\int_0^\infty | -k_i' |^r \leq c \int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for } i = 1, 2.
\]

Using this estimate in Lemma 4.2, we get

\[
\int_I | -k_1' |^{1+1/(p+1)} \frac{\partial u}{\partial v} \, d\Gamma \geq c E(0)^{-1/((1-r)(p+1))} \\
\times \left( \int_I | -k_1' | \frac{\partial u}{\partial v} \, d\Gamma \right)^{1+1/((1-r)(p+1))},
\]

(4.5)

\[
\int_I | -k_2' |^{1+1/(p+1)} \frac{\partial u}{\partial v} \, d\Gamma \geq c E(0)^{-1/((1-r)(p+1))} \left( \int_I | -k_2' | \frac{\partial u}{\partial v} \, d\Gamma \right)^{1+1/((1-r)(p+1))}.
\]

(4.6)
On the other hand, since the energy is bounded we have

$$\mathcal{N}(t)^{1+1/(1-r)(p+1)} \leq c E(0)^{1/(1-r)(p+1)} \mathcal{N}(t). \quad (4.7)$$

Substituting of (4.5)–(4.7) into (4.4) we arrive at

$$\frac{d}{dt} \mathcal{L}(t) \leq -c E(0)^{-1/(1-r)(p+1)} \mathcal{N}(t)^{1+1/(1-r)(p+1)}$$

$$+ c N R^2(t) E(0) - c E(0)^{-1/(1-r)(p+1)}$$

$$\times \left\{ \left( \int_{\Gamma} [-k'_1] \Box u \, d\Gamma \right)^{1+1/(1-r)(p+1)} \right.$$

$$+ \left( \int_{\Gamma} [-k'_2] \Box v \, d\Gamma \right)^{1+1/(1-r)(p+1)} \right\}. \quad (4.8)$$

Taking into account inequality (3.8), we obtain

$$\frac{d}{dt} \mathcal{L}(t) \leq - \frac{c}{\mathcal{L}(0)^{1/(1-r)(p+1)}} \mathcal{L}(t)^{1+1/(1-r)(p+1)} + c N R^2(t) E(0). \quad (4.8)$$

Therefore, from Lemma 4.3 we conclude that

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}} \mathcal{L}(0). \quad (4.9)$$

Since $(1 - r)(p + 1) > 1$ we get, for $t \geq 0$, the following estimations:

$$t \| u(t, .) \|^2_{L^2(\Gamma)} + t \left\| \frac{\partial u(t, .)}{\partial v} \right\|^2_{L^2(\Gamma)} \leq c t \mathcal{L}(t) < \infty,$$

$$\int_0^t \| u(s, .) \|^2_{L^2(\Gamma)} + \left\| \frac{\partial u(s, .)}{\partial v} \right\|^2_{L^2(\Gamma)} \leq c \int_0^\infty \mathcal{L}(t) < \infty.$$

Using the above estimates in Lemma 4.2 with $r = 0$, we get

$$\int_{\Gamma} [-k'_1]^{1+1/(p+1)} \frac{\partial u}{\partial v} \, d\Gamma \geq \frac{c}{E(0)^{1/(p+1)}} \left( \int_{\Gamma} [-k'_1] \Box u \, d\Gamma \right)^{1+1/(p+1)},$$

$$\int_{\Gamma} [-k'_2]^{1+1/(p+1)} u \, d\Gamma \geq \frac{c}{E(0)^{1/(p+1)}} \left( \int_{\Gamma} [-k'_2] \Box u \, d\Gamma \right)^{1+1/(p+1)}.$$
Using these inequalities and the same arguments as in the derivation of (4.8), we have

\[
\frac{d}{dt} \mathcal{L}(t) \leq - \frac{c}{\mathcal{L}(0)^{1/(p+1)}} \mathcal{L}(t)^{1+1/(p+1)} + cN R^2(t) E(0).
\]

Applying Lemma 4.3 we obtain

\[
\mathcal{L}(t) \leq \frac{c}{(1 + t)^{p+1}} \mathcal{L}(0).
\]

Using inequality (3.8) we conclude that

\[
E(t) \leq \frac{c}{(1 + t)^{p+1}} E(0),
\]

which completes the proof. □

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References


