Polynomial Decay for the Energy with an Acoustic Boundary Condition

J. E. Muñoz Rivera
National Laboratory for Scientific Computation (LNCC)
Rua Getulio Vargas 333, Quitandinha 25651-070
Petrópolis-RJ, Brazil
rivera@lncc.br

Yuming Qin*
Department of Mathematics, Henan University
Kaifeng 475001, P.R. China
qinyuming@mail.henu.edu.cn
and
Rua Getulio Vargas 333, Quitandinha 25651-070, Petrópolis-RJ, Brazil
yuming@lncc.br.

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Abstract—In this paper, we establish the polynomial decay for the energy of a wave motion in a bounded domain \( \Omega \subset \mathbb{R}^3 \) with a smooth boundary \( \partial \Omega = \Gamma \), on a part \( \Gamma_0 \) of which an acoustic boundary condition is subjected. The multiplicative techniques and energy method are used. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we study the wave equation in theoretical acoustics. We denote by \( \Omega \) a domain of \( \mathbb{R}^3 \) filled with fluid which is at rest except for acoustic wave motion. If \( \phi \) is the potential velocity, so that \( -\nabla \phi \) is the particle velocity, then \( \phi \) satisfies the wave equation

\[
\phi_{tt} = c^2 \Delta \phi, \quad \text{in } \Omega, \tag{1.1}
\]

where \( c \) is the speed of sound in the medium. We assume that the boundary \( \partial \Omega = \Gamma \) is divided into two parts,

\[
\Gamma = \Gamma_0 \cup \Gamma_1,
\]

*Postdoctoral Researcher at the National Laboratory for Scientific Computation (LNCC/CNPq).
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such that
\[ \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset, \quad \Gamma_0 \neq \emptyset. \]
We suppose that \( \Gamma_0 \) is not rigid. Finally, we assume that each point reacts to excess pressure of the acoustic wave like a resistive harmonic oscillator, and that the different parts of the boundary \( \Gamma \) do not influence each other, that is, the surface is \textit{locally reacting}, but subject to small oscillations. Under the above conditions, the normal displacement \( \delta \) of \( \Gamma_0 \) into the domain satisfies an equation of the form
\[
m(x)\delta_{tt}(x,t) + d(x)\delta_t(x,t) + k(x)\delta(x,t) = -\rho\phi_t(x,t), \quad \text{on } \Gamma_0, \tag{1.2}
\]
where \( \rho \) is the density of the fluid, \( m, d, \) and \( k \) are mass per unit area, resistivity, and spring constant on \( \Gamma_0 \), respectively. If we also assume that \( \Gamma_0 \) is impenetrable, we obtain a third equation from the continuity of the velocity at the boundary \( \Gamma_0 \),
\[
\delta_t(x,t) = \frac{\partial \phi(x,t)}{\partial \nu}, \quad \text{on } \Gamma_0, \tag{1.3}
\]
where \( \frac{\partial \phi(x,t)}{\partial \nu} = \nabla \phi(x,t) \cdot \nu \) denotes the outward normal velocity at \( x \in \Gamma_0 \) and \( \nu = \nu(x) \) stands for the outward normal vector at \( x \in \Gamma \).

We assume that \( \Gamma_1 \) is rigid and on it, \( \phi \) satisfies the Dirichlet boundary condition, that is,
\[
\phi(x,t) = 0, \quad \text{on } \Gamma_1. \tag{1.4}
\]
For more details on the model, we refer to [1–4]. Moreover, we assume that there is a point \( x_0 \in \mathbb{R}^3 \) such that
\[
\Gamma_1 = \{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) \leq 0 \},
\]
\[
\Gamma_0 = \{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) \geq a > 0 \},
\]
for some constant \( a > 0 \).

As a typical example for the existence of the point \( x_0 \), we can see the following domain.

Additionally, we prescribe the initial conditions
\[
\phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x), \quad \forall x \in \Omega, \tag{1.5}
\]
\[
\delta(x,0) = \delta_0(x), \quad \delta_t(x,0) = \delta_1(x), \quad \forall x \in \Gamma_0. \tag{1.6}
\]
We assume that \( m(x), d(x), \) and \( k(x) \) are positive sufficiently smooth functions on \( \Gamma_0 \), and hence, they satisfy
\[
0 < m_0 \equiv \inf_{x \in \Gamma_0} m(x) \leq m(x) \leq m_1 \equiv \sup_{x \in \Gamma_0} m(x) < \infty, \tag{1.7}
\]
\[
0 < k_0 \equiv \inf_{x \in \Gamma_0} k(x) \leq k(x) \leq k_1 \equiv \inf_{x \in \Gamma_0} k(x) < \infty, \tag{1.8}
\]
\[
0 < d_0 \equiv \inf_{x \in \Gamma_0} d(x) \leq d(x) \leq d_1 \equiv \inf_{x \in \Gamma_0} d(x) < \infty. \tag{1.9}
\]
When boundary conditions (1.2), (1.3) are prescribed on the whole boundary \( \Gamma \), Beale [1–3] proved the global existence and regularity of solutions in a Hilbert space of data with finite energy by means of semigroup methods. The asymptotic behaviour was obtained in [2, Theorem 2.6], but no decay rate was given there. This model is used in [5, p. 263] for waves assumed to be at a definite frequency. To the best of the authors’ knowledge, there is no result on the decay rate of solutions to problem (1.1)–(1.6) (and to that in [1–3]). The aim of the present paper is to establish the polynomial decay of the energy of problem (1.1)–(1.6). The global existence and regularity of solutions can be shown in the same method as in [1–3] (see Theorems 1.1 and 1.2). We would like to refer to the works by Li and Chen [6–8], Qin [9], and those cited therein for the Cauchy problem of nonlinear wave equations. Moreover, we would also like to refer to the works by Rivera and Racke [10,11], Rivera and Oliveira [12] and Rivera and Andrade [4] in the fields of elasticity, thermoelasticity, and (thermo-)magneto-elasticity.

Let us define the following space:

\[
\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Gamma_0) \times L^2(\Gamma_0),
\]

with

\[
H^1_0(\Omega) = \{ u : u \in H^1(\Omega), u|_{\Gamma_1} = 0 \}.
\]

It is not difficult to show that \( \mathcal{H} \), together with the inner product

\[
\langle u, w \rangle = \int_\Omega (\rho \nabla u_1 \cdot \nabla w_1 + \rho c^{-2} u_2 w_2) \, dx + \int_{\Gamma_0} (ku_3 w_3 + mu_4 w_4) \, ds
\]

is a Hilbert space, where \( u = (u_1, u_2, u_3, u_4)^T \), \( w = (w_1, w_2, w_3, w_4)^T \in \mathcal{H} \). Thus, the induced norm on \( \mathcal{H} \) by the above inner product is

\[
|u|_\mathcal{H}^2 = \int_\Omega \left( \rho \|
abla u_1\|^2 + \rho c^{-2} |u_2|^2 \right) \, dx + \int_{\Gamma_0} \left( k |u_3|^2 + m |u_4|^2 \right) \, ds,
\]

for any \( u = (u_1, u_2, u_3, u_4)^T \in \mathcal{H} \).

Next we define an operator \( -A \) on \( \mathcal{H} \) so that for smooth \( u = (\phi, \phi_t, \delta, \delta_t)^T \), (1.1)–(1.4) are equivalent to \( u(t) \in D(A) \) and \( u_t = -Au \). We define

\[
Au = [u_2, c^2 \Delta u_1, u_4, -m^{-1} (\rho u_2 + ku_3 + du_4)]^T,
\]

for \( u = (u_1, u_2, u_3, u_4)^T \in D(A) \), where

\[
D(A) = \left\{ u : \Delta u_1 \in L^2(\Omega), u_2 \in H^1_0(\Omega), \frac{\partial u_1}{\partial n} = u_4 \right\}.
\]

Here \( u_2 \) in the last component of \( Au \) is understood as the trace in \( H^{1/2}(\Gamma_0) \) and \( \frac{\partial u_1}{\partial n} = u_4 \) is meant in the weak sense

\[
\int_\Omega ((\Delta u_1) \psi + \nabla u_1 \cdot \nabla \psi) \, dx = \int_{\Gamma_0} u_4 \psi \, ds, \quad \forall \psi \in H^1_0(\Omega).
\]

The relation \( u_1 \in H^2(\Omega) \) is equivalent to the condition that \( u_4 \) is the normal derivative of \( u_1 \) as a trace. Thus, similar to the proofs in [1–3], we readily obtain the following results on the global existence and regularity of solutions.

**Theorem 1.1.** \( A \) is closed, densely defined, and dissipative. It is the generator of a \( C_0 \)-semigroup. If \( d \equiv 0 \), \( A \) is skew-adjoint and generates a unitary group.
THEOREM 1.2. Assume that \( u_0 \in \mathcal{H} \) is \( C^\infty \) and vanishes near \( \partial \Omega \); let \( u(t) \) be the solution of \( u'(t) = Au(t), \ t \geq 0, \) with \( u(0) = u_0 \). Then \( u_1(t), u_2(t) \in C^\infty(\Omega) \) and \( u_3(t), u_4(t) \in C^\infty(\Gamma_0) \) for any \( t \geq 0 \).

REMARK 1.1. The statement "\( u \) vanishes near \( \partial \Omega \)" means \( u_3 = u_4 = 0 \) on \( \Gamma_0 \), and \( u_1 \) and \( u_2 \) vanishes near \( \partial \Omega \).

To continue our analysis, we define energy functions

\[
E_0(t; \phi, \delta) = \frac{1}{2} \int_\Omega (\rho |\nabla \phi|^2 + \rho c^{-2} \phi^2) \, dx + \frac{1}{2} \int_{\Gamma_0} (k(x) \delta^2 + m(x) \delta_t^2) \, ds,
\]

(1.10)

\[
E_j(t) = E_j(t; \phi, \delta) = E_0 \left( t; \partial_t^j \phi, \partial_t^j \delta \right), \quad j = 1, 2, \ldots.
\]

(1.11)

The novelty of this paper is the following results on the asymptotic behaviour of solutions.

THEOREM 1.3. Under the above assumptions and with smooth initial data \( (\phi_0, \phi_1, \delta_0, \delta_1) \) such that

\[
\sum_{j=0}^{k+1} E_j(0) < \infty,
\]

(1.12)

for \( k \geq 0 \). Then the summation of energies up to \( k \) order, \( \sum_{j=0}^{k} E_j(t) \), decays polynomially, more precisely, there is a positive constant \( \tilde{C} \) such that

\[
\sum_{j=0}^{k} E_j(t) \leq \frac{\tilde{C}^{k+1}}{t} \sum_{j=0} E_j(0), \quad \forall t > 0.
\]

(1.13)

REMARK 1.2. Clearly, (1.13) involves the higher order of initial energies than those estimated energies. Indeed, it is still open if there is a uniform exponential decay of the associated semigroup. The techniques presented here are not helpful for this topic.

The notation in this paper is standard and follows Lions and Magenes's book [8]. We put \( \| \cdot \| = \| \cdot \|_{L^2} \). We use \( C \) (sometimes \( C_1, C_2 \ldots \)) to stand for the universal positive constant independent of time \( t > 0 \).

The rest of this paper is organized as follows. In Section 2, we give some lemmas to establish energy estimates and finish the proof of Theorem 1.3.

## 2. ENERGY ESTIMATES

In this section, we use multiplicative techniques to establish some energy estimates. By (1.1)-(1.6) and Green's formula, it is not hard to verify

\[
\frac{d}{dt} E_0(t; \phi, \delta) = - \int_{\Gamma_0} d(x) \delta_t^2 \, ds.
\]

(2.1)

Similarly, noting equation (1.1) and boundary conditions (1.2),(1.3) are all linear, we have that for \( j = 0, 1, \ldots, k + 1 \),

\[
\frac{d}{dt} E_j(t; \phi, \delta) = - \int_{\Gamma_0} d(x) \left| \partial_t^{j+1} \delta \right|^2 \, ds.
\]

(2.2)

Define

\[
q(x) = x - x_0, \quad F_0(t; \phi, \delta) = \int_\Omega (\phi_t q \cdot \nabla \phi + \phi \phi_t) \, dx,
\]

(2.3)

\[
F_j(t) \equiv F_j(t; \phi, \delta) \equiv F_0 \left( t; \partial_t^j \phi, \partial_t^j \delta \right), \quad j = 1, 2, \ldots, k.
\]

(2.4)

Under the above notations, we have the following.
Lemma 2.1. For \( j = 0, 1, \ldots, k \), we obtain the following identity:

\[
\frac{d}{dt} F_j (t; \phi, \delta) = -\frac{1}{2} \int_\Omega \left( \left| \partial_t^{j+1} \phi \right|^2 + c^2 \left| \nabla \partial_t^j \phi \right|^2 \right) dx - \frac{c^2}{2} \int_{\Gamma_0} q \cdot \nu \left| \nabla \partial_t^j \phi \right|^2 ds \\
+ c^2 \int_{\Gamma_0} \partial_t^{j+1} \delta q \cdot \nabla \partial_t^j \phi ds + \frac{c^2}{2} \int_{\Gamma_1} q \cdot \nu \left| \nabla \partial_t^j \phi \right|^2 ds \\
+ \frac{1}{2} \int_{\Gamma_0} q \cdot \nu \left| \partial_t^{j+1} \phi \right|^2 ds + c^2 \int_{\Gamma_0} \partial_t^{j+1} \delta \partial_t^j \phi ds.
\]  

(2.5)

Proof. By (1.1)-(1.6), we easily derive

\[
\frac{d}{dt} \int_\Omega \partial_t^{j+1} \phi \cdot \nabla \partial_t^j \phi dx = \int_\Omega \left( c^2 \Delta \partial_t^j \phi q \cdot \nabla \partial_t^j \phi + \frac{1}{2} q \cdot \nabla \left| \partial_t^{j+1} \phi \right|^2 \right) dx
\]

\[= c^2 \int_{\Gamma_0} \partial_t^j \phi \cdot \nabla \partial_t^j \phi ds + c^2 \int_{\Gamma_1} \nabla \partial_t^j \phi \cdot \nu q \cdot \nabla \partial_t^j \phi ds
\]

\[- c^2 \int_\Omega \left| \nabla \partial_t^j \phi \right|^2 dx - \frac{c^2}{2} \int_{\Omega} q \cdot \nabla \left| \partial_t^{j+1} \phi \right|^2 dx
\]

\[+ \frac{1}{2} \int_{\Gamma_0} q \cdot \nu \left| \partial_t^{j+1} \phi \right|^2 ds.
\]  

(2.6)

By virtue of (1.4) (hence, \( \partial_t^j \phi \mid_{\Gamma_1} = 0 \)), we get on \( \Gamma_1 \),

\[\nabla \partial_t^j \phi \cdot \nu q \cdot \nabla \partial_t^j \phi = q \cdot \nu \left| \nabla \partial_t^j \phi \right|^2.
\]  

(2.7)

Thus, it follows from (1.5), (1.6) and (2.6), (2.7) that

\[
\frac{d}{dt} \int_\Omega \partial_t^{j+1} \phi \cdot \nabla \partial_t^j \phi dx = - \frac{3}{2} \int_\Omega \left| \partial_t^{j+1} \phi \right|^2 dx
\]

\[+ \frac{c^2}{2} \int_{\Gamma_0} \left| \nabla \partial_t^j \phi \right|^2 dx + c^2 \int_{\Gamma_0} \partial_t^{j+1} \delta q \cdot \nabla \partial_t^j \phi ds
\]

\[+ \frac{1}{2} \int_{\Gamma_0} q \cdot \nu \left| \partial_t^{j+1} \phi \right|^2 ds.
\]  

(2.8)

Similarly, by (1.1)-(1.6), we conclude

\[
\frac{d}{dt} \int_\Omega \partial_t^j \phi \partial_t^{j+1} \phi dx = \int_\Omega \left| \partial_t^{j+1} \phi \right|^2 dx + c^2 \int_{\Gamma_0} \nabla \partial_t^j \phi \cdot \nu \partial_t^{j+1} \phi ds - c^2 \int_{\Gamma_0} \left| \nabla \partial_t^j \phi \right|^2 dx
\]

\[= \int_\Omega \left| \partial_t^{j+1} \phi \right|^2 dx + c^2 \int_{\Gamma_0} \partial_t^{j+1} \delta \partial_t^j \phi ds - c^2 \int_{\Gamma_0} \left| \nabla \partial_t^j \phi \right|^2 dx.
\]  

(2.9)

Thus, adding (2.9) to (2.8) yields the desired estimate (2.5). The proof is complete.

If we now denote by

\[
G_0 (t; \phi, \delta) = \int_{\Gamma_0} \left( \frac{m(x) \delta \delta + d(x) \delta^2}{2} + \rho \phi \delta \right) ds,
\]

\[
G_j (t) = G_j (t; \phi, \delta) = G_0 \left( t; \partial_t^j \phi, \partial_t^j \delta \right), \quad j = 1, 2, \ldots, k,
\]

then it is easy to verify from (1.2) the following lemma.

Lemma 2.2. For \( j = 0, 1, 2, \ldots, k \), we obtain

\[
\frac{d}{dt} G_j (t; \phi, \delta) = - \int_{\Gamma_0} k(x) \left| \partial_t^j \delta \right|^2 ds + \int_{\Gamma_0} \left( m(x) \left| \partial_t^{j+1} \delta \right|^2 + \rho \partial_t^j \phi \partial_t^{j+1} \delta \right) ds.
\]

(2.12)
Now we define the following Liapunov functional:

$$L_k(t) = N^2 \sum_{j=0}^{k+1} E_j(t) + N^{1/2} \sum_{j=0}^{k} G_j(t) + \sum_{j=0}^{k} F_j(t),$$

(2.13)

where $N$ is a large positive number specified later on.

**Lemma 2.3.** For $N$ large enough, there are positive constants $C_0, C_1, C_2,$ and $C_3$ such that

$$0 \leq C_0 \sum_{j=0}^{k+1} E_j(t) \leq L_k(t) \leq C_1 \sum_{j=0}^{k+1} E_j(t), \quad \forall t \geq 0,$$

(2.14)

and

$$\frac{d}{dt} L_k(t) \leq -C_2 \sum_{j=0}^{k+1} \int_{\Gamma_o} \left( \partial_t^{j+1} \phi \right)^2 ds - \frac{1}{4} \sum_{j=0}^{k} \int_{\Omega} \left( \partial_t^{j+1} \phi \right)^2 + c^2 \left| \nabla \partial_t^{j+1} \phi \right|^2 \right) dx$$

$$- C_2 \sum_{j=0}^{k} \int_{\Gamma_o} \left| \nabla \partial_t^j \phi \right|^2 ds$$

$$\leq -C_3 \sum_{j=0}^{k} E_j(t), \quad \forall t > 0.$$

**Proof.** By (1.4) and the Poincaré inequality, we infer that for $j = 0, 1, \ldots, k,$

$$\int_{\Omega} \left| \partial_t^j \phi \right|^2 dx \leq C \int_{\Omega} \left| \nabla \partial_t^j \phi \right|^2 dx.$$

(2.16)

Thus, in view of (2.16) and the trace theorem, we get

$$\int_{\Gamma_o} \left| \partial_t^j \phi \right|^2 ds \leq C \left\| \partial_t^j \phi \right\|_{H^{1/2} (\Gamma)} \leq C \left\| \partial_t^j \phi \right\|_{H^1 (\Omega)} \leq C \int_{\Omega} \left| \nabla \partial_t^j \phi \right|^2 dx.

(2.17)

Using (1.7)–(1.9), (2.13)–(2.17), and Schwartz’s inequality, we infer that for $N$ large enough, there exists a constant $C_0$ such that

$$L_k(t) \geq C N^2 \sum_{j=0}^{k+1} \left[ \int_{\Omega} \left( \left| \nabla \partial_t^j \phi \right|^2 + \left| \partial_t^{j+1} \phi \right|^2 \right) dx + \int_{\Gamma_o} \left( \left| \partial_t^j \phi \right|^2 + \left| \partial_t^{j+1} \phi \right|^2 \right) ds \right]$$

$$- C N^{1/2} \sum_{j=0}^{k} \left[ \int_{\Gamma_o} \left( \left| \partial_t^j \phi \right|^2 + \left| \partial_t^{j+1} \phi \right|^2 \right) ds + \int_{\Omega} \left| \nabla \partial_t^j \phi \right|^2 dx \right]$$

$$- C \sum_{j=0}^{k} \int_{\Omega} \left( \left| \partial_t^j \phi \right|^2 + \left| \partial_t^{j+1} \phi \right|^2 \right) dx$$

$$\geq C_0 \sum_{j=0}^{k+1} E_j(t).$$

(2.18)

Similarly, for $N$ large enough, we have a constant $C_1 > 0$ such that

$$L_k(t) \leq C_1 \sum_{j=0}^{k+1} E_j(t),$$

which, combined with (2.18), gives (2.14).
On the other hand, by (1.2), (1.3), (1.7)–(1.9), and (2.17), we have that for \( j = 0, 1, \ldots, k \).

\[
N^{1/2} \int_{\Gamma_0} \left( m(x) \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 + \rho \frac{\partial^j \phi \partial^{j+1} \delta}{\partial t^{j+1}} \right) \ dx \leq C N \int_{\Gamma_0} \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 \ ds + \frac{c^2}{8} \int_{\Omega} \left| \nabla \frac{\partial^j \phi}{\partial t^j} \right|^2 \ dx, \tag{2.19}
\]

\[
c^2 \int_{\Gamma_0} \frac{\partial^j \phi \partial^{j+1} \delta}{\partial t^{j+1}} \ ds \leq C \int_{\Gamma_0} \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 \ ds + \frac{c^2}{8} \int_{\Omega} \left| \nabla \frac{\partial^j \phi}{\partial t^j} \right|^2 \ dx, \tag{2.20}
\]

\[
\int_{\Omega} q \cdot \nabla \frac{\partial^{j+1} \phi}{\partial t^{j+1}} \ ds \leq C \int_{\Gamma_0} \left( \left| \frac{\partial^{j+2} \delta}{\partial t^{j+2}} \right|^2 + \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 + \left| \frac{\partial^j \delta}{\partial t^j} \right|^2 \right) \ ds, \tag{2.21}
\]

\[
c^2 \int_{\Gamma_0} \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \ n \cdot \nabla \phi \ ds \leq C \int_{\Gamma_0} \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 \ ds + \frac{c^2}{4} \int_{\Gamma_0} q \cdot \nabla \frac{\partial^j \phi}{\partial t^j} \ ds. \tag{2.22}
\]

Thus, it follows from (1.10), (1.11), Lemmas 2.1 and 2.2, (2.17), and (2.19)–(2.22) that for \( N \) large enough, there are constants \( C_2, C_3 > 0 \) such that

\[
\frac{d}{dt} \mathcal{L}_k(t) \leq -C_2 \sum_{j=0}^{k+1} \int_{\Gamma_0} \left| \frac{\partial^{j+1} \delta}{\partial t^{j+1}} \right|^2 \ ds - \frac{1}{2} \sum_{j=0}^{k} \int_{\Omega} \left( \left| \frac{\partial^{j+1} \phi}{\partial t^{j+1}} \right|^2 + c^2 \left| \nabla \frac{\partial^{j+1} \phi}{\partial t^{j+1}} \right|^2 \right) \ dx
\]

\[
- C_2 \sum_{j=0}^{k} \int_{\Gamma_0} \left| \nabla \frac{\partial^j \phi}{\partial t^j} \right|^2 \ ds
\]

\[
\leq -C_3 \sum_{j=0}^{k} E_j(t).
\]

The proof is complete.

Based on the estimates obtained above, we are able to finish the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By (1.12) and Lemma 2.3, we get

\[
\int_{0}^{t} \sum_{j=0}^{k} E_j(\tau) \ d\tau \leq C_3^{-1}(\mathcal{L}_k(0) - \mathcal{L}_k(t)) \leq C_3^{-1} C_1 \sum_{j=0}^{k+1} E_j(0) < \infty. \tag{2.23}
\]

Clearly, by (2.2), we easily get that for any \( t > 0 \),

\[
\frac{d}{dt} \left[ t \sum_{j=0}^{k} E_j(t) \right] = \sum_{j=0}^{k} E_j(t) + t \sum_{j=0}^{k} \frac{d}{dt} E_j(t) \leq \sum_{j=0}^{k} E_j(t), \tag{2.24}
\]

when, together with (2.23),

\[
\sum_{j=0}^{k} E_j(t) \leq \hat{C} \sum_{j=0}^{k+1} E_j(0), \tag{2.25}
\]

with \( \hat{C} = C_3^{-1} C_1 \). The proof of Theorem 1.3 is now complete.

**References**