Polynomial stability in two-dimensional magneto-elasticity

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For a class of bounded reference configurations that are of partially rectangular type, the linear system of magneto-elasticity in two space dimensions is considered and a polynomial decay rate of the energy as time tends to infinity is proved.

Keywords: isotropic elasticity; Lyapunov functional; uniform stability.

1. Introduction

We consider the following linear initial boundary-value problem in magneto-elasticity which describes, for a homogenous isotropic medium with bounded reference configuration \( \Omega \subset \mathbb{R}^2 \), the interaction between elastic movements and a magnetic field. The governing differential equations for the displacement vector \( u = (u^1, u^2, 0)' = u(t, x) \) depending on the time variable \( t \geq 0 \) and on the space variable \( x \in \Omega \), and for the magnetic field \( h = (h^1, h^2, 0)' = h(t, x) \) are

\[
\begin{align*}
\ddot{u} - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u - \alpha [\nabla \times \vec{h}] \times \vec{H} &= 0, \\
\dot{h} - \Delta h - \beta \nabla \times [u \times \vec{H}] &= 0,
\end{align*}
\]

cp. Eringen & Maugin (1990). Here \( \lambda, \mu \) and \( \kappa \) are positive constants. The coupling constants \( \alpha, \beta \) satisfy \( \alpha \beta > 0 \); \( \vec{H} = (H, 0, 0)' \) is a constant vector with \( H \neq 0 \).

Additionally, one has initial conditions

\[
\begin{align*}
u(0, x) &= u_0(x), & u_t(0, x) &= u_1(x), & h(0, x) &= h_0(x)
\end{align*}
\]

and the following classical Dirichlet-type boundary conditions:

\[
\begin{align*}
u = 0, & \quad \nu \times (\nabla \times h) = 0, & \nu \cdot h &= 0 \quad \text{on } \Gamma := \partial \Omega.
\end{align*}
\]

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Moreover,
\[ \text{div } h = 0, \tag{1.5} \]
which follows from (1.2) if \( \text{div } h_0 = 0 \). The vector \( v = (v_1, v_2, 0)' = v(x) \) denotes the exterior normal vector in \( x \in \Gamma \), the boundary of \( \Omega \).

Note that the system above is very similar to the isotropic thermoelastic system. The main difference is that now the dissipation is given by the magnetic field instead of the temperature. In thermoelasticity it is well known by now that for materials configured in bounded domains of \( \mathbb{R}^n \), for \( n > 1 \), there is no decay in general. This ineffectiveness of the thermal dissipation is due essentially to the degree of freedom of the displacement vector field, which is greater than the degree of freedom of the temperature when the dimension is greater than \( n = 1 \). In magneto-elasticity the situation is different, because the dissipative mechanism and the displacement have the same degree of freedom. But now, this system introduces an additional difficulty given by the coupling term. Making a closer analysis of the system (see (2.3)–(2.6) of Section 2) we see that the dissipative term is coupled only to the gradient of the second component of the displacement, while the first component is coupled to the second one only by (2.3), (2.4) through its mixed derivatives, which is very subtle. The main question here is whether the dissipation given by the magnetic field is strong enough to produce a uniform rate of decay for the whole system. If so, what type of rate of decay can we expect? It seems to us that there is not yet any result about this topic. To fill part of this gap we study this topic here.

The main result of this paper is to show that the solution of the magneto-elastic system decays polynomially as time goes to infinity. To show this we assume that the boundary \( \Gamma \) is smooth with the exception of at most finitely many points. This is in accordance with the following assumption on the two-dimensional domain \( \Omega \). Without loss of generality we may assume that \( 0 \in \Omega \) and additionally that \( \Omega \) is connected and of partially rectangular type which means that it is homeomorphic to the unit ball and one of the following three conditions (i)–(iii) is satisfied.

(i) \( \Omega \) is the union of finitely many rectangles with axes parallels to the \( x_1 \)- and \( x_2 \)-axes, respectively; see Fig. 1a.

(ii) \( \Omega \) satisfies \( v_1 v_2 = 0 \) in the first quadrant (where \( x_1 \geq 0 \) and \( x_2 \geq 0 \)) and in the third quadrant (where \( x_1 \leq 0 \) and \( x_2 \leq 0 \)). In the second and fourth quadrants \( \Omega \) satisfies \( x v \geq \delta_0 > 0 \), for some \( \delta_0 \); see Fig. 1b.

(iii) \( \Omega \) satisfies \( v_1 v_2 = 0 \) in the second and fourth quadrants. In the first and third quadrants \( \Omega \) satisfies \( x v \geq \delta_0 > 0 \), for some \( \delta_0 \); see Fig. 1c.

REMARK By domains of partially rectangular type (i), all sufficiently smoothly bounded, connected domains can be exhausted; also all connected Jordan measurable sets.

The energy \( E = E(t) \) (of first order) associated to the equations (1.1), (1.2) is given by
\[ E(t) := E(t; u, h) := \frac{1}{2} \int_{\Omega} \left( |u_1|^2 + \mu |\nabla u|^2 + (\mu + \lambda) \text{div } u|^2 + \frac{\alpha}{\beta} |h|^2 \right)(t, x) \, dx. \tag{1.6} \]

We shall also use energy terms of higher order given for \( j \in \mathbb{N} \) by
\[ E_j(t) := E(t; \partial^j_1 u, \partial^j_1 h). \tag{1.7} \]
Then it will be proved that the energy $E_1(t)$ decays like $t^{-1}$. More precisely, the main theorem is the following.

**THEOREM 1.1** Let $(u, h)$ be the solution to the initial boundary-value problem (1.1)–(1.5). Then the energy $E_1$ defined in (1.6), (1.7) decays polynomially,

$$
\exists d > 0 \quad \forall t > 0 : \quad E_1(t) \leq \frac{d}{t} \sum_{j=0}^{q} E_j(0).
$$

This result presents a polynomial decay that is uniform with respect to initial data but involves derivatives at time $t = 0$ higher than those estimated for $t > 0$. Indeed, it is open whether there is a uniform exponential decay of the associated semigroup, and our calculations do not assist this possibility.

The method we use is an energy method, looking for appropriate multipliers and Lyapunov functionals.

**REMARK** The existence of solutions to the initial boundary-value problem is simply assumed. It is a standard procedure to obtain solutions in Sobolev spaces—for example,
via semigroup theory, cp. Perla Menzala & Zuazua (1998). The assumed smoothness of
the initial data is described by the finiteness of the right-hand side in the estimate in
Theorem 1.1.

The time-asymptotic behaviour for the initial boundary-value problem has been studied
by Perla Menzala & Zuazua (1998). They proved the decay of $E(t)$ to zero for fixed initial
data; no uniformity was given, but more general domains, also in three space dimensions,
were considered. For a damping boundary condition of memory type, replacing $u = 0$, the
authors (1999) proved the exponential stability. We also mention the results on polynomial
decay for the corresponding Cauchy problem contained in the papers by Andreou &
Dassios (1997) and by the authors (1999). For earlier papers on magneto-elasticity, for
example, for plane waves see the references in Perla Menzala & Zuazua (1998) and in the
paper by the authors (1999).

**REMARK.** We remark that an additional damping as in magneto-thermo-elasticity (see the
authors’ (1999) paper) is of course likely to lead to a similar result. The damping will
certainly not give worse decay, but we conjecture that it will also not improve the decay
essentially.

In connection with the question of optimal decay rates it should be mentioned that
for the related thermo-elastic system exponential decay can only be expected in special
situations like radial symmetry, for example, see Jiang et al. (1998), but not in general, see

In Section 2 we shall present appropriate multipliers and the essential estimates for the
components of the final Lyapunov functional. Theorem 1.1 is then proved in Section 3.

2. Multipliers and energy estimates

It is easy to verify that

$$
\frac{d}{dt} E(t; u, h) = -\frac{\alpha}{\beta} \int_{\Omega} |\nabla \times h|^2 \, dx
$$

and, in general,

$$
\frac{d}{dt} E_j(t) = -\frac{\alpha}{\beta} \int_{\Omega} |\nabla \times \partial^j h|^2 \, dx.
$$

Using the fact that $\Omega$ is simply connected, we conclude that

$$
\int_{\Omega} |h|^2 \, dx + \int_{\Omega} |\nabla h|^2 \, dx \leq c \int_{\Omega} |\nabla \times h|^2 \, dx
$$

(cf. Duvaut & Lions (1976, p. 356) or Leis (1986, p. 157)), hence

$$
\frac{d}{dt} E_j(t) \leq -c \int_{\Omega} |\nabla \partial^j h|^2 \, dx, \quad j = 0, \ldots, 7. \quad (2.1)
$$

Here and in the sequel $c$ will denote various positive constants not depending on $t$ or
on the initial data. Using the notation

$$
\partial_j := \frac{\partial}{\partial x_j}, \quad j = 1, 2,
$$
simple calculations give
\[ \nabla \times \mathbf{h} = (0, 0, \partial_1 h^2 - \partial_2 h^1)'. \]

\[ (\nabla \times \mathbf{h}) \times \mathbf{H} = H (0, \partial_1 h^2 - \partial_2 h^1, 0)', \]
\[ u_1 \times \mathbf{H} = H (0, 0, -u_1^2)', \]
\[ \nabla \times (u_1 \times \mathbf{H}) = H (-\partial_2 u_1^2, \partial_1 u_1^2, 0)', \]
\[ \nu \times (\nabla \times \mathbf{h}) = (\nu_2 (\partial_1 h^2 - \partial_2 h^1), -\nu_1 (\partial_1 h^2 - \partial_2 h^1), 0)', \]

In particular, the boundary condition \( \nabla \times (\nabla \times \mathbf{h}) |_{\Gamma} = 0 \) turns into
\[ (\partial_1 h^2 - \partial_2 h^1) |_{\Gamma} = 0. \quad (2.2) \]

The function \( \omega := \partial_1 h^2 - \partial_2 h^1 \) denotes the two-dimensional rotation of \( (h^1, h^2)' \), and the differential equations (1.1), (1.2) can be written as
\[ u_1''' - \mu \Delta u_1' - (\mu + \lambda) \partial_1 \partial_2 u_1^2 = 0, \quad (2.3) \]
\[ u_2''' - \mu \Delta u_2' - (\mu + \lambda) \partial_1 \partial_2 u_2^2 = 0, \quad (2.4) \]
\[ h_1' - \Delta h^1 = -\beta H \partial_2 u_1^2, \quad (2.5) \]
\[ h_2' - \Delta h^2 = \beta H \partial_1 u_2^2. \quad (2.6) \]

We introduce the following functionals as abbreviations:
\[ A_1 u_1 := -\mu \Delta u_1' - (\mu + \lambda) \partial_1^2 u_1^4, \]
\[ A_2 u_2 := -\mu \Delta u_2' - (\mu + \lambda) \partial_1^2 u_2^4, \]
\[ E^1(t) := \frac{1}{2} \int_{\Omega} |u_1|^2 + \mu |\nabla u_1|^2 + (\mu + \lambda)|\partial_1 u_1|^2 \, dx. \]

In the next lemma we define the functional which shows the dissipative properties of the second component of the displacement.

**Lemma 2.1** Let
\[ \Phi_1 := \Phi_1(t) := \int_{\Omega} \omega u_2^2 + \omega \Delta u_2' - \beta H (\mu + \lambda) \partial_2 u_2^2 \partial_1 u_1 \, dx - \beta H E^1(t). \]

Then
\[ \frac{d}{dt} \Phi_1 = -\beta H \int_{\Omega} |\nabla u_2|^2 \, dx + \int_{\Omega} \omega u_2 \Delta u_2' \, dx + \alpha H \int_{\Omega} |\omega|^2 \, dx + \int_{\Omega} \omega A_2 u_2^2 \, dx + (\mu + \lambda) \int_{\Omega} h_1' \partial_1 u_1 \, dx. \]
Proof. From the equations (2.5) and (2.6) we obtain
\[
\frac{d}{dt} \int_{\Omega} h^1 \partial_2 u^2_t \, dx = \int_{\Omega} h^1 \partial_2 u^2_t + h^1 \partial_2 u^2_t \, dx
\]

\[
= \int_{\Omega} \Delta h^1 \partial_2 u^2_t \, dx - \beta H \int_{\Omega} |\partial_2 u^2_t|^2 \, dx - \int_{\Omega} \partial_2 h^1 u^2_t \, dx
\]

\[
= \int_{\Omega} \Delta h^1 \partial_2 u^2_t \, dx - \beta H \int_{\Omega} |\partial_2 u^2_t|^2 \, dx - \int_{\Omega} \partial_2 h^1 A_2 u^2 \, dx
\]

\[
- (\mu + \lambda) \int_{\Omega} \partial_1 \partial_2 u^1 \partial_2 h^1 \, dx - \alpha H \int_{\Omega} \partial_2 h^1 \omega \, dx.
\]

Similarly, using (2.3), (2.6), we get
\[
- \frac{d}{dt} \int_{\Omega} h^2 \partial_1 u^2_t \, dx = - \int_{\Omega} \Delta h^2 \partial_1 u^2_t \, dx - \beta H \int_{\Omega} |\partial_1 u^2_t|^2 \, dx
\]

\[
+ \int_{\Omega} \partial_1 h^2 A_2 u^2 \, dx + (\mu + \lambda) \int_{\Omega} \partial_1 \partial_2 u^1 \partial_1 h^2 \, dx + \alpha H \int_{\Omega} \partial_1 h^2 \omega \, dx.
\]

Summing up the last two identities we obtain
\[
\frac{d}{dt} \int_{\Omega} \omega u^2_t \, dx = \frac{d}{dt} \int_{\Omega} h^1 \partial_2 u^2_t - h^2 \partial_1 u^2_t \, dx
\]

\[
= \int_{\Omega} \nabla \omega \nabla u^2_t \, dx - \beta H \int_{\Omega} |\nabla u^2_t|^2 \, dx
\]

\[
= I_1
\]

\[
+ \int_{\Omega} \omega A_2 u^2 \, dx + (\mu + \lambda) \int_{\Omega} \omega \partial_1 \partial_2 u^1 \, dx + \alpha H \int_{\Omega} \omega^2 \, dx. \quad (2.7)
\]

Note that, since \(\omega_{\Gamma} = 0\),

\[
I_1 = - \int_{\Omega} \omega \Delta u^2_t \, dx = - \frac{d}{dt} \int_{\Omega} \omega \Delta u^2_t \, dx + \int_{\Omega} \omega \Delta u^2 \, dx \quad (2.8)
\]

and

\[
I_2 = - (\mu + \lambda) \int_{\Omega} \partial_2 \omega \partial_1 u^1 \, dx.
\]

Since

\[
\text{div } h = 0
\]

we have

\[
\partial_2 \omega = - \Delta h^1,
\]
therefore we get

\[
I_2 = (\mu + \lambda) \int_{\Omega} \Delta h^1 \partial_1 u^1 \, dx
\]

\[
= (\mu + \lambda) \int_{\Omega} \left( h^1 \partial_1 u^1 + \beta H \partial_2 u^2 \right) \partial_1 u^1 \, dx
\]

\[
= (\mu + \lambda) \int_{\Omega} h^1 \partial_1 u^1 \, dx + \beta H (\mu + \lambda) \int_{\Omega} \partial_2 u^2 \partial_1 u^1 \, dx
\]

\[
= (\mu + \lambda) \int_{\Omega} \partial_1 u^1 \partial_2 u^2 \, dx.
\]

Multiplying equation (2.3) by \( u^1_t \) we get

\[
\frac{d}{dt} E^1(t) = -(\mu + \lambda) \int_{\Omega} \partial_1 u^1 \partial_2 u^2 \, dx.
\]

Substituting this identity into (2.9) we obtain

\[
I_2 = (\mu + \lambda) \int_{\Omega} h^1 \partial_1 u^1 \, dx + \frac{d}{dt} \left( \beta H (\mu + \lambda) \int_{\Omega} \partial_2 u^2 \partial_1 u^1 \, dx + \beta H (\mu + \lambda) E^1(t) \right).
\]

Recalling the definition of \( \Phi_1 \) the conclusion follows from (2.7), (2.8), (2.10). \( \square \)

Following Lemma 2.1 and since \((u_t, h_t)\) also satisfies (1.1), (1.2) and (1.4), (1.5) we can introduce functionals \( \Phi_2 \) and \( \Phi_3 \) in the following way. Writing

\[
\Phi_1(t) = \Phi_1(t; u, h)
\]

we have (see Lemma 2.1)

\[
\frac{d}{dt} \Phi_1(t; u_t, h_t) = -\beta H \int_{\Omega} |\nabla u^1_t|^2 \, dx + \int_{\Omega} \omega_{1t} \Delta u^2_t \, dx
\]

\[
+ \alpha H \int_{\Omega} |\omega_t|^2 \, dx + \int_{\Omega} \omega_t A_2 u^2_t \, dx + (\mu + \lambda) \int_{\Omega} h^1 \partial_1 u^1_t \, dx
\]

\[
= -\beta H \int_{\Omega} |\nabla u^1_t|^2 \, dx + \frac{d}{dt} \int_{\Omega} \omega_{1t} \Delta u^2_t \, dx
\]

\[
- \int_{\Omega} \omega_{1tt} \Delta u^2_t \, dx + \alpha H \int_{\Omega} |\omega_t|^2 \, dx + \frac{d}{dt} \int_{\Omega} \omega_t A_2 u^2 \, dx
\]

\[
- \int_{\Omega} \omega_{1tt} A_2 u^2 \, dx + (\mu + \lambda) \frac{d}{dt} \int_{\Omega} h^1 \partial_1 u^1 \, dx - (\mu + \lambda) \int_{\Omega} \partial_1 u^1 \, dx
\]

which implies for

\[
\Phi_2(t) := \Phi_2(t; u, h) := \Phi_1(t; u_t, h_t) - \int_{\Omega} \omega_{1tt} \Delta u^2 + \omega_t A_2 u^2 - (\mu + \lambda) h^1 \partial_1 u^1 \, dx
\]
the identity
\[
\frac{d}{dt} \Phi_d(t) = -\beta h \int_{\Omega} |\nabla u_t|^2 \, dx - \int_{\Omega} \omega u_t \Delta u^2 \, dx + \alpha H \int_{\Omega} |\omega_t|^2 \, dx 
- \int_{\Omega} \omega_t A_2 u^2 \, dx - (\mu + \lambda) \int_{\Omega} h^1_{tt} \partial_1 u^1 \, dx.
\] (2.11)

Similarly we can define
\[
\Phi_3(t) := \Phi_d(t; u_t, h_t) - \int_{\Omega} \omega_{ttt} \Delta u^2 + \omega_{ttt} A_2 u^2 + (\mu + \lambda) h^1_{tttt} \partial_1 u^1 \, dx.
\]

Then we have
\[
\frac{d}{dt} \Phi_3(t) = -\beta H \int_{\Omega} |\nabla u_t|^2 \, dx + \int_{\Omega} \omega_{ttt} \Delta u^2 \, dx 
+ \alpha H \int_{\Omega} |\omega_t|^2 \, dx + \int_{\Omega} \omega_{ttt} A_2 u^2 \, dx + (\mu + \lambda) \int_{\Omega} h^1_{tttt} \partial_1 u^1 \, dx.
\] (2.12)

The following lemma plays an important role in the sequel because it will connect the dissipative properties of the magnetic field with the first component of the displacement vector field.

**Lemma 2.2** Let
\[
q(x) := (x_2 - \delta x_1, x_1 - \delta x_2)'
\]
for some \(\delta \in \mathbb{R}\), and let
\[
F(t) := \int_{\Omega} u_t^2_q \nabla u_t + A_2 u_t^2_q \nabla u^1 \, dx.
\]

Then for \(|\delta|\) sufficiently large—only depending on the domain \(\Omega\)—we have
\[
\frac{d}{dt} F(t) \leq -(\mu + \lambda) \int_{\Omega} |\nabla u_t^1|^2 \, dx + \int_{\Omega} |\partial u_t^1| \left( \frac{\partial u^1}{\partial v} \right) \, ds 
+ c \int_{\Omega} |\nabla u_t^2| |\Delta u^1| \, dx + c \int_{\Omega} |\omega_t| |\nabla u_t^1| \, dx.
\]

**Proof.** The main idea to show the above inequality is to use (2.3)–(2.6) to connect the dissipation given by \(h\) to \(u^1\). Then we will use the geometrical properties of the domain to
eliminate unpleasant boundary terms in $u^1$. In fact let us consider

$$
\frac{d}{dt} \int_\Omega u_{t_1}^2 q_1 \partial_1 u_t^1 \, dx = \int_\Omega u_{t_1}^2 q_1 \partial_1 u_t^1 \, dx + \int_\Omega u_{t_1}^2 q_1 \partial_1 u_t^1 \, dx \\
= - \int_\Omega A_2 u_t^2 q_1 \partial_1 u_t^1 \, dx + \mu + \lambda \int_\Omega q_1 \partial_1 u_t^1 \, dx \\
+ \alpha H \int_\Omega \omega q_1 \partial_1 u_t^1 \, dx \\
= - \int_\Omega A_2 u_t^2 q_1 \partial_1 u_t^1 \, dx + \int_\Omega A_2 u_t^2 q_1 \partial_1 u_t^1 \, dx \\
+ \mu + \lambda \int_\Gamma q_1 \nu \frac{\partial}{\partial n} q_1 \partial_1 u_t^1 \, ds - \mu + \lambda \int_\Omega \nu q_1 \partial_1 u_t^1 \, dx \\
+ \alpha H \int_\Omega \omega q_1 \partial_1 u_t^1 \, dx.
$$

(2.13)

Note that

$$
\int_\Omega A_2 u_t^2 q_1 \partial_1 u_t^1 \, dx = -\mu \int_\Gamma \frac{\partial u_t^2}{\partial n} q_1 \partial_1 u_t^1 \, ds - (\mu + \lambda) \int_\Gamma v_2 \partial_2 u_t^2 q_1 \partial_1 u_t^1 \, ds \\
+ (\mu + \lambda) \int_\Omega v_2 \partial_2 u_t^2 [\partial_2 q_1 \partial_1 u_t^1 + \partial_1 \partial_2 u_t] \, dx \\
+ \mu \int_\Omega \nabla u_t^2 [\nabla q_1 \partial_1 u_t^1 + q_1 \partial_1 \nabla u_t] \, dx
$$

from which we conclude that

$$
\frac{d}{dt} \left\{ \int_\Omega u_{t_1}^2 q_1 \partial_1 u_t^1 \, dx + \int_\Omega A_2 u_t^2 q_1 \partial_1 u_t^1 \, dx \right\} \\
= \mu + \lambda \int_\Gamma q_1 v_2 [\partial_1 u_t^1 \, dx - \mu \int_\Gamma \frac{\partial u_t^2}{\partial n} q_1 \partial_1 u_t^1 \, ds \\
- (\mu + \lambda) \int_\Gamma v_2 \partial_2 q_1 \partial_1 u_t^1 \, ds + (\mu + \lambda) \int_\Omega \partial_2 u_t^2 [\partial_2 q_1 \partial_1 u_t^1 + \partial_1 \partial_2 u_t] \, dx \\
+ \mu \int_\Omega \nabla u_t^2 [\nabla q_1 \partial_1 u_t^1 + q_1 \partial_1 \nabla u_t] \, dx - (\mu + \lambda) \int_\Omega \partial_2 q_1 [\partial_1 u_t^1 \, dx \\
+ \alpha H \int_\Omega \omega q_1 \partial_1 u_t^1 \, dx
$$
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Summing up (2.14) and (2.15) we get

Similarly we obtain

\[\frac{d}{dt} \left\{ \int_{\Omega} u^2_1 q_2 \partial_2 u^1 \, dx + \int_{\Omega} A_2 u^2_q q_2 \partial_2 u^1 \, dx \right\} \]

\[= \frac{\mu + \lambda}{2} \int_{\Gamma} q_1 v_1 v_2 \left| \frac{\partial u^1_1}{\partial v} \right| ds - \mu \int_{\Gamma} \frac{\partial u^2_1}{\partial v} q_1 v_1 \frac{\partial u^1_1}{\partial v} \, ds \]

\[- (\mu + \lambda) \int_{\Omega} q_2 v_1 v_2 \frac{\partial u^2_1}{\partial v} \, ds + (\mu + \lambda) \int_{\Omega} \partial_2 u^2_q [\partial_2 q_1 \partial_1 u^1 + q_1 \partial_1 \partial_2 u^1] \, dx \]

\[= \mu \int_{\Omega} \nabla u^2_q [\nabla q_1 \partial_1 u^1 + q_2 \partial_2 \nabla u^1] \, dx - (\mu + \lambda) \int_{\Omega} \partial_1 q_2 [\partial_2 u^1] \, dx \]

\[+ \alpha H \int_{\Omega} \omega q_1 \partial_1 u^1_1 \, dx. \] (2.14)

Summing up (2.14) and (2.15) we get

\[\frac{d}{dt} F(t) = \frac{d}{dt} \left\{ \int_{\Omega} u^2_q q \nabla u_1 \, dx + \int_{\Omega} A_2 u^2_q q \nabla u^1 \, dx \right\} \]

\[= \frac{\mu + \lambda}{2} \int_{\Gamma} v_1 v_2 [v_1 q_1 + v_2 q_2] \left| \frac{\partial u^1_1}{\partial v} \right| ^2 ds \]

\[- \mu \int_{\Gamma} \frac{\partial u^2_q}{\partial v} q v \frac{\partial u^1_1}{\partial v} \, ds - (\mu + \lambda) \int_{\Gamma} v_1 v_2 [q_1 v_2 + q_2 v_1] \frac{\partial u^2_q}{\partial v} \frac{\partial u^1_1}{\partial v} \, ds \]

\[+ (\mu + \lambda) \int_{\Omega} \partial_2 u^2_q [\partial_2 q_1 \partial_1 u^1 + \partial_2 q_2 \partial_2 u^1 + q_1 \partial_1 \partial_2 u^1 + q_2 \partial_2^2 u^1] \, dx \]

\[+ \mu \int_{\Omega} \nabla u^2_q [\nabla q_1 \partial_1 u^1 + \nabla q_2 \partial_2 u^1 + q_1 \partial_1 \nabla u^1 + q_2 \partial_2 \nabla u^1] \, dx \]

\[- (\mu + \lambda) \int_{\Omega} \partial_1 q_2 [\partial_2 u^1] ^2 + \partial_1 q_2 [\partial_2 u^1] \, dx \]

\[+ \alpha H \int_{\Omega} \omega q_1 \partial_1 u^1_1 + q_2 \partial_2 u^1_1 \, dx. \]
which implies that

\[
\frac{d}{dt} F(t) \leq \frac{\mu + \lambda}{2} \int_{\Gamma} v_1 v_2 v \cdot q \left| \frac{\partial u_1^1}{\partial v} \right|^2 \, ds \\
= I_3 \\
- (\mu + \lambda) \int_{\Omega} |\nabla u_1^1|^2 \, dx + c \int_{\Gamma} \left| \frac{\partial u_1^2}{\partial v} \right|^2 \left| \frac{\partial u_1^1}{\partial v} \right| \, ds \\
+ c \int_{\Omega} |\nabla u_1^2|^2 |\Delta u_1^1| \, dx + c \int_{\Omega} |\omega| |\nabla u_1^1| \, dx.
\]  
(2.16)

Now we use that \( \Omega \) is of partial rectangular type (i), (ii) or (iii).
For type (i) we have \( v_1 v_2 = 0 \), hence

\[ I_3 = 0. \]

For type (ii) we have for \( \delta \) positive and sufficiently large (only depending on \( \Omega \))

\[ I_3 \leq 0. \]

For type (iii) we have for \( \delta \) negative and \( |\delta| \) sufficiently large (only depending on \( \Omega \))

\[ I_3 \leq 0. \]

That is, for all types (i), (ii), (iii) we have

\[ I_3 \leq 0, \]
(2.17)

provided \( \delta \) is chosen appropriately. The assertion of the lemma now follows from (2.16) and (2.17).

\[ \square \]

Lemma 2.2 gives a good estimate for \( u_1^1 \), except for the boundary term which contains the normal derivative on \( u_2^1 \). To estimate this surface integral we will use the following lemma applied to \( u_1^1 \) and \( u_2^2 \).

**Lemma 2.3** Let \( g \in C^1(\mathbb{R}^2) \), \( g = (g_1, g_2)' \) such that \( g v \geq \gamma > 0 \) for some \( \gamma \). Then

\[
- \frac{d}{dt} \int_{\Omega} u_1^2 g_k \partial_k u_2^2 \, dx \leq - \frac{\mu}{2} \int_{\Gamma} g v k \left| \frac{\partial u_2^2}{\partial v} \right| \, ds \\
+ c \int_{\Omega} |u_1^2|^2 + |\nabla u_2^2|^2 \, dx - (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_2^1 \partial_k u_2^2 \, dx \\
+ \alpha H \int_{\Omega} \omega g_k \partial_k u_2^1 \, dx.
\]
Proof.

\[ \frac{d}{dt} \int_{\Omega} u_t^2 g_k \partial_k u^2 \, dx = \int_{\Omega} u_t^2 g_k \partial_k u^2 \, dx + \int_{\Omega} u_t^2 g_k \partial_k u_t^2 \, dx \]
\[ = \mu \int_{\Omega} \Delta u^2 g_k \partial_k u^2 \, dx + (\mu + \lambda) \int_{\Omega} \partial_x^2 u^2 g_k \partial_k u^2 \, dx \]
\[ + \frac{1}{2} \int_{\Omega} g_k \partial_k |u_t^2|^2 \, dx + (\mu + \lambda) \int_{\Omega} \partial_1 \partial^t u^1 g_k \partial_k u^2 \, dx \]
\[ + \alpha H \int_{\Omega} \omega g_k \partial_k u^2 \, dx \]
\[ = \mu \int_{\Gamma} \frac{\partial u^2}{\partial v} g_k \partial_k u^2 \, ds + (\mu + \lambda) \int_{\Gamma} v_2 \partial_x^2 g_k \partial_k u^2 \, ds \]
\[ - \mu \int_{\Omega} \nabla u^2 \nabla g_k \partial_k u^2 \, dx - (\mu + \lambda) \int_{\Omega} \partial_x^2 u^2 g_k \partial_k u^2 \, dx \]
\[ - (\mu + \lambda) \int_{\Omega} \partial_x^2 u^2 \partial_2 g_k \partial_k u^2 \, dx + (\mu + \lambda) \int_{\Omega} \partial_2 u^2 g_k \partial_2 g_k \partial_k u^2 \, dx \]
\[ - \frac{1}{2} \int_{\Omega} g_k \partial_k |u_t^2|^2 \, dx + \frac{1}{2} \int_{\Gamma} g_k v_k |u_t^2|^2 \, ds \]
\[ + (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial^t u^1 \partial_k u^2 \, dx + \alpha H \int_{\Omega} \omega g_k \partial_k u^2 \, dx. \]

This implies that

\[ \frac{d}{dt} \int_{\Omega} u_t^2 g_k \partial_k u^2 \, dx = \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u^2}{\partial v} \right|^2 \, ds + \frac{\mu + \lambda}{2} \int_{\Gamma} g_k v_k |\partial_2 u^2|^2 \, ds \]
\[ - \frac{1}{2} \int_{\Omega} g_k \partial_k |u_t^2|^2 - \mu |\nabla u^2|^2 - (\mu + \lambda) |\partial_2 u^2|^2 \, dx \]
\[ - \mu \int_{\Omega} \nabla u^2 \nabla g_k \partial_k u^2 \, dx - (\mu + \lambda) \int_{\Omega} \partial_x^2 u^2 g_k \partial_k u^2 \, dx \]
\[ + (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial^t u^1 \partial_k u^2 \, dx + \alpha H \int_{\Omega} \omega g_k \partial_k u^2 \, dx \]
\[ + \frac{1}{2} \int_{\Gamma} g_k v_k |u_t^2|^2 \, ds. \]

This implies that

\[ - \frac{d}{dt} \int_{\Omega} u_t^2 g_k \partial_k u^2 \, dx \leq - \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u^2}{\partial v} \right|^2 \, ds + c \int_{\Omega} |u_t^2|^2 + |\nabla u^2|^2 \, dx \]
\[ - (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial^t u^1 \partial_k u^2 \, dx - \alpha H \int_{\Omega} \omega g_k \partial_k u^2 \, dx \]

which completes the proof of Lemma 2.3.

□

Let us extend Lemma 2.3 to time derivatives of \( u \) and \( h \). Since \((u_t, h_t)\) satisfy
essentially the same equations as \((u, h)\), we have
\[
- \frac{d}{dt} \int_{\Omega} u_t^2 g_k \partial_k u_t^2 \, dx \leq - \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u_t}{\partial v} \right|^2 \, ds + c \int_{\Omega} |u_t^2|^2 + |\nabla u_t|^2 \, dx \\
- (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_t^2 \, dx - \alpha H \int_{\Omega} \omega_t g_k \partial_k u_t^2 \, dx. \quad (2.18)
\]

Defining
\[
J(t; u, h) := - \int_{\Omega} u_t^2 g_k \partial_k u_t^2 \, dx
\]
we conclude from (2.18) that
\[
\frac{d}{dt} J(t; u_t, h_t) \leq - \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u_t}{\partial v} \right|^2 \, ds \\
+ c \int_{\Omega} |u_t^2|^2 + |\nabla u_t|^2 \, dx - (\mu + \lambda) \frac{d}{dt} \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_t^2 \, dx \\
+ (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_t^2 \, dx - \alpha H \int_{\Omega} \omega_t g_k \partial_k u_t^2 \, dx.
\]

which implies for the functional
\[
J_2(t) := J_2(t; u, h) := J(t; u_t, h_t) + (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_t^2 \, dx
\]
that
\[
\frac{d}{dt} J_2(t) \leq - \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u_t}{\partial v} \right|^2 \, ds + c \int_{\Omega} |u_t^2|^2 + |\nabla u_t|^2 \, dx \\
+ (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_t^2 \, dx - \alpha H \int_{\Omega} \omega_t g_k \partial_k u_t^2 \, dx. \quad (2.19)
\]

Similarly, defining
\[
J_3(t) := J_2(t; u_t, h_t) + (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_{t\tau} \, dx,
\]
we conclude that
\[
\frac{d}{dt} J_3(t) \leq - \frac{\mu}{2} \int_{\Gamma} g_k v_k \left| \frac{\partial u_t}{\partial v} \right|^2 \, ds + c \int_{\Omega} |u_t^2|^2 + |\nabla u_t|^2 \, dx \\
- (\mu + \lambda) \int_{\Omega} g_k \partial_1 \partial_2 u_t^1 \partial_k u_{t\tau} \, dx - \alpha H \int_{\Omega} \omega_t g_k \partial_k u_t^2 \, dx. \quad (2.20)
\]

If we define the functional (cp. (2.12))
\[
\Phi_4(t) := \Phi_3(t; u_t, h_t) - \int_{\Omega} \omega_{t\tau} \Delta u^2 \, dx - \int_{\Omega} \omega_{t\tau} A_2 u^2 \, dx - (\mu + \lambda) \int_{\Omega} \partial_1 u_t^1 \, dx,
\]
we get for $N > N_1 > 0$ sufficiently large (observe that $g_k v_k \geq \gamma > 0$):

\[
\frac{d}{dt} \left[ N_1 J_3(t) + N \Phi_3(t) + N \Phi_4(t) \right] \\
\leq -\frac{N_1 \mu}{4} \int_{\Omega} \left| \frac{\partial u_{i_1}}{\partial v} \right|^2 ds - \frac{N}{2} \int_{\Omega} |u_{i_1}|^2 + |\nabla u_{i_1}|^2 + |\nabla u_{i_1}|^2 dx \\
+ c_\varepsilon N \left( \int_{\Omega} |\Delta u|^2 + |\Delta u|^2 dx \right) + C_\varepsilon N \int_{\Omega} \left( \sum_{j=2}^{7} |x^j w|^2 + \sum_{j=3}^{5} |x^j h|^2 \right) dx,
\]

(2.21)

where $\varepsilon > 0$ is small and $C_\varepsilon = C(\varepsilon)$ is a positive constant. Using Lemma 2.1 and Lemma 2.2 we conclude that

\[
\frac{d}{dt} \left[ F(t) + N_1 J_3(t) + N \Phi_3(t) + N \Phi_4(t) \right] \\
\leq -\frac{N_1 \mu \gamma}{4} \int_{\Omega} \left| \frac{\partial u_{i_1}}{\partial v} \right|^2 ds - \frac{N}{4} \int_{\Omega} |u_{i_1}|^2 + |\nabla u_{i_1}|^2 + |\nabla u_{i_1}|^2 dx \\
- \mu + \lambda \int_{\Omega} |\nabla u|^2 dx + 2 \varepsilon \int_{\Omega} |\Delta u|^2 dx - \frac{c_\varepsilon N}{2} \int_{\Omega} |\nabla u|^2 dx \\
+ C_\varepsilon \int_{\Omega} \left( \sum_{j=0}^{7} |x^j w|^2 + \sum_{j=3}^{5} |x^j h|^2 \right) dx.
\]

(2.22)

Finally we obtain, using the differential equations once more, and writing $Au := (A_1 u^1, A_2 u^2)^T$,

\[
\frac{d}{dt} \int_{\Omega} u_i Au \ dx = \int_{\Omega} \mu |\nabla u_i|^2 + (\mu + \lambda) \text{div} u_i \ dx - \int_{\Omega} |Au|^2 \ dx + \alpha H \int_{\Omega} \omega Au \ dx
\]

from which we get

\[
\frac{d}{dt} \int_{\Omega} u_i Au \ dx \leq c \int_{\Omega} |\nabla u_i|^2 \ dx - \frac{1}{2} \int_{\Omega} |Au|^2 \ dx + c \int_{\Omega} |\omega|^2 \ dx.
\]

(2.23)

3. Proof of Theorem 1.1

Let $L$ denote the following Lyapunov functional:

\[
L(t) := \sum_{k=0}^{7} M_k E_k(t) + N(\Phi_1(t) + \Phi_3(t) + \Phi_4(t)) + F(t) + N_1 J_3(t) + \int_{\Omega} u_i Au \ dx,
\]

with $M_k > 0$. Then we obtain ($\varepsilon > 0$ sufficiently small in (2.22)) for sufficiently large $M_k$, combining (2.1), Lemma 2.1, (2.22) and (2.23), that there exists $k_0 > 0$ such that for $t \geq 0$:

\[
\frac{d}{dt} L(t) \leq -k_0 E_1(t) - k_0 \int_{\Gamma} \left| \frac{\partial u_{i_1}}{\partial v} \right|^2 ds - k_0 \sum_{j=0}^{7} \int_{\Omega} |x^j \omega|^2 \ dx \\
\leq -k_0 E_1(t).
\]
which implies, since $L(t) \geq 0$ for large $M_k$, that

$$\int_0^t E_1(s) \, ds \leq \frac{1}{k_0} L(0) \leq d \sum_{j=0}^7 E_j(0)$$

(3.1)

for some $d > 0$. Observing (2.1) we have

$$\frac{d}{dt} (t E_1(t)) = E_1(t) + t \frac{d}{dt} E_1(t) \leq E_1(t)$$

which implies, using (3.1), that

$$E_1(t) \leq \frac{d}{t} \sum_{j=0}^7 E_j(0).$$

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REFERENCES


