Asymptotic behaviour in inhomogeneous linear thermoelasticity*

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1 Introduction

In this paper we will study the asymptotic behaviour of solution of classic thermoelastic equations for inhomogeneous and unidimensional material supporting different types of boundary conditions.

Constantine Dafermos studied this pattern of problems and showed in [3] that the solution of a plenty class of thermoelasticity equations are asymptotically stable with respect to the strong topology of the space C^k . In particular, for the one dimensional case, this result implies that the displacement vector field decays to zero as time goes to infinity. In other dimensions the convergence also holds but not necessarily to zero. This fact can be seen also for materials which occupy the whole space R^3 . In this direction the work of Dassiou and Grillakis [4] shows that the displacement vector field can be decomposed into two orthogonal parts. One of them called selenoidal, which conserves its energy, and the other part called irrotational which decays algebraically as t^{-m} as time goes to infinity, with rate m depending on the symmetry of the initial data. Similar result was obtained in [8] for n -dimensional materials which occupy the whole space R^n . The fact that one of the components conserves its energy implies, in general, that the total thermoelastic energy does not converge to zero. For that reason the decay of the total thermoelastic energy is not expected in R^n ($n > 1$).

For bounded domains the situation is more complicated than for the R^n case, because of the boundary conditions. In this context Chirita's result in [2] is very significant and confirms some properties obtained for material which occupies the whole space R^n . Chirita proved that the asymptotic equipartition occurs between the Cesàro means of the kinetic and strain energies, which implies that the thermal effects do not influence explicitly the asymptotic equipartition of the mean kinetic and strain energies.

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To the best of my knowledge, there is no result concerning the existence of a rate of decay for the thermal energy or for the strain or stress energy stored in a component of the displacement vector field for n-dimensional thermoelastic models (bounded materials). However there exists some particular result as Carvalho and Mensala [7] who showed the exponential decay of the total thermoelastic energy, provided the restoring force is proportional to the velocity vector field; and Rivera's result in [9] for thermo-visco-elastic materials. For general thermoelastic models the question about rate of decay remains open.

The one dimensional thermoelastic equations are given by:

$$\rho u_{tt} - (au_x)_x + (m\theta)_x = 0, \quad]0, L[\times]0, T[\quad (1.1)$$

$$\eta \theta_t - (k\theta_x)_x + \beta m u_{xt} = 0, \quad]0, L[\times]0, T[\quad (1.2)$$

$$u(x, 0) = u_0(x), \quad u_t = u_1(x), \quad \theta(x, 0) = \theta_0(x) \quad \text{in} \quad]0, L[$$

By ρ we are denoting the mass density, a is the elasticity coefficient, m the stress-temperature, η the specific heat, k the heat conductivity and by β we denote the absolute temperature.

The main result of this paper is that the total thermoelastic energy decays to zero exponentially as time goes to infinity, when the material is submited to Dirichlet's Newmann's or mixed boundary conditions. More precisely, let us define the following functionals:

$$\begin{aligned} E_1(t) &= \frac{1}{2} \int_0^L \rho |u_t|^2 + a |u_x|^2 + \frac{1}{\beta} k |\theta|^2 dx \\ E_2(t) &= \frac{1}{2} \int_0^L \rho |u_{tt}|^2 + a |u_{xt}|^2 + \frac{1}{\beta} k |\theta_t|^2 dx \\ E_3(t) &= \frac{1}{2} \int_0^L \rho |u_{xt}|^2 + |(au_x)_x|^2 + \frac{1}{\beta} k |\theta_x|^2 dx \end{aligned}$$

We will prove for Dirichlet's Newmann's and Mixed boundary condition that

$$E_1(t) + E_2(t) + E_3(t) \leq C e^{-\gamma t}, \quad (1.3)$$

For some positive constant C and γ . It is not difficult to see that, in general, there is no decay of E_1 for Newmann boundary conditions. In fact, let us consider $u(x, t) = t + 1$ and $\theta(x, t) = 0$, it is easy to see that the above couple satisfies equations (1.1) and (1.2) for the following initial data: $u_0(x) = 1$, $u_1(x) = 1$, and $\theta_0 = 0$, with Newmann boundary conditions on u . In this case E_1 is constant and therefore there is no decay. So to eliminate this type of solutions we impose the normalization condition on the initial data. That is:

$$\int_0^L \rho u_0(x) dx = 0, \quad \int_0^L \rho u_1(x) dx = 0 \quad (1.4)$$

Note that this conditions implies

$$\int_0^L \rho u(x, t) dx = 0, \quad \int_0^L \rho u_t(x, t) dx = 0 \quad \forall t \geq 0 \quad (1.5)$$

With this assumptions we will show that the exponential decay also holds for Newmann's boundary condition.

The fundamental domain is bounded is responsible for the exponential decay since there is a continuous mode conversion on the boundaries which transform part of the uncoupled transverse wave to the coupled dissipative longitudinal wave. On the other hand for the Cauchy problem where no boundaries are present the energy decay algebraically (see [4] and [8]). For dimensions larger than one, the dissipation given by the heat flux is not enough to yield the total thermoelastic energy to zero. This because the dissipation affects only the component of the displacement vector field in the direction of the flux, that is, the corresponding component, which is a gradient (the irrotational part). The other component called selenoidal part, can be uncoupled of the thermoelastic system and satisfies the elastic equation of motion which is not influenced by the thermal effect. This, of course, is not the case for the unidimensional case. The method we use here, to obtain the exponential decay is based on the construction of a new Liapunov's Functional, using multiplicatives techniques and regularities results for the wave equation. The existence, uniqueness and regularity of the solution of equations (1.1) - (1.2) is well known by now and was shown in [3].

2 Asymptotic behaviour

Let us denote by K the operator in $L^2(0, L)$ defined by $Kw = -(kw_x)_x$ with domain the Sobolev's space $H_0^1(0, L) \cap H^2(0, L)$. By A we will denote the operator on $L^2(0, L)$ defined by $Aw = -(aw_x)_x$ with domain $D(A)$ given by one of the following sets:

$$D(A) = H_0^1(0, L) \cap H^2(0, L)$$

$$D(A) = \{w \in H^2(0, L); \quad w_x(0) = w_x(L) = 0\}$$

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It is well known that A and K are positives self adjoints operator over $L^2(0, L)$. Let us denote by B_i ($i = 1, 2$) the operator defined by

$$B_1w(x) = m(x)w_x(x); \quad B_2w(x) = (m(x)w(x))_x$$

with corresponding domains given by

$$D(B_1) = \text{Closure of } D(A) \text{ with respect to the norm } \|v\|_{\mathcal{H}}^2 = \int_0^L |v|^2 + vAv \, dx,$$

$$D(B_2) = H_0^1(0, L)$$

We denote by $\mathcal{H}$ the space $\mathcal{H} = D(B_1) \times L^2(0, L) \times L^2(0, L)$, with norm given by

$$\|V\|_{\mathcal{H}}^2 = \int_0^L |u|^2 + uAu + |v|^2 + |\theta|^2 dx$$

where $V = (u, v, \theta)^\tau$. Let us define the operator $\mathcal{A}$ as follows:

$$D(\mathcal{A}) = D(A) \times D(B_1) \times H_0^1(0, L) \cap H^2(0, L); \quad \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$$

$$\mathcal{A}V = -\frac{1}{\rho} \begin{pmatrix} 0 & -\rho & 0 \\ A & 0 & \alpha B_2 \\ 0 & \beta B_1 & K \end{pmatrix}$$

Finally from now on ρ, a, m, k and η will be functions of $C^3(0, L)$ satisfying the following conditions:

$$0 < \rho_0 \leq \rho(x) \leq \rho_1; \quad 0 < a_0 \leq a(x) \leq a_1$$

$$0 < m_0 \leq m(x) \leq m_1; \quad 0 < k_0 \leq k(x) \leq k_1; \quad 0 < \eta_0 \leq \eta(x) \leq \eta_1.$$

For any x in $[0, L]$. With this notations we get the following theorem, which is a consequence of Dafermos's existence theorem in [3].

Theorem 2.1 *Let $(u_0, u_1, \theta_0) \in D(\mathcal{A})$ and $T > 0$. Then there exist only one strong solution of system (1.1)-(1.2) satisfying:*

$$u \in C(0, T; D(A)) \cap C^1(0, T; D(B_1)) \cap C^2(0, T; L^2(0, L))$$

$$\theta \in C(0, T; D(K)) \cap C^1(0, T; L^2(0, L))$$

Remark 2.1 *Let us define $D(\mathcal{A}^2)$ as*

$$D(\mathcal{A}^2) = \{V \in D(\mathcal{A}); \mathcal{A} \in D(\mathcal{A})\}$$

then it is easy to see that whenever $U_0 = (u_0, u_1, \theta_0) \in D(\mathcal{A}^2)$, we have

$$U \in C^1(0, T; D(\mathcal{A})) \cap C^2(0, T; D(\mathcal{H})),$$

in particular

$$u \in C^2(0, T; D(B_1)) \cap C^3(0, T; L^2(0, L)); \quad \theta \in C^1(0, T; H_0^1(0, L)).$$

Lemma 2.1 *Let v be a strong solution of the scalar wave equation*

$$\rho v_{tt} - (av_x)_x = F(x, t) \quad \text{in} \quad]0, L[\times]0, T[\tag{2.1}$$

$$v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x) \quad \text{in} \quad]0, L[\times]0, T[$$

where v_0, v_1 and F belongs to $D(A)$, $D(B_1)$ and $L^2(0, T; L^2(0, L))$ respectively. Then for $q \in C^1(0, L)$ the following identity holds:

$$\begin{aligned} & \frac{1}{2} \left\{ a(L)q(L)|v_x(L, t)|^2 - a(0)q(0)|v_x(0, t)|^2 \right\} = \\ & \frac{d}{dt} \int_0^L \rho q v_t v_x dx + \frac{1}{2} \int_0^L \left[a^2 \left(\frac{q}{a} \right)_x |v_x|^2 + (\rho q)_x |v_t|^2 \right] dx - \int_0^L q F v_x dx - \frac{1}{2} \rho q |v_t|^2 \Big|_{x=0}^{x=L} \end{aligned}$$

Proof.- Multiplying equation (2.1) by $q v_x$ and integrating in x we have

$$\int_0^L \rho q v_{tt} v_x dx - \int_0^L q (a v_x)_x v_x dx = \int_0^L F q v_x dx \quad (2.2)$$

Direct calculations give us the following identities

$$\begin{aligned} \int_0^L \rho q v_{tt} v_x dx &= \frac{d}{dt} \int_0^L \rho q v_t v_x dx - \int_0^L \rho q v_t v_{xt} dx \\ &= \frac{d}{dt} \int_0^L \rho q v_t v_x dx - \frac{1}{2} \rho q |v_t|^2 \Big|_{x=0}^{x=L} + \frac{1}{2} \int_0^L (\rho q)_x v_t^2 dx \end{aligned} \quad (2.3)$$

On the other hand we have

$$\begin{aligned} \int_0^L q (a v_x)_x v_x dx &= \int_0^L \left(\frac{q}{a} \right) \frac{d}{dx} |a v_x|^2 dx \\ &= \frac{1}{2} \left\{ q(L) a(L) v_x^2(L, t) - q(0) a(0) v_x^2(0, t) \right\} - \frac{1}{2} \int_0^L a^2 \left(\frac{q}{a} \right)_x v_x^2 dx \end{aligned} \quad (2.4)$$

From equations (2.2), (2.3) and (2.4) our result follows.

With this tools we are in conditions to prove the main result of this paper. For this, we proceed as follows: We define three functionals which we will call of I , J and H , with the help of the energies function defined in the introduction of this work, we will prove that there exist a linear combination of this all functionals, which we will denote by $\mathcal{E}$, whose derivative is negative proportional to itself. This means that the functional $\mathcal{E}$, decays exponentially to zero as time goes to infinity. We will do this in the next theorem.

Theorem 2.2 *Under the same hypotheses of Theorem 2.1, for Dirichlet's, Neumann's or mixed boundary conditions on u , we have that there exist positive constants C and γ such that the solution of system (1.1) and (1.2) satisfies (1.3).*

Proof.- Let us take $(u_0, u_1, \theta_0) \in D(\mathcal{A}^2)$. Multiplying equations (1.1) and (1.2) by u_t and θ respectively we have

$$\frac{1}{2} \frac{d}{dt} \int_0^L \rho |u_t|^2 + a |u_x|^2 dx = \int_0^L m \theta u_{tx} dx \quad (2.5)$$

and

$$\int_0^L m\theta u_{xt} dx = -\frac{1}{\beta} \left\{ \frac{1}{2} \frac{d}{dt} \int_0^L \eta |\theta|^2 dx + \int_0^L k |\theta_x|^2 dx \right\}$$

Substitution of this identity in (2.5) implies

$$\frac{d}{dt} E_1(t) = -\frac{1}{\beta} \int_0^L k |\theta_x|^2 dx \quad (2.6)$$

Next we will find the derivative of E_2 . Differentiating equations (1.1) and (1.2) with respect to time variable, and multiplying by u_{tt} and θ_t respectively, we obtain (using essentially the same calculations as above)

$$\frac{d}{dt} E_2(t) = -\frac{1}{\beta} \int_0^L k |\theta_{xt}|^2 dx \quad (2.7)$$

First we will consider Newmann's boundary condition on u . To show the exponential decay we will introduce the following functions

$$f(x) = \frac{1}{m} \int_0^x \rho u_t dx; \quad G_1(t) = \int_0^L \rho u_t u dx; \quad G_2(t) = \int_0^L \eta \theta f dx$$

Note that the normalized condition implies that f satisfies $f(0) = f(L) = 0$. Using equation (1.1) it is easy to verify that

$$\begin{aligned} \frac{d}{dt} G_1(t) &= \int_0^L \rho |u_t|^2 dx - \int_0^L a |u_x|^2 dx - \int_0^L (m\theta)_x u dx \\ &\leq \int_0^L \rho |u_t|^2 dx - \frac{1}{2} \int_0^L a |u_x|^2 dx - c_1 \int_0^L |\theta_x|^2 dx \end{aligned} \quad (2.8)$$

On the other hand,

$$\frac{d}{dt} G_2(t) = \int_0^L \eta \theta_t f dx + \int_0^L \eta \theta f_t dx.$$

Using equation (1.2) we obtain

$$\int_0^L \eta \theta_t f dx = \int_0^L \frac{\rho k}{m} \theta_x u_t dx + \int_0^L \frac{k m_x}{m} \theta_x f dx - \beta \int_0^L \rho |u_t|^2 dx$$

Similarly using equation (1.1) we get

$$\int_0^L \eta \theta f_t dx = \int_0^L \frac{a \eta}{m} \theta u_x dx + \int_0^L \eta |\theta|^2 dx$$

The last three equations yields

$$\frac{d}{dt} G_2(t) = -\beta \int_0^L \rho |u_t|^2 dx + \int_0^L \frac{k m_x}{m} \theta_x f dx + \int_0^L \frac{\rho k}{m} \theta_x u_t dx + \int_0^L \frac{a \eta}{m} \theta u_x dx + \int_0^L \eta |\theta|^2 dx$$

which implies

$$\frac{d}{dt} G_2(t) \leq -\frac{1}{2} \beta \int_0^L \rho |u_t|^2 dx + \frac{\beta}{16} \int_0^L a |u_x|^2 dx + c_3 \int_0^L |\theta_x|^2 dx \quad (2.9)$$

From inequalities (2.8) and (2.9) we have

$$\frac{d}{dt} \left\{ \frac{\beta}{4} G_1(t) + G_2(t) \right\} \leq -\frac{\beta}{4} \int_0^L \rho |u_t|^2 dx - \frac{\beta}{16} \int_0^L a |u_x|^2 dx + c_3 \int_0^L |\theta_x|^2 dx$$

So using (2.6) and taking N big enough the following inequalities holds

$$\frac{d}{dt} \left\{ N E_1(t) + \frac{\beta}{4} G_1(t) + G_2(t) \right\} \leq -\frac{\beta}{16} E_1(t);$$

$$\frac{N}{2} E_1(t) \leq N E_1(t) + \frac{\beta}{4} G_1(t) + G_2(t) \leq 2 N E_1(t)$$

From where we get

$$\frac{d}{dt} \left\{ N E_1(t) + \frac{\beta}{4} G_1(t) + G_2(t) \right\} \leq -\frac{\beta}{32N} \left\{ N E_1(t) + \frac{\beta}{4} G_1(t) + G_2(t) \right\}$$

Using the last two inequalities we obtain

$$E_1(t) \leq 4 E_1(0) e^{-\frac{\beta}{32N} t}$$

Since (u_t, θ_t) satisfies (1.1), (1.2) and the same boundary conditions as (u, θ) we also conclude that

$$E_2(t) \leq 4 E_2(0) e^{-\frac{\beta}{16N} t}$$

Then our result follows for Newmann's boundary conditions. To show the exponential decay for Dirichlet and Mixed boundary conditions let us find the time derivative of E_3 . To do this let us multiply equation (1.1) and (1.2) by $-\frac{1}{\rho}(au_x)_{xt}$ and $-\frac{a}{\rho}\theta_{xx}$ respectively. Integrating we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^L a |u_{xt}|^2 + \frac{1}{\rho} |(au_x)_x|^2 dx = au_{tt}u_{xt}|_{x=0}^{x=L} - \int_0^L \frac{1}{\rho} (m\theta)_x (au_x)_{xt} dx \\ & = au_{tt}u_{xt}|_{x=0}^{x=L} - \frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} + \int_0^L au_{xt} \left\{ \left[\frac{m_x}{\rho} \right]_x \theta + \left[\frac{m_x}{\rho} + \left(\frac{m}{\rho} \right)_x \right] \theta_x \right\} dx + \int_0^L \frac{am}{\rho} u_{xt} \theta_{xx} dx \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^L \frac{a}{\rho} \eta |\theta_x|^2 dx + \int_0^L k \frac{a}{\rho} |\theta_{xx}|^2 dx = \eta \frac{a}{\rho} \theta_t \theta_x |_{x=0}^{x=L} - \int_0^L \left(\frac{a}{\rho} \eta \right)_x \theta_t \theta_x dx \\ & \quad - \int_0^L \frac{a}{\rho} k_x \theta_x \theta_{xx} dx - \beta \int_0^L \left[\frac{a}{\rho} \theta_x \right]_x u_{xt} dx \end{aligned}$$

the above two identities imply that

$$\begin{aligned} \frac{d}{dt} E_3(t) &= -\frac{1}{\beta} \int_0^L \frac{a}{\rho} m k |\theta_{xx}|^2 dx - \frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} + \frac{1}{\beta} m \eta \theta_t \theta_x |_{x=0}^{x=L} \\ & \quad - \frac{1}{\beta} \int_0^L \frac{a}{\rho} k_x \theta_x \theta_{xx} dx - \int_0^L au_{xt} \left\{ \left[\frac{m_x}{\rho} \right]_x \theta + \left[\frac{m_x}{\rho} + \left(\frac{m}{\rho} \right)_x \right] \theta_x \right\} dx - \frac{1}{\beta} \int_0^L \left(\frac{a}{\rho} \eta \right)_x \theta_t \theta_x dx \end{aligned}$$

Using Remark 2.1 we conclude that

$$u_{tt}u_{xt}|_{x=0}^{x=L} = m\eta\theta_t\theta_x|_{x=0}^{x=L} = 0,$$

for Dirichlet, Newmann or Mixed boundary conditions. Hence the derivative of E_3 can be expressed as:

$$\begin{aligned} \frac{d}{dt}E_3(t) = & -\frac{1}{\beta} \int_0^L \frac{a}{\rho} k |\theta_{xx}|^2 dx - \frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} \\ & -\frac{1}{\beta} \int_0^L \frac{a}{\rho} k_x \theta_x \theta_{xx} dx - \frac{1}{\beta} \int_0^L \left(\frac{a}{\rho}\eta\right)_x \theta_t \theta_x dx \end{aligned} \quad (2.10)$$

Let us define the functions

$$I(t) = - \int_0^L (\eta\theta)_x u_t dx; \quad J(t) = \int_0^L a u_x u_{xt} dx.$$

Clearly we have

$$|I(t)| \leq \frac{1}{2} \int_0^L \eta^2 |\theta|^2 + |u_{xt}|^2 dx \quad (2.11)$$

$$|J(t)| \leq \frac{1}{2} \int_0^L a^2 |u_x|^2 + |u_{xt}|^2 dx \quad (2.12)$$

Differentiating J and I with respect to t and using equations (1.1)-(1.2) we have

$$\frac{d}{dt}I(t) = - \int_0^L \frac{1}{\rho} (\eta\theta)_x (a u_x)_x dx + \quad (2.13)$$

$$+ \int_0^L \frac{1}{\rho} (\eta\theta)_x (m\theta)_x dx + \int_0^L (k\theta_x)_x u_{xt} dx - \beta \int_0^L m |u_{xt}|^2 dx,$$

$$\frac{d}{dt}J(t) = \int_0^L a |u_{xt}|^2 dx - \int_0^L \frac{1}{\rho} |(a u_x)_x|^2 dx + \int_0^L \frac{1}{\rho} (m\theta)_x (a u_x)_x dx \quad (2.14)$$

because $(\eta\theta)_x u_t|_{x=0}^{x=L} = a u_x u_{tt}|_{x=0}^{x=L} = 0$. From equation (2.13) and (2.14) it follows

$$\begin{aligned} \frac{d}{dt}I(t) & \leq C(\epsilon) \int_0^L |(\eta\theta)_x|^2 dx \\ & + \epsilon m_0 \frac{\beta}{a_1} \int_0^L \frac{1}{\rho} |(a u_x)_x|^2 dx + \frac{c}{\epsilon} \int_0^L a k |\theta_{xx}|^2 dx - (\beta - \epsilon) m_0 \int_0^L |u_{xt}|^2 dx, \\ \frac{d}{dt}J(t) & \leq a_1 \int_0^L |u_{xt}|^2 dx - (1 - \epsilon) \int_0^L \frac{1}{\rho} |(a u_x)_x|^2 dx + C(\epsilon) \int_0^L |(m\theta)_x|^2 dx \end{aligned}$$

where $C(\epsilon)$ denotes various constants depending on ϵ , and c is a positive constant which does not depend on ϵ . By means of the above inequalities, the following relation is found:

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{m_0 \beta}{2} J(t) + a_1 I(t) \right\} & \leq -\frac{m_0 \beta}{2} (1 - 3\epsilon) \int_0^L \frac{1}{\rho} |(a u_x)_x|^2 dx \\ & - a_1 \left(\frac{\beta}{2} - \epsilon \right) m_0 \int_0^L |u_{xt}|^2 dx + \frac{c}{\epsilon} \int_0^L \frac{a}{\rho} m k |\theta_{xx}|^2 dx + C(\epsilon) \int_0^L |(m\theta)_x|^2 dx. \end{aligned} \quad (2.15)$$

On the other hand, from (2.10) we have

$$\frac{d}{dt}E_3(t) \leq \frac{1 - \epsilon}{\beta} \int_0^L \frac{a}{\rho} k |\theta_{xx}|^2 dx - \frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} + C(\epsilon) \int_0^L m |\theta_x|^2 dx - \int_0^L |\theta_t|^2 dx \quad (2.16)$$

Multiplying equation (2.15) by $\frac{\epsilon}{2\beta c}$ and adding up to (2.16) we obtain

$$\begin{aligned} \frac{d}{dt} \left\{ E_3(t) + \frac{m_0\epsilon}{4c} J(t) + \frac{a_1\epsilon}{2\beta c} I(t) \right\} &\leq -\frac{1-2\epsilon}{2\beta} \int_0^L \frac{a}{\rho} m k |\theta_{xx}|^2 dx - \frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} \\ &+ C(\epsilon) \int_0^L |(m\theta)_x|^2 dx + \int_0^L |\theta_t|^2 dx - \frac{1-3\epsilon}{4c} m_0 \epsilon \int_0^L |(au_x)_x|^2 dx - \frac{\epsilon a_1}{2\beta c} \left(\frac{\beta}{2} - \epsilon \right) m_0 \int_0^L |u_{xt}|^2 dx \end{aligned}$$

Therefore taking ϵ small enough we conclude the existence of positive constants μ and c_1 such that

$$\begin{aligned} \frac{d}{dt} \left\{ E_3(t) + \frac{m_0\epsilon}{4c} J(t) + \frac{a_1\epsilon}{2\beta c} I(t) \right\} &\leq -\alpha \frac{1}{\beta} (m\theta)_x u_{tx}|_{x=0}^{x=L} \\ &- \mu \left\{ \int_0^L k |\theta_{xx}|^2 dx + \int_0^L |(au_x)_x|^2 dx + \int_0^L |u_{xt}|^2 dx \right\} + c_1 \int_0^L |(m\theta)_x|^2 dx + \int_0^L |\theta_t|^2 dx \end{aligned}$$

Let us denote by $\mathcal{E}$ the following function

$$\mathcal{E}(t) = N_1 E_1(t) + N_2 E_2(t) + E_3(t) + \frac{m_0\epsilon}{4c} J(t) + \frac{a_1\epsilon}{2\beta c} I(t)$$

Taking N_1 and N_2 big enough we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq -\frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} - c_2 N_1 \int_0^L |\theta_x|^2 dx - c_2 N_2 \int_0^L |\theta_{xt}|^2 dx \\ &- \mu \left\{ \int_0^L k |\theta_{xt}|^2 dx + \int_0^L k |\theta_{xx}|^2 dx + \int_0^L |u_{xt}|^2 dx + \int_0^L |(au_x)_x|^2 dx \right\} \end{aligned} \quad (2.17)$$

Choosing q such that $q(L) \geq 0$ and $q(0) \leq 0$ we have

$$\begin{aligned} &\frac{a}{\rho} (m\theta)_x u_{tx}|_{x=0}^{x=L} \leq \\ &\leq \frac{2m_1 c_2}{\delta} \sup \left\{ |\theta_x|^2; x \in [0, L] \right\} + \frac{\delta}{4} \left\{ a(L) q(L) u_{xt}^2(L, t) - a(0) q(0) u_{xt}^2(0, t) \right\} \end{aligned} \quad (2.18)$$

From Glagiaro-Niremborg's inequality we easily obtain a positive constant c satisfying:

$$\begin{aligned} &\sup \left\{ |\theta_x|^2; x \in [0, L] \right\} \leq \\ &\leq c \left\{ \int_0^L |\theta_x|^2 dx \right\}^{\frac{1}{2}} \left\{ \int_0^L |\theta_x|^2 dx + \int_0^L |\theta_{xx}|^2 dx \right\}^{\frac{1}{2}} \leq [c + (\frac{c}{\delta})^2] \int_0^L |\theta_x|^2 dx + \frac{\delta^2}{4} \int_0^L |\theta_{xx}|^2 dx \end{aligned} \quad (2.19)$$

Relations (2.17), (2.18) and (2.19) guarantee that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq C(\delta) \int_0^L |\theta_x|^2 dx + \frac{\delta m_1 c_2}{2} \int_0^L |\theta_{xx}|^2 dx - c_2 N_1 \int_0^L |\theta_x|^2 dx - c_2 N_2 \int_0^L |\theta_{xt}|^2 dx \\ &- \mu \left\{ \int_0^L k |\theta_{xt}|^2 dx + \int_0^L k |\theta_{xx}|^2 dx + \int_0^L |u_{xt}|^2 dx + \int_0^L |(au_x)_x|^2 dx \right\} \\ &+ \frac{\delta}{4} \left\{ a(L) q(L) u_{xt}^2(L, t) - a(0) q(0) u_{xt}^2(0, t) \right\} \end{aligned} \quad (2.20)$$

For Dirichlet boundary condition we will use $q(x) = x - \frac{1}{2}L$ and for Mixed we will use $q(x) = L - x$ if $u(L) = 0$ or $q(x) = x$ if $u(0) = 0$. Letting $v = u_t$ and $F = -m\theta_{xt}$ in Lemma 2.1 and since $(u_0, u_1, \theta_0) \in D(\mathcal{A}^2)$ we conclude that $v(0) = u_1 \in D(\mathcal{A})$ and $v_t(0) \in D(B_2)$. Then we have

$$\begin{aligned} \frac{d}{dt}H(t) &= -\frac{1}{2} \left\{ a(L)q(L)u_{xt}^2(L, t) - a(0)q(0)u_{xt}^2(0, t) \right\} \\ &\quad - \frac{1}{2} \int_0^L \left\{ a^2\left(\frac{q}{a}\right)_x |u_{xt}|^2 + (pq)_x |u_{tt}|^2 \right\} dx + \int_0^L q(m\theta_x)_t u_{xt} dx - \frac{1}{2} \rho q u_{tt}^2 \Big|_{x=0}^{x=L} \end{aligned} \quad (2.21)$$

where H stands for

$$H(t) = - \int_0^L q \rho u_{tt} u_{xt} dx$$

By the choose of q with respect to the boundary conditions we have that $\rho q u_{tt}^2 \Big|_{x=0}^{x=L} = 0$, while from equations (2.20) and (2.21) we get

$$\begin{aligned} \frac{d}{dt} \left\{ \mathcal{E}(t) + \frac{\delta}{2} H(t) \right\} &\leq C(\delta) \int_0^L |\theta_x|^2 dx + \frac{\delta}{4} m_1 c_2 \int_0^L |\theta_{xx}|^2 dx \\ \mu \left\{ \int_0^L k |\theta_{xt}|^2 + k |\theta_{xx}|^2 + |u_{xt}|^2 + |(au_x)_x|^2 dx \right\} &- c_2 N_1 \int_0^L |\theta_x|^2 dx - c_2 N_2 \int_0^L |\theta_{xt}|^2 dx \\ &+ \frac{\delta}{2} \int_0^L \left\{ \left(\frac{q}{a}\right)_x |u_{xt}|^2 + (pq)_x |u_{tt}|^2 \right\} dx - \delta \int_0^L q(m\theta_x)_t u_{xt} dx, \end{aligned} \quad (2.22)$$

Sustitution of u_{tt} given by relation (1.1) in (2.22) implies, for N_1, N_2 big enough and δ small enough, that there exist a positive constant μ_0 satisfying

$$\frac{d}{dt} \left\{ \mathcal{E}(t) + \frac{\delta}{2} H(t) \right\} \leq -\mu_0 \left\{ \int_0^L k |\theta_{xt}|^2 + k |\theta_{xx}|^2 + |u_{xt}|^2 + |(au_x)_x|^2 dx \right\}$$

Since

$$\mathcal{E}(t) + \frac{\delta}{2} H(t) \leq \nu_0 \left\{ \int_0^L k |\theta_{xt}|^2 + k |\theta_{xx}|^2 + |u_{xt}|^2 + |(au_x)_x|^2 dx \right\}$$

taking $\gamma = \frac{\mu_0}{\nu_0}$ we obtain

$$\frac{d}{dt} \left\{ \mathcal{E}(t) + \frac{\delta}{2} H(t) \right\} + \gamma \left\{ \mathcal{E}(t) + \frac{\delta}{2} H(t) \right\} \leq 0.$$

Which implies that

$$\mathcal{E}(t) + \frac{\delta}{2} H(t) \leq \left\{ \mathcal{E}(0) + \frac{\delta}{2} H(0) \right\} \exp -\gamma t \quad (2.23)$$

Taking N_1 and N_2 big enough it follows

$$E_1(t) + E_2(t) + E_3(t) \leq 2\mathcal{E}(t) + \delta H(t)$$

Then from (2.23) our conclusion follows. The proof is now complete

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