A Linear Thermoelastic Plate Equation with Dirichlet Boundary Condition

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We consider the initial-boundary value problem for a linear thermoelastic plate equation and we prove that the energy associated to the system decays exponentially to zero as time goes to infinity. © 1997 by B. G. Teubner Stuttgart–John Wiley & Sons Ltd.

1. Introduction

This paper is concerned with the exponential decay of the first energy of solutions of the initial-boundary value problem of a linear thermoelastic plate equation of the form

\[ u_{tt} - h \Delta u_{tt} + \Delta^2 u + x \Delta \theta = 0 \quad \text{in} \ (0, \infty) \times \Omega, \]

\[ \theta_t - \beta \Delta \theta - z \Delta u_t = 0 \quad \text{in} \ (0, \infty) \times \Omega, \]

\[ u = \partial u / \partial \nu = \theta = 0 \quad \text{on} \ (0, \infty) \times \partial \Omega, \]

\[ u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ \theta(0, x) = \theta_0(x) \quad \text{in} \ \Omega, \]

where \( \alpha \neq 0, \beta > 0 \) and \( h > 0 \) are real constants, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( C^\infty \) boundary \( \partial \Omega \), identified with a thin plate of height \( h \), \( v = (v_1, \ldots, v_n) \) stands for the unit outer normal to \( \partial \Omega \) and \( \partial / \partial \nu = \sum_{j=1}^n v_j \partial / \partial x_j \). By \( u \) and \( \theta \) we denote the vertical deflection and the temperature of the plate, respectively. The derivation of (1.1) and (1.2) can be found in Lagnese [2], where the author discusses stability of various plate models and shows that the energy of a linear thermoelastic plate decays exponentially fast with a certain dissipative boundary condition. When \( h = 0 \), Kim [1] proved that the exponential decay can be achieved with a homogeneous Dirichlet boundary...
condition, namely (1.3). Recently Liu and Renardy [6] improved Kim’s result, showing that the semigroup associated to the elliptic part of system (1.1)–(1.4) \((h = 0)\) is of analytic type, which in particular implies that the couple \((u, \theta)\) decays uniformly as time goes to infinity and reveals the parabolic character of the system. Unfortunately, this approach cannot be applied to show the uniform decay of the solution of the above system, even the semigroup approach to prove the existence result is more delicate.

Our purpose is to extend the result in [1, 6] to the case that \(h > 0\). In particular we also improve the work in [2] in the sense that we show that any additional damping term to show the exponential decay of the solution is not necessary. Introducing the term \(\Delta u_n\) not only makes the problem more physically meaningful but also more interesting from the mathematical point of view, because to prove the decay of solutions we need more regularity, which is not a simple question. For example, the compatibility conditions are different from the solutions we need more regularity, which is not a simple question. For example, the compatibility conditions are different from the \(h = 0\) case and more complicated. Also, we meet several technical problems even in proving the existence of solutions, which did not appear in the \(h = 0\) case. Moreover, when \(h = 0\) the first equation is of parabolic type, while when \(h > 0\) it is hyperbolic, so somehow we need a different argument than that used in Kim’s work.

To state our main result, let us introduce some notations which will be used throughout the paper. Let \(L^2(\Omega)\) be a usual \(L^2\) space on \(\Omega\) and let \((\cdot, \cdot)\) and \(\| \cdot \|\) denote its innerproduct and norm, respectively. Let \(H^s(\Omega)\) be a usual Sobolev space in the \(L^2\) sense of order \(s\) and let \(\| \cdot \|_s\) denote its norm. Put

\[
H^s_0(\Omega) = \{ u \in H^s(\Omega) \mid u = 0 \text{ on } \partial \Omega \}, \quad s \geq 1,
\]

\[
H^s_{0,0}(\Omega) = \{ u \in H^s(\Omega) \mid u = \frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega \}, \quad s \geq 2.
\]

If

\[
\begin{align*}
 u &\in C^0([0, \infty); H^3_{0,0}(\Omega)) \cap C^1([0, \infty); H^3(\Omega)), \\
 \theta &\in C^0([0, \infty); L^2(\Omega)) \cap L^2((0, \infty); H^3(\Omega)),
\end{align*}
\]

(1.5)

(1.6)

where \(u\) and \(\theta\) satisfy (1.1) and (1.2) in the sense of distributions in \((0, \infty) \times \Omega\), and (1.4) holds, then \(u\) and \(\theta\) are said to be weak solutions of (1.1)–(1.4) (the precise definition will be given in section 2). The goal of this paper is to prove the following theorem.

**Theorem 1.1.** Given \(u_0 \in H^3_{0,0}(\Omega), u_1 \in H^3(\Omega)\) and \(\theta_0 \in L^2(\Omega), (1.1)\)–(1.4) admit a unique pair of weak solutions \(u\) and \(\theta\).

Moreover, there exist positive constants \(M\) and \(\gamma\) independent of \(u\) and \(\theta\) such that

\[
e^{2\gamma t} E_1((u, \theta), t) + \int_0^t e^{2\gamma s} \{ E_1((u, \theta), s) + \| \nabla \theta(s, \cdot) \|_s^2 \} \, ds
\leq M \left\{ \| u_0 \|_2^2 + \| u_1 \|_2^2 + \| \theta_0 \|_2^2 \right\},
\]

(1.7)

where \(\nabla \theta = (\partial \theta / \partial x_1, \ldots, \partial \theta / \partial x_n)\) and

\[
E_1((u, \theta), t) = \| u(t, \cdot) \|_2^2 + \| u_t(t, \cdot) \|_2^2 + \| \theta(t, \cdot) \|_2^2.
\]
If weak solutions \( u \) and \( h \) satisfy the regularity conditions

\[
\begin{align*}
&u \in \bigcap_{j=0}^{2} C^{j}([0, \infty); H^{3-j}_{00}(\Omega)) \cap C^{3}([0, \infty); H^{1}_{0}(\Omega)), \\
&\theta \in \bigcap_{j=0}^{1} C^{j}([0, \infty); H^{2-j}_{0}(-\Omega)) \cap C^{2}([0, \infty); L^{2}(\Omega)), \theta_{u} \in L^{2}((0, \infty); H^{1}_{0}(\Omega)),
\end{align*}
\]

then \( u \) and \( \theta \) are said to be strong solutions.

We start with proving the energy identity for weak solutions.

**Lemma 2.2.** If \( u \) and \( \theta \) are weak solutions, then

\[
e((u, \theta), t) + \beta \int_{0}^{t} \| \nabla \theta(s, \cdot) \|^{2} \, ds = e((u, \theta), 0),
\]

where

\[
e((u, \theta), t) = \frac{1}{2} \left\{ \| u_{t}(t, \cdot) \|^{2} + h \| \nabla u_{t}(t, \cdot) \|^{2} + \| \Delta u(t, \cdot) \|^{2} + \| \theta(t, \cdot) \|^{2} \right\}.
\]

**Proof.** Let \( \rho(t) \) be a function in \( C_{0}^{\infty}(\mathbb{R}) \) satisfying

\[
\rho \geq 0, \quad \text{supp} \, \rho \subset [-2, -1] \quad \text{and} \quad \int_{-\infty}^{\infty} \rho(s) \, ds = 1.
\]

Put \( \rho_{\varepsilon}(s) = \varepsilon^{-1} \rho(\varepsilon^{-1}s) \) for \( \varepsilon > 0 \). Let us take \( \varphi \in H^{1}_{00}(\Omega) \) and \( \psi \in H^{1}_{0}(\Omega) \) and let us denote by

\[
f^{\varepsilon}(s, x) = \rho_{\varepsilon}(t - s) \varphi(x), \quad g^{\varepsilon}(s, x) = \rho_{\varepsilon}(t - s) \psi(x),
\]
then \( f' \in C^\infty_0([0, \infty); H^2_{00}(\Omega)) \) and \( g' \in C^\infty_0([0, \infty); H^1_{0}(\Omega)) \) for any \( t \geq 0 \). Substituting \( f' \) and \( g' \) into (2.1) and (2.2) and noting that \( \rho_s(t) = (d/dt) \rho_s(t) = 0 \) for \( t \geq 0 \), we have

\[
(u_{\text{str}}(t, \cdot), \varphi) + h(\nabla u_{\text{str}}(t, \cdot), \nabla \varphi) + (\Delta u_s(t, \cdot), \Delta \varphi) - \varepsilon(\nabla \theta_s(t, \cdot), \nabla \varphi) = 0, \tag{2.3}
\]

\[
(\theta_u(t, \cdot), \psi) + \beta(\nabla \theta_s(t, \cdot), \nabla \psi) + \varepsilon(\nabla u_{\text{str}}(t, \cdot), \nabla \psi) = 0 \tag{2.4}
\]

for any \( t \geq 0 \), \( \varphi \in H^2_{00}(\Omega) \) and \( \psi \in H^1_{0}(\Omega) \), where

\[
u_e(t, x) = \int_{-\infty}^{\infty} \rho_s(t-s)u(s, x)\,ds \quad \text{and} \quad \theta_e(t, x) = \int_{-\infty}^{\infty} \rho_s(t-s)\theta(s, x)\,ds. \tag{2.5}
\]

It follows from (1.5) and (1.6) that \( u_s \in C^\infty([0, \infty); H^2_{00}(\Omega)) \) and \( \theta_s \in C^\infty([0, \infty); H^1_{0}(\Omega)) \). Putting \( \varphi = u_{\text{str}}(t, \cdot) \) in (2.3) and \( \psi = \theta_s(t, \cdot) \) in (2.4), and integrating the resulting relations with respect to \( t \), we have

\[
e((u_e, \theta_e), t) + \beta \int_0^t \| \nabla \theta_e(s, \cdot) \|^2\,ds = e((u_e, \theta_e), 0). \tag{2.6}
\]

Letting \( \varepsilon \to 0 \) in (2.6), we have the lemma.

Now, we are able to show the existence and uniqueness of weak solutions as well as the corresponding regularity result. This will be summarized in the next theorem.

**Theorem 2.3.** (1) If \( u_0 \in H^3_{00}(\Omega) \), \( u_1 \in H^2_{0}(\Omega) \) and \( \theta_0 \in L^2(\Omega) \), then (1.1)–(1.4) admits a unique pair of weak solutions \( u \) and \( \theta \).

Moreover, if \( u_0, u_1 \) and \( \theta_0 \) satisfy the regularity assumption,

\[
u_0 \in H^3_{00}(\Omega), \quad u_1 \in H^2_{0}(\Omega) \quad \text{and} \quad \theta_0 \in H^1_{0}(\Omega), \tag{2.7}
\]

then

\[
\nu \in \bigcap_{j=0}^{1} C^j([0, \infty); H^3_{00-j}(\Omega)) \cap C^2([0, \infty); H^1_{0}(\Omega)), \tag{2.8}
\]

\[
\theta \in C^0([0, \infty); H^3_{0}(\Omega)) \cap C^1([0, \infty); L^2(\Omega)), \quad \theta_t \in L^2((0, \infty); H^1_{0}(\Omega)). \tag{2.9}
\]

**Proof.** In view of Lemma 2.2, the uniqueness of weak solutions holds, hence our attention will be focused on to prove the existence and the regularity of the solution. To do it we first take the initial data \( u_0, u_1 \) and \( \theta_0 \) satisfying one more regularity condition, namely (2.7), then we use the Galerkin method, which is well known as a method of constructing solutions through suitable approximate solutions in the finite dimensional spaces (cf. [3]), but usually it works in terms of one basis. In our case, we have to use two different bases corresponding to \(-\Delta\) and \(\Delta^2\) with Dirichlet zero conditions, so that we shall give a proof below.

To facilitate our analysis, we introduce the following notations:

\[
(u, v)_\Lambda = (\Delta u, \Delta v), \quad \| u \|_\Lambda = \| \Delta u \|, \quad (u, v)_h = (u, v) + h(\nabla u, \nabla v), \quad \| u \|_h = (u, u)_h^{1/2}.
\]

It is not difficult to see that \( H^2_{00}(\Omega) \) becomes a Hilbert space in terms of the innerproduct \((\cdot, \cdot)_\Lambda\). This follows from the well-known inequalities

\[
c_1^{-1} \| w \|_2 \leq \| \Delta w \| \leq c_1 \| w \|_2 \quad \text{for any} \ w \in H^2_{00}(\Omega) \quad \text{(2.10)}
\]
with some positive constant $c_1$, which is obtained from
\[ c_2^{-1} \| w \|_1 \leqslant \| \nabla w \| \leqslant c_2 \| w \|_1 \quad \text{for any } w \in H_0^1(\Omega), \] (2.11)
and the following two relations:
\[ \| \nabla w \|^2 = - (w, \Delta w) \quad \text{and} \quad \| w \|_2 \leqslant c_3 (\| \Delta w \| + \| w \|_1). \]

Also, we note that $H_0^1(\Omega)$ becomes a Hilbert space equipped with the inner product $(\nabla u, \nabla v)$ as well as the inner product $(u, v)_h$.

Let $\{ \phi_n \}_{n=1, 2, \ldots}$ and $\{ \psi_n \}_{n=1, 2, \ldots}$ be eigenfunctions of $\Delta^2$ and $-\Delta$ with Dirichlet zero condition corresponding to eigenvalues $\lambda_n$ and $\mu_n$, respectively, that is
\[ \Delta^2 \phi_n = \lambda_n \phi_n \text{ in } \Omega \quad \text{and} \quad \phi_n = \frac{\partial \phi_n}{\partial v} = 0 \text{ on } \partial \Omega, \]
\[ -\Delta \psi_n = \mu_n \psi_n \text{ in } \Omega \quad \text{and} \quad \psi_n = 0 \text{ on } \partial \Omega. \]

We may assume that both $\{ \phi_n \}$ and $\{ \psi_n \}$ form orthonormal bases of $L^2(\Omega)$. Put
\[ u_i(x) = \sum_{j=1}^\infty u_{ij} \phi_j(x), \quad i = 0, 1, \quad \text{and} \quad \theta_0(x) = \sum_{j=1}^\infty \theta_{0j} \phi_j(x). \]

For any integer $m \geqslant 1$, let $u_j^n(t), \theta_j^n(t), \; j = 1, \ldots, m$, be solutions to the $m$-system of ordinary differential equations:
\[ \sum_{i=1}^m \{ u_j^{n''}(t)(\phi_i, \phi_j)_h + u_j^n(t)(\phi_i, \phi_j)_\Delta - x\theta_j^n(t)(\nabla \phi_i, \nabla \phi_j) \} = 0, \tag{2.12} \]
\[ \sum_{i=1}^m \{ \theta_j^{n''}(t)(\psi_i, \psi_j) + \beta \theta_j^n(t)(\nabla \psi_i, \nabla \psi_j) + x u_j^n(t)(\nabla \psi_i, \nabla \phi_j) \} = 0, \tag{2.13} \]
\[ u_j^n(0) = u_{0j}, \quad u_j^n(t) = u_{1j}, \quad \theta_j^n(0) = \theta_{0j} \tag{2.14} \]
for $i = 1, 2, \ldots, m$, where the prime denotes $d/dt$. The coefficient of $u_j^{n''}(t), \; j = 1, 2, \ldots, m$, is an $m \times m$ matrix $A_m = (\phi_i, \phi_j)_h$, which is positive definite. In fact, for any $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$,
\[ \eta A_m \eta = \left| \sum_{i=1}^m \eta_i \phi_i \right|^2 + h \left| \sum_{i=1}^m \eta_i \nabla \phi_i \right|^2, \]
where $\eta$ denotes the transposed vector of $\eta$, so that
\[ \eta A^*_m \eta \geqslant 0 \quad \text{and} \quad \eta A_m A^*_m \eta = 0 \iff \eta = 0, \]
because $\eta_k = (\sum_{i=1}^m \eta_i \phi_i, \phi_k).$ The coefficient matrix of $\theta_j^{n''}(t), \; j = 1, 2, \ldots, m$, is the identity matrix, because $(\psi_i, \psi_j) = 1$ for $i = j$ and $n = 0$ for $i \neq j$. Therefore, solutions $u_j^n(t), \theta_j^n(t), \; j = 1, \ldots, m$, exist and $u_j^n(t), \theta_j^n(t) \in C^\infty([0, \infty)).$ Put
\[ u^n(t, x) = \sum_{j=1}^m u_j^n(t) \phi_j(x) \quad \text{and} \quad \theta^n(t, x) = \sum_{j=1}^m \theta_j^n(t) \psi_j(x). \]

Multiplying (2.12) by $u_j^n(t)$ and (2.13) by $\theta_j^n(t)$ and summing up the resulting formulae, we have
\[ e((u^n, \theta^n), t) + \beta \int_0^t \| \nabla \theta^n(s, \cdot) \|^2 \, ds = \frac{1}{2} \left( \| u^n_1 \|_h^2 + \| u^n_0 \|_\Delta^2 + \| \theta^n_0 \|^2 \right), \tag{2.15} \]
where

\[ u^n = \sum_{j=1}^{m} u_j \varphi_j \quad \text{and} \quad \theta^n_0 = \sum_{j=0}^{m} \theta_{0j} \psi_j. \]

Since \( u_0 \in H^2_0(\Omega) \), \( u_1 \in H^1(\Omega) \) and \( \theta_0 \in L^2(\Omega) \), we have

\[ \lim_{m \to \infty} \| u^n - u_i \|_{2-i} = 0, \quad i = 0, 1, \quad \text{and} \quad \lim_{m \to \infty} \| \theta_0^n - \theta_0 \| = 0, \quad (2.16) \]

hence (2.15) implies that

\[ e((u^n, \theta^n), t) + \beta \int_0^t \| \nabla \theta^n(s, \cdot) \|^2 \, ds \leq c_4, \quad \forall m \geq 1, \quad (2.17) \]

with a suitable constant \( c_4 > 0 \).

Differentiating (2.12) and (2.13) once with respect to \( t \) and multiplying the resulting formulae by \( u_i^{m'}(t) \) and \( \theta_i^{m'} \) we have

\[ e((u^n, \theta^n), t) + \beta \int_0^t \| \nabla \theta^n(s, \cdot) \|^2 \, ds = \frac{1}{2} \left\{ \| u^n(0, \cdot) \|^2 + \| u^n_1 \|^2 + \| \theta^n(0, \cdot) \|^2 \right\}. \quad (2.18) \]

By using (2.7), we shall show that

\[ \lim_{m \to \infty} \| u^n_0(0, \cdot) - u_2 \|_{H} = 0 \quad \text{and} \quad \lim_{m \to \infty} \| \theta^n_0(0, \cdot) - \theta_1 \| = 0, \quad (2.19) \]

which together with (2.18) implies that

\[ e((u^n, \theta^n), t) + \beta \int_0^t \| \nabla \theta^n(s, \cdot) \|^2 \, ds \leq c_5, \quad \forall m \geq 1, \quad (2.20) \]

for a suitable constant \( c_5 > 0 \). To show (2.19), we shall use the following two formulae:

\[ \| u^n_0(0, \cdot) \|^2 - (\nabla u^n_0(0, \cdot), \nabla u^n_0) - \alpha(\nabla u^n_0(0, \cdot), \nabla \theta^n_0) = 0, \quad (2.21) \]

\[ \| \theta^n_0(0, \cdot) \|^2 - \beta(\theta^n_0(0, \cdot), \Delta \theta^n_0) - \alpha(\theta^n_0(0, \cdot), \Delta u^n_0) = 0, \quad (2.22) \]

which are obtained by the multiplication of (2.12) with \( t = 0 \) by \( u^n_0(0) \) and the multiplication of (2.13) with \( t = 0 \) by \( \theta^n_0(0) \). By (2.7) we know that

\[ \lim_{m \to \infty} \| u^n_0 - u_0 \|_{H} = 0, \quad \lim_{m \to \infty} \| u^n_1 - u_1 \|_{2} = 0, \quad \lim_{m \to \infty} \| \theta^n_0 - \theta_0 \|_{2} = 0, \quad (2.23) \]

which together with (2.21) and (2.22) implies the boundedness of \( \{ u^n_0(0, \cdot) \} \) in \( H^1(\Omega) \) and \( \{ \theta^n_0(0, \cdot) \} \) in \( L^2(\Omega) \), hence passing to a subsequence if necessary, we see that \( u^n_0(0, \cdot) \) and \( \theta^n_0(0, \cdot) \) converge to some \( u_2 \in H^1(\Omega) \) and \( \theta_1 \in L^2(\Omega) \) in the weak sense, respectively. By (2.21), the boundedness of \( \{ u^n_0(0, \cdot) \} \) in \( H^1(\Omega) \), (2.23) and the weak convergence of \( \{ u^n_0(0, \cdot) \} \) in \( H^1(\Omega) \) to \( u_2 \), we have

\[ \lim_{m \to \infty} \| u^n_0(0, \cdot) \|^2 = (\nabla u_2, \nabla u_0) + \alpha(\nabla u_2, \nabla \theta_0). \quad (2.24) \]

On the other hand, passing (2.12) with \( t = 0 \) to the limit as \( m \to \infty \), we have

\[ (\varphi_i, u_2)_H = (\nabla \varphi_i, \Delta u_0) + \alpha(\nabla \varphi_i, \nabla \theta_0), \quad \forall i \geq 1, \]
which together with (2.24) implies that

$$\lim_{m \to \infty} \|u'''_m(0, \cdot)\|_H^2 = \|u_2\|_H^2,$$

because we can write

$$u_2 = \sum_{i=1}^{\infty} (\varphi_i, u_2) \varphi_i$$

and the summation converges in the strong topology of $H^1(\Omega)$. Therefore, we have the first part of (2.19). Similarly, by (2.22) and (2.13) with $t = 0$ we can deduce the second part of (2.19). Since any bounded set in a Hilbert space in the strong topology is also a relatively compact set in the weak topology, passing to a subsequence if necessary, by (2.17) and (2.20) we see that there exist $u_t, u \in L^2_{loc}(0, \infty; H^2_0(\Omega)), u_{tt} \in L^2_{loc}(0, \infty; H^2_0(\Omega))$ and $t, \theta \in L^2((0, \infty); H^1_0(\Omega))$ such that

$$\lim_{m \to \infty} \int_0^t (\mathcal{C}^l(u^m(s, \cdot) - u(s, \cdot)), f(s, \cdot))_H ds = 0, \quad \forall f \in L^2((0, t); H^2_0(\Omega)), \quad t > 0$$

for any $t > 0$ where $l = 0$ and 1. For any $g \in C^{1}_0([0, \infty); H^2_0(\Omega))$, choosing $\eta \in C^{1}_0([0, \infty); H^2_0(\Omega))$ and $\zeta \in C^{1}_0([0, \infty); H^2_0(\Omega))$ so that $-\Delta \eta = g$ and $(1 - h\Delta)\zeta = \Delta g$ in $\Omega$ for any $t \geq 0$, by (2.25) we have

$$\lim_{m \to \infty} \int_0^t (\mathcal{C}^l(u^m(s, \cdot) - \theta(s, \cdot)), \nabla \eta(s, \cdot)) ds = 0,$$

$$\lim_{m \to \infty} \int_0^t (u'''_m(s, \cdot) - u_{ss}(s, \cdot), \phi(s, \cdot))_H ds = 0,$$

which together with (2.25), (2.12)–(2.14) and (2.16) implies that $u$ and $\theta$ satisfy (2.1) and (2.2), because for $f \in C^{1}_0([0, \infty); H^2_0(\Omega))$ and $g \in C^{1}_0([0, \infty); H^2_0(\Omega))$ we have

$$\frac{\partial^k}{\partial t^k} f(t, x) = \sum_{j=1}^{\infty} \frac{d^k}{dt^k} (f(t, \cdot), \varphi_j) \varphi_j(x) \quad \text{and}$$

$$\frac{\partial^l}{\partial t^l} g(t, x) = \sum_{j=1}^{\infty} \frac{d^l}{dt^l} (g(t, \cdot), \psi_j) \psi_j(x),$$

where $k = 0, 1, 2$, $l = 0, 1$. The first summation converges in the strong topology of $H^2(\Omega)$ uniformly in $t \in [0, \infty)$ and the second does in the strong topology of $H^1(\Omega)$. Since

$$u^m(t, x) - u'''_0(x) = \int_0^t u'''_m(s, x) ds, \quad u'''_t(t, x) - u'''_m(x) = \int_0^t u'''_m(s, x) ds,$$

$$\theta^m(t, x) - \theta'''_0(x) = \int_0^t \theta'''_m(s, x) ds,$$

(2.26)
Our task is to prove that \( \chi_{(0,t)}(s) \varphi(x) \) and \( \chi_{(0,t)}(s) \psi(x) \) in (2.25) where \( \chi_{(0,t)}(s) \) is the characteristic function of the interval \((0,t)\), using (2.25), passing (2.26) to the limit as \( m \to \infty \) and changing the values of \( u(t,x) \), \( u_i(t,x) \) and \( \theta(t,x) \) on a set of measure zero if necessary, we can deduce that

\[
(u(t,\cdot) - u_0, \varphi)_\lambda = \int_0^t (u_i(s,\cdot), \varphi)_\lambda \, ds, \quad \forall \varphi \in H^2_{00}(\Omega),
\]

\[
(u_i(t,\cdot) - u_1, \varphi)_h = \int_0^t (u_{ss}(s,\cdot), \varphi)_h \, ds, \quad \forall \varphi \in H^1_0(\Omega),
\]

\[
(\theta(t,\cdot) - \theta_0, \psi) = \int_0^t (\theta_s(s,\cdot), \psi) \, ds, \quad \forall \psi \in L^2(\Omega),
\]

for all \( t \geq 0 \), which immediately implies that \( u \) and \( \theta \) satisfy (1.4), (1.5) and (1.6). Therefore, we can prove the existence of weak solutions under the assumption (2.7).

To remove the additional regularity assumption (2.7), we use the fact that \( C^0_0(\Omega) \) is dense in \( H^2_{00}(\Omega) \), \( H^1_0(\Omega) \) and \( L^2(\Omega) \). Namely, given \( u_0 \in H^2_{00}(\Omega) \), \( u_1 \in H^1_0(\Omega) \) and \( \theta_0 \in L^2(\Omega) \), let us choose sequences \( \{u_p^0\}, \{u_1^p\} \) and \( \{\theta_p^0\} \) in \( C^0_0(\Omega) \) such that \( \|u_p^0 - u_0\|_2 \to 0 \), \( \|u_1^p - u_1\|_1 \to 0 \) and \( \|\theta_p^0 - \theta_0\| \to 0 \) as \( p \to \infty \). Let \( u^p(t,x) \) and \( \theta^p(t,x) \) be weak solutions corresponding to the initial data \( u_0^p, u_1^p \) and \( \theta_0^p \), the existence of which was already proved. Considering the difference \( u^p - u^q \) and \( \theta^p - \theta^q \) and applying Lemma 2.2, we see that \( \{u^p\} \) and \( \{\theta^p\} \) are Cauchy sequences in \( C^0(\Omega; C^1([0,\infty); H^1_0(\Omega))) \) and \( C^0([0,\infty); L^2(\Omega)) \) respectively. This immediately implies the existence of desired weak solutions, which completes the proof of the existence of weak solutions.

Now, we shall show (2.8) and (2.9) under the assumption (2.7). Let \( u_2 \in H^1_0(\Omega) \) and \( \theta_1 \in L^2(\Omega) \) be the same as in (2.19). In particular, passing (2.12) and (2.13) with \( t = 0 \) to the limit as \( m \to \infty \) and using (2.14), (2.16) and (2.19), we have

\[
(u_2, \varphi)_h + (u_0, \varphi)_\lambda - \alpha(\nabla \theta_0, \nabla \varphi) = 0,
\]

\[
(\theta_1, \psi) + \beta(\nabla \theta_0, \nabla \psi) + \alpha(\nabla u_2, \nabla \psi) = 0
\]

for any \( \varphi \in H^2_{00}(\Omega) \) and \( \psi \in H^1_0(\Omega) \). Since \( u_1 \in H^2_{00}(\Omega) \), \( u_2 \in H^1_0(\Omega) \) and \( \theta_1 \in L^2(\Omega) \), we already knew the existence of \( v \in C^0([0,\infty); H^2_{00}(\Omega)) \cap C^1([0,\infty); H^1_0(\Omega)) \) and \( \kappa \in C^0([0,\infty); L^2(\Omega)) \cap L^2((0,\infty); H^1_0(\Omega)) \) which satisfy (2.1) and (2.2), replacing \( u_1 \) by \( u_2 \) and \( \theta_0 \) by \( \theta_1 \), and the initial condition: \( v(0,x) = u_1(x), \quad v_t(0,x) = u_2(x) \) and \( \kappa(0,x) = \theta_1(x) \). Put

\[
w(t,x) = u_0(x) + \int_0^t v(s,x) \, ds \quad \text{and} \quad \mu(t,x) = \theta_0(x) + \int_0^t \kappa(s,x) \, ds.
\]

Our task is to prove that \( w = u \) and \( \mu = \theta \), which together with the fact that \( v \) and \( \kappa \) satisfy (1.5) and (1.6) implies

\[
u \in C^1([0,\infty); H^2_{00}(\Omega)) \cap C^2([0,\infty); H^1_0(\Omega)),
\]

\[
\theta \in C^1([0,\infty); L^2(\Omega)), \quad \theta_t \in L^2((0,\infty); H^1_0(\Omega)).
\]
Since \( w = v \) and \( \mu = k \), by (2.1) and (2.2) we have
\[
-[w_t, F_t] - h[\nabla w_t, \nabla F_t] + [\Delta w_t, \Delta F] - z[\nabla \mu, \nabla F] = (u_2, F(0, \cdot))_h,
\]
\[
\left[ \mu_t, - G_t \right] + \beta[\nabla \mu, \nabla G] + z[\nabla w_t, \nabla G] = (\theta_1, G(0, \cdot))
\]
for any \( F \in C^2_{\beta}(0, \infty); H^2_{\beta}(\Omega) \) and \( G \in C^1_{\beta}(0, \infty); H^1_{\beta}(\Omega) \). Putting
\[
F(t, x) = -\int_0^t f(s, x) \, ds \quad \text{and} \quad G(t, x) = -\int_0^t g(s, x) \, ds
\]
for any given \( f \in C^2_{\beta}(0, \infty); H^2_{\beta}(\Omega) \) and \( g \in C^1_{\beta}(0, \infty); H^1_{\beta}(\Omega) \), by integration by parts we have
\[
-[w_t, f_t] - h[\nabla w_t, \nabla f_t] + [\Delta w, \Delta f] - z[\nabla \mu, \nabla f] = (u_1, f(0, \cdot))_h,
\]
\[
[\mu_t, - g_t] + \beta[\nabla \mu, \nabla g] + z[\nabla w_t, \nabla g] = (\theta_0, g(0, \cdot)),
\]
because \( w(0, x) = u_0(x), w_t(0, x) = u_1(x), \mu(0, x) = \theta_0(x), F_t = f, G_t = g, \)
\[
(u_2, F(0, \cdot))_h + (u_0, F(0, \cdot))_h - \Delta(\nabla \theta_0, \nabla G(0, \cdot)) = 0,
\]
\[
(\theta_1, G(0, \cdot)) + \beta(\nabla \theta_0, \nabla G(0, \cdot)) + z(\nabla u_1, \nabla G(0, \cdot)) = 0.
\]
The final two facts follow from (2.28). Therefore, by the uniqueness of the weak solutions guaranteed by Lemma 2.2 we have \( u = w \) and \( \theta = \mu \), and then \( u \) and \( \theta \) satisfy (2.29) and (2.30).

Passing (2.3) and (2.4) to the limit as \( \varepsilon \to 0 \) and using (1.5), (1.6), (2.29) and (2.30), we have
\[
(u_n(t, \cdot), \varphi) + h(\nabla u_n(t, \cdot), \nabla \varphi) + (\Delta u(t, \cdot), \Delta \varphi) + z(\theta(t, \cdot), \Delta \varphi) = 0, \tag{2.31}
\]
\[
(\theta(t, \cdot), \psi) - \beta(\theta(t, \cdot), \Delta \psi) - z(\Delta u(t, \cdot), \psi) = 0 \tag{2.32}
\]
for any \( t \geq 0, \varphi \in H^2_{\beta}(\Omega) \) and \( \psi \in H^2_{\beta}(\Omega) \). Since \( \{\Delta v | v \in H^2_{\beta}(\Omega)\} = L^2(\Omega) \) and since \( \theta(t, \cdot) - \Delta u(t, \cdot) \in C^0([0, \infty); L^2(\Omega)) \) as follows from (2.29) and (2.30), (2.32) implies that
\[
\theta \in C^0([0, \infty); H^2_{\beta}(\Omega)), \quad \| \theta(t, \cdot) \|_2 \leq C \{ \| \dot{\theta}(t, \cdot) \| + \| u(t, \cdot) \|_2 \}. \tag{2.33}
\]
According to Theorem A.1 in the appendix below, we know that
\[
\{\Delta^2 v | v \in H^3_{\beta}(\Omega)\} = H^{-1}(\Omega), \quad \| v \|_3 \leq C \| \Delta^2 v \|_{-1} \text{ for } v \in H^3_{\beta}(\Omega). \tag{2.34}
\]
Since \( u_t - h\Delta u_t + z\Delta \theta \in C^0([0, \infty); H^{-1}(\Omega)) \) as follows from (2.30) and (2.33), by (2.31) and (2.34) we have
\[
u \in C^0([0, \infty); H^3_{\beta}(\Omega)), \quad \| u(t, \cdot) \|_3 \leq C \{ \| u_n(t, \cdot) \|_1 + \| \theta(t, \cdot) \|_1 \}, \tag{2.35}
\]
which completes the proof of the theorem.

Now, we are going to prove the existence of strong solutions. To do this, let us denote by
\[
u_j(x) = \partial_j^l u(0, x), \quad 0 \leq j \leq 3 \quad \text{and} \quad \theta_j(x) = \partial_j^l \theta(0, x), \quad j = 0, 1.
\]
So, the compatibility conditions imply that
\[ \theta_1 = \beta \Delta \theta_0 + \alpha \Delta u_1, \quad (2.36) \]
\[ u_{2+j} - h \Delta u_{2+j} + \Delta^2 u_j + \alpha \Delta \theta_j = 0, \quad j = 0, 1. \quad (2.37) \]
Moreover, we have that
\[ u_j \in H^{3-j}_0(\Omega), \quad 0 \leq j \leq 2, \quad u_3 \in H^1_0(\Omega), \quad \theta_j \in H^{3-j}_0(\Omega), \quad j = 0, 1. \quad (2.38) \]
Note that when \( h = 0 \) the above regularity easily follows for \( u_2 \) under the restriction:
\[ \Delta^2 u_0 + \alpha \Delta \theta_0 \in H^{3}_0(\Omega). \]

Obviously, for \( h > 0 \), it is not true that for any \( u_0 \in H^{4}_{00}(\Omega), \ u_1 \in H^{2}_{00}(\Omega), \ \theta_0 \in H^{3}_{0}(\Omega), \ \theta_1, u_2 \) and \( u_3 \) always satisfy (2.37) and (2.38). Therefore, let us introduce the set \( \mathcal{D} \) of all \( u_0 \in H^{4}_{00}(\Omega), \ u_1 \in H^{2}_{00}(\Omega) \) and \( \theta_0 \in H^{3}_{0}(\Omega) \) for which (2.37) and (2.38) hold. \( \mathcal{D} \) contains plenty of elements. For example, given \( \theta_1 \in H^{2}_{0}(\Omega) \) and \( u_1 \in H^{2}_{00}(\Omega), \ let \ \theta_0 \in H^{3}_{0}(\Omega) \) be a solution of the boundary value problem: \( \beta \Delta \theta_0 = \theta_1 - \alpha \Delta u_1 \in H^2(\Omega) \) and \( \theta_0 = 0 \) on \( \partial \Omega \). Given \( u_2 \in H^{2}_{00}(\Omega), \ let \ u_0 \in H^{3}_{0}(\Omega) \) be a solution of the boundary value problem:
\[ \Delta^2 u_0 = -(u_2 - h \Delta u_2 + \alpha \Delta \theta_0) \in L^2(\Omega), \ u_0 = \partial u_0/\partial v = 0 \ on \ \partial \Omega, \ \text{the existence of such} \ u_0 \ \text{being guaranteed by Theorem A.1 in the appendix below. If we consider the coercive bilinear form,} \ (v, w) + h(\nabla v, \nabla w) \ \text{on} \ H^{2}_{0}(\Omega) \times H^0_{0}(\Omega), \ \text{then there exists a} \ u_3 \in H^{2}_{0}(\Omega) \ \text{satisfying the relation} \ u_3 - h \Delta u_3 = -(\Delta^2 u_1 + \alpha \Delta \theta_1) \in H^{-1}(\Omega). \ \text{This is one of the ways of constructing the elements of} \ \mathcal{D}. \]

**Theorem 2.4.** For any \((u_0, u_1, \theta_0) \in \mathcal{D}, \ there \ exists \ a \ unique \ pair \ of \ strong \ solutions \ u \ and \ \theta \ \text{of} \ (1.1)-(1.4). \]

**Proof.** Let \( v \) and \( \kappa \) be weak solutions of (1.1)–(1.3) satisfying (2.8), (2.9) and the initial condition,
\[ v(0, x) = u_1(x) \in H^3_0(\Omega), \quad v_t(0, x) = u_2(x) \in H^{2}_{00}(\Omega), \quad \kappa(0, x) = \theta_1(x) \in H^2_0(\Omega), \]
the existence of \( v \) and \( \kappa \) being guaranteed by Theorem 2.3, and put
\[ u(t, x) = u_0(x) + \int_0^t v(s, x) \, ds \quad \text{and} \quad \theta(t, x) = \theta_0(x) + \int_0^t \kappa(s, x) \, ds. \]

Then, by employing the same argument as in the final part of the proof of Theorem 2.3, we see that \( u \) and \( \theta \) satisfy (1.4), (2.1) and (2.2). Obviously, by (2.8) and (2.9)
\[ u \in C^1([0, \infty); H^3_0(\Omega)) \cap C^2([0, \infty); H^{3}_0(\Omega)) \cap C^3([0, \infty); H^{3}_0(\Omega)), \]
\[ \theta \in C^1([0, \infty); H^2_0(\Omega)) \cap C^2([0, \infty); L^2(\Omega)), \quad \theta_t \in L^2((0, \infty); H^2_0(\Omega)). \]

Note that \( u \) and \( \theta \) satisfy (2.31) and (2.32). Since \( \theta_1 - \alpha \Delta u_1 \in C^0([0, \infty); H^1(\Omega)) \) and \( \{\Delta v, v \in H^0_0(\Omega)\} = \mathcal{H}_1(\Omega), \) by (2.32) we see that \( \theta \in C^0([0, \infty); H^3_0(\Omega)). \) Similarly, since \( u_t - h \Delta u_t - \alpha \Delta \theta \in C^0([0, \infty); L^2(\Omega)) \) and \( \{\Delta^2 v, v \in H^0_0(\Omega)\} = L^2(\Omega) \) (see Theorem A.1 in the appendix below). From (2.31) we see that \( u \in C^0([0, \infty); H^4_{00}(\Omega)), \) which completes the proof of the theorem.

**Remark.** We can extend Theorem 2.4 to a more regular class of solutions. Inductively we can say that the initial data \( u_i, \theta_i \) and \( k \)-regular if
\[ u_j \in H^{3-j}_0(\Omega) \quad \text{for} \ j = 1, \ldots, k - 2, \quad u_{k} \in H^2_0(\Omega), \]
\[ \theta_j \in H^{3-j}_0 \quad \text{for} \ j = 1, \ldots, k - 2. \]

---

and there exists a solution for the iterated equations:
\[ \theta_i = \beta \Delta \theta_{i-1} + \alpha \Delta u_i \quad \text{for} \quad i = 1, \ldots, k, \]
\[ u_{2+j} - h \Delta u_{2+j} + \Delta^2 u_j + \alpha \Delta \theta_j = 0, \quad j = 0, \ldots, k - 2. \]
So, using the same argument as in Theorem 2.4, we are able to prove that the solutions satisfy
\[ u \in C^i([0, T]; H_h^{k-j}), \quad j = 0, \ldots, k, \quad \theta \in C^i([0, T]; H_h^{k-1-j}), \quad j = 0, \ldots, k - 1. \]

3. Exponential decay of the energy

Our starting point is to establish the energy identity for weak solutions

**Theorem 3.1.** Given \( u_0 \in H_0^3(\Omega), \ u_1 \in H_0^{10}(\Omega) \) and \( \theta_0 \in H_0^2(\Omega), \) let \( u \) and \( \theta \) be weak solutions of (1.1)–(1.4) satisfying (2.8) and (2.9). Then, there exist constants \( M \) and \( \gamma > 0 \) such that

\[ e^{2\gamma s} E_2((u, \theta), t) + \int_0^t e^{2\gamma s} (E_2((u, \theta), s) + \| \nabla \theta_s(s, \cdot) \|^2) \, ds \leq ME_2((u, \theta), 0), \quad (3.1) \]

where

\[ E_2((u, \theta), t) = \| u_t(t, \cdot) \|^2 + \| u(t, \cdot) \|^2 + \| \theta(t, \cdot) \|^2 + \| \theta_t(t, \cdot) \|^2. \]

**Remark.** Since \( u_t(t, \cdot) \in H_0^1(\Omega) \) and \( \theta(t, \cdot) \in H_0^2(\Omega) \) for \( t \geq 0, \) in view of (2.31) and (2.32) with \( t = 0, \) by integration by parts we see easily that

\[ \| \theta_t(0, \cdot) \| \leq C \{ \| \theta_0 \|_2 + \| u_1 \|_2 \} \quad \text{and} \quad \| u_t(0, \cdot) \|_1 \leq C \{ \| u_0 \|_3 + \| \theta_0 \|_1 \}, \]

which implies that

\[ E_2((u, \theta), 0) \leq c \{ \| u_0 \|^2 + \| u_1 \|^2 + \| \theta_0 \|^2 \}. \quad (3.2) \]

**Proof.** Let \( u_\varepsilon \) and \( \theta_\varepsilon \) be the same as in the proof of Lemma 2.2 (cf. (2.5)). Since \( (u, \theta) \) satisfies the regularity conditions (2.8) and (2.9), to show inequality (3.1) it is enough to show that

\[ e^{2\gamma s} E_2((u_\varepsilon, \theta_\varepsilon), t) + \int_0^t e^{2\gamma s} (E_2((u_\varepsilon, \theta_\varepsilon), s) + \| \nabla \theta_{ts}(s, \cdot) \|^2) \, ds \leq ME_2((u_\varepsilon, \theta_\varepsilon), 0); \]

our conclusion will follow letting \( \varepsilon \to 0. \) To simplify notations, we write \( u = u_\varepsilon \) and \( \theta = \theta_\varepsilon. \) Under these conditions we have that \( u \) and \( \theta \) satisfy

\[ u \in C^\infty([0, \infty); H_0^3(\Omega)) \quad \text{and} \quad \theta \in C^\infty([0, \infty); H_0^2(\Omega)). \]

Moreover, in view of (2.3) and (2.4), (1.1) holds in the sense of \( H^{-1}(\Omega) \) and (1.2) holds in the sense of \( L^2(\Omega) \) for all \( t \geq 0. \) Put \( \varphi = u_{tr} \) and \( \psi = \theta_\varepsilon \) in (2.3) and (2.4), we have

\[ \frac{d}{dt} e((u, \theta), t) + \beta \| \nabla \theta(t, \cdot) \|^2 = 0, \quad (3.3) \]
where \( e((u, \theta), t) \) is the same as in Lemma 2.2. Differentiating (2.3) and (2.4) once with respect to \( t \), we have also
\[
\frac{d}{dt} e((u, \theta), t) + \beta \| \nabla \theta(t, \cdot) \|^2 = 0. \tag{3.4}
\]
Multiplying (3.3) and (3.4) by \( e^{2\gamma t} \) and integrating the resulting formulae, we have
\[
e^{2\gamma t} F_2(t) + \beta \int_0^t e^{2\gamma s} \left( \| \nabla \theta(s, \cdot) \|^2 + \| \nabla \theta(s, \cdot) \|^2 \right) ds = F_2(0) + 2\gamma \int_0^t e^{2\gamma s} F_2(s) ds, \tag{3.5}
\]
where
\[
F_2(t) = e((u, \theta), t) + e((u, \theta), t).
\]
By (2.33), (2.35) and (2.10), we see that
\[
F_2(t) \leq E_2((u, \theta), t) \leq C F_2(t). \tag{3.6}
\]
From now on, we use the letter \( C \) to denote various constants. For notational simplicity, below we write \( E_2(t) = E_2((u, \theta), t) \). By (3.5) and (3.6), we have
\[
e^{2\gamma t} E_2(t) + \int_0^t e^{2\gamma s} \left( \| \nabla \theta(s, \cdot) \|^2 + \| \nabla \theta(s, \cdot) \|^2 \right) ds
\[
\leq C \left\{ E_2(0) + \gamma \int_0^t e^{2\gamma s} E_2(s) ds \right\}. \tag{3.7}
\]
Our problem reduces to show
\[
\int_0^t e^{2\gamma s} E_2(s) ds \leq CM_\gamma(t), \tag{3.8}
\]
where
\[
M_\gamma(t) = e^{2\gamma t} E_2(t) + E_2(0) + \int_0^t e^{2\gamma s} \left( \| \nabla \theta(s, \cdot) \|^2 + \| \nabla \theta(s, \cdot) \|^2 \right) ds
\[
+ \gamma \int_0^t e^{2\gamma s} E_2(s) ds.
\]
In fact, if inequality (3.8) holds, then combining (3.7) and (3.8) implies that
\[
e^{2\gamma t} E_2(t) + \int_0^t e^{2\gamma s} \left( E_2(s) + \| \nabla \theta(s, \cdot) \|^2 \right) ds
\[
\leq C \left\{ E_2(0) + \gamma \int_0^t e^{2\gamma s} E_2(s) ds \right\}.
\]
Choosing \( \gamma > 0 \) so small that \( C \gamma \leq \frac{1}{2} \), we have (3.1), which completes the proof of the theorem. To get (3.8), we start estimating the term \( \| \Delta u \|^2 \). Since (1.2) holds in the
sense of $L^2(\Omega)$ for all $t \geq 0$, we have
\[
x\|\Delta u_t\|^2 = (\theta_t - \beta \Delta \theta, \Delta u_t)
\]
\[
= (\theta_t, \Delta u_t) - \beta \left( \frac{\partial \theta}{\partial v}, \Delta u_t \right) + \beta \frac{d}{dt} (\nabla \theta, \nabla u_t) - \beta (\nabla \theta_t, \Delta u_t).
\]
(3.9)

Here, we have put
\[
(u, v)_{\Omega} = \int_{\Omega} u(x)v(x) \, dx,
\]
do being the surface element of $\partial \Omega$.

Put $\langle u \rangle^0 = \langle u, u \rangle_{\Omega}$. Since $u_t \in C^\infty([0, \infty); H^3_{00}(\Omega))$, by Lemma A.2 in the appendix below we see that
\[
\langle \Delta u_t \rangle^0 \leq 2 \left( \nabla \Delta u_t, \nabla \frac{\partial}{\partial N} u_t \right) + C \| \Delta u_t \|^2
\]
\[
= 2 \frac{d}{dt} \left( \nabla \Delta u, \nabla \frac{\partial}{\partial N} u_t \right) - \left( \nabla \Delta u, \nabla \frac{\partial}{\partial N} u_t \right) + C \| \Delta u_t \|^2.
\]
(3.10)

Here and hereafter, $\partial / \partial N = \sum_{j=1}^{n} N_j(x) \partial / \partial x_j$ and each $N_j(x)$ is a function in $C^\infty(\bar{\Omega})$ such that $N_j(x) = v_j(x)$, $x \in \partial \Omega$ (that is, $N_j$ is a $C^\infty$ extension of $v_j$ to whole $\Omega$). Since $u(t, \cdot), u_t(t, \cdot) \in H^3_{00}(\Omega)$ and $\theta(t, \cdot) \in H^2_{00}(\Omega)$ for any $t \geq 0$, it follows from (2.3) and the fact that $C^\infty(\Omega) \subset H^3_{00}(\Omega)$ and $C^\infty(\Omega)$ is dense in $H^2_{00}(\Omega)$ that
\[
(u_t(t, \cdot), \varphi) - h(\Delta u_t(t, \cdot), \varphi) - (\nabla \Delta u(t, \cdot), \nabla \varphi) + \alpha(\Delta \theta(t, \cdot), \varphi) = 0
\]
for any $\varphi \in H^1_{00}(\Omega)$. Therefore, since $\partial u_t / \partial N \in H^1_{00}(\Omega)$, we have
\[
\left( \nabla \Delta u, \nabla \frac{\partial}{\partial N} u_t \right) = (u_t - h \Delta u_t + \alpha \Delta \theta, \frac{\partial}{\partial N} u_t).
\]
(3.11)

Since
\[
\left| \left( v, \frac{\partial}{\partial N} u \right) \right| \leq C \| u \|^2 \quad \forall v \in H^1_{00}(\Omega),
\]
(3.12)

and observing that
\[
(\Delta u_t, \frac{\partial}{\partial N} u_t) = - (\nabla u_t, \nabla \frac{\partial}{\partial N} u_t) - (\nabla u_t, \frac{\partial}{\partial N} \nabla u_t),
\]
we have that (3.12) implies
\[
\left| u_t - h \Delta u_t, \frac{\partial}{\partial N} u_t \right| \leq C \| u_t \|^2,
\]
(3.13)

hence combining (3.10), (3.11) and (3.13) we get
\[
\langle \Delta u_t \rangle^0 \leq 2 \frac{d}{dt} \left( \nabla \Delta u, \nabla \frac{\partial}{\partial N} u_t \right) + CE_2(t).
\]
(3.14)

Since
\[
\left| \left( \nabla \Delta u, \nabla \frac{\partial}{\partial N} u \right) \right| \leq CE_2(t),
\]

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We have
\[
\left| \int_0^t e^{2ts} \left( \nabla \Delta u(s, \cdot), \nabla \frac{\partial}{\partial N} u_s(s, \cdot) \right) ds \right| \leq CM(t).
\] (3.15)

In fact, for any function \( f \) such that \( |f(t)| \leq CE_2(t) \), we have
\[
\left| \int_0^t e^{2ts} \frac{d}{ds} f(s) ds \right| = \left| e^{2ts} f(t) - f(0) - 2\gamma \int_0^t e^{2ts} f(s) ds \right| \leq CM(t).
\] (3.16)

Let \( \varepsilon > 0 \) be a small positive constant; then it follows that
\[
\left| \left( \frac{\partial \theta}{\partial v}, \Delta u \right)_{\varepsilon\Omega} \right| \leq \varepsilon \langle \Delta u \rangle_0^2 + C(\varepsilon) \left( \frac{\partial \theta}{\partial v} \right)_0^2,
\] (3.17)
where \( C(\varepsilon) \) denotes various constants depending on \( \varepsilon \) which may blow up when \( \varepsilon \to 0 \). The well-known boundary estimate
\[
\langle v \rangle_0^2 \leq C \| v \|_1 \| v \| \quad \forall v \in H^1(\Omega)
\]
implies that
\[
C(\varepsilon) \left( \frac{\partial \theta}{\partial v} \right)_0^2 \leq \frac{\varepsilon}{2} \left\| \frac{\partial \theta}{\partial N} \right\|_1^2 + \frac{C^2 C(\varepsilon)^2}{2\varepsilon} \left\| \frac{\partial \theta}{\partial N} \right\|_1^2,
\]
which together with (3.17) implies that
\[
\left| \left( \frac{\partial \theta}{\partial v}, \Delta u \right)_{\varepsilon\Omega} \right| \leq C \varepsilon \{ \| \theta \|_2^2 + \langle \Delta u \rangle_0^2 \} + C(\varepsilon) \| \theta \|_1.
\] (3.18)

It follows from (3.14), (3.15) and (3.18) that
\[
\frac{\beta}{\varepsilon} \int_0^t e^{2ts}\left( \frac{\partial \theta}{\partial v}(s, \cdot), \Delta u_s(s, \cdot) \right)_{\varepsilon\Omega} ds \leq C \varepsilon \int_0^t e^{2ts} E_2(s) ds + C(\varepsilon)M_s(t).
\] (3.19)

Since \( |\nabla \theta, \nabla \Delta u| \leq CE_2(t) \) and since
\[
\frac{1}{\varepsilon} \left( \nabla \theta, \nabla \Delta u \right) \leq \varepsilon E_2(t) + C(\varepsilon) \| \nabla \theta \|_2^2,
\]
by (3.9), (3.16) and (3.19) we get
\[
\int_0^t e^{2ts} \| u_s(s, \cdot) \|_2^2 ds \leq C \varepsilon \int_0^t e^{2ts} E_2(s) ds + C(\varepsilon)M_s(t).
\] (3.20)

Using equation (1.2) and inequalities (2.10), (3.20), we arrive at
\[
\int_0^t e^{2ts} \| \theta(s, \cdot) \|_2^2 ds \leq C \varepsilon \int_0^t e^{2ts} E_2(s) ds + C(\varepsilon)M_s(t).
\] (3.21)

Putting \( \phi = u_{tt} \) in (2.3), we have
\[
\| u_{tt} \|^2 + \rho \| \nabla u_{tt} \|^2 + \frac{d}{dt} \left( \Delta u, \Delta u_t \right) - \| \Delta u_t \|^2 - \alpha(\nabla \theta, \nabla u_{tt}) = 0,
\]
which together with (3.20), (3.21) and (3.16) implies that
\[
\int_0^t e^{2 \gamma s} \| u_2(s, \cdot) \|^2 \, ds \leq C \varepsilon \int_0^t e^{2 \gamma s} E_2(s) \, ds + C(\varepsilon) M_j(t),
\]
(3.22)
because \(|(\Delta u, \Delta u)| \leq C E_2(t)|. Finally, using (2.35) and (3.22), we find that
\[
\int_0^t e^{2 \gamma s} \| u(s, \cdot) \|^2 \, ds \leq C \varepsilon \int_0^t e^{2 \gamma s} E_2(s) \, ds + C(\varepsilon) M_j(t).
\]
(3.23)
Combining (3.20), (3.21), (3.22) and (3.23), it follows that
\[
\int_0^t e^{2 \gamma s} E_2(s) \, ds \leq C \varepsilon \int_0^t e^{2 \gamma s} E_2(s) \, ds + C(\varepsilon) M_j(t).
\]
(3.24)
Choosing \( \varepsilon > 0 \) so small that \( C \varepsilon \leq \frac{1}{2} \) in (3.24), we have (3.8), which completes the proof of the theorem.

**Proof of Theorem 1.1.** The unique existence of weak solutions has been proved by Theorem 2.3, so that we shall prove (1.7) only. To prove (1.7), we shall use Theorem 3.1. To do this, we shall represent \( u = v, \theta = \kappa, \) Put \( v_1 = u_0 \in H_0^1(\Omega), v_2 = u_1 \in H_0^1(\Omega) \) and \( \kappa_1 = \theta_0 \in L^2(\Omega), \) and then let \( \kappa_0 \in H_0^1(\Omega) \) be a solution of the elliptic boundary value problem: \(-\beta \Delta \kappa_0 = \alpha \Delta v_1 - \kappa_1 \in L^2(\Omega) \) and \( \kappa_0 = 0 \) on \( \partial \Omega. \) And also, let \( v_0 \in H_0^1(\Omega) \) be a solution of the elliptic boundary value problem:
\[
\Delta^2 v_0 = -(1-h\Delta)v_2 - \alpha \Delta v_0 \in H^{-1}(\Omega), \quad v_0 = (\partial/\partial \nu)v_0 = 0 \quad \text{on} \quad \partial \Omega,
\]
the existence of such a \( v_0 \) being guaranteed by Theorem A.1 in the appendix below. We know that
\[
\| v_0 \|_3 + \| v_1 \|_2 + \| \kappa_0 \|_2 \leq C \{ \| u_0 \|_2 + \| u_1 \|_1 + \| \theta_0 \| \}.
\]
(3.25)
By Theorem 2.3, we know the existence of weak solutions \( v \) and \( \kappa \) satisfying (2.8),(2.9) and the initial condition \( v(0, x) = v_0(\cdot), v_1(0, x) = v_1(\cdot), \) and \( \kappa(0, x) = \kappa(\cdot) \) being guaranteed by Theorem A.1 in the appendix below. We know that
\[
\| \kappa \|_3 \leq C \| \Delta \theta \|_1 \quad \forall \theta \in H_0^1(\Omega) \quad \text{and} \quad \| u \|_4 \leq C \| \Delta^2 u \| \quad \forall u \in H_0^4(\Omega),
\]
we have the following corollary.

**Corollary 3.2.** Let \( u \) and \( \theta \) be strong solutions of (1.1)–(1.4) and put
\[
E_3((u, \theta), t) = \| u(t, \cdot) \|^2 + \| u_0(t, \cdot) \|^2 + \| u_1(t, \cdot) \|^2
\]
\[
+ \| u(t, \cdot) \|^2 + \| \theta_0(t, \cdot) \|^2 + \| \theta_1(t, \cdot) \|^2 + \| \theta(t, \cdot) \|^2.
\]
Then we have
\[ e^{2\gamma s} E_3(u, \theta, t) + \int_0^t e^{2\gamma s} (E_3(u, \theta), s) + \| \nabla \theta_n(s, \cdot) \|^2 \, ds \leq ME_3((u, \theta), 0) \]
for suitable positive constants \( M \) and \( \gamma \) independent of \( u \) and \( \theta \).

Appendix

In this appendix, we shall show two facts which play important roles in the context.

**Theorem A.1.** Let \( s \) be a real number such that \( 2 \leq s \leq 4 \). For any \( f \in H^{s-4}(\Omega) \), \( g_1 \in H^{s-1/2}(\partial \Omega) \) and \( g_2 \in H^{s-3/2}(\partial \Omega) \), there exists a unique solution \( u \in H^s(\Omega) \) of the boundary value problem:
\[
\Delta^2 u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g_1, \quad \frac{\partial u}{\partial v} = g_2 \quad \text{on} \quad \partial \Omega. \tag{A.1}
\]
Moreover, there exists a constant \( M_s \) depending on \( s \) such that
\[
\| u \| \leq M_s \{ \| f \|_{s-4} + \langle g_1 \rangle_{s-1/2} + \langle g_2 \rangle_{s-3/2} \}, \tag{A.2}
\]
where \( \langle \cdot \rangle_m \) denotes the norm of \( H^m(\partial \Omega) \).

**Remark.** Instead of \( H^{s-4}(\Omega) \), the space \( \Xi^{s-4}(\Omega) \) is used in Theorem 7.4 of Lions and Magenes [5, pp. 188, 189]. But, as stated there, \( \Xi^{s-4}(\Omega) ( \subset H^{s-4}(\Omega) ) \) is not optimal space. Theorem A.1 is a slight improvement of Theorem 7.4. For the sake of completeness, we shall give a proof below.

**Proof of Theorem A.1.** Let us define the map by the relation
\[
\mathcal{P} : H^s(\Omega) \to H^{s-4}(\Omega) \times H^{s-1/2}(\partial \Omega) \times H^{s-3/2}(\partial \Omega); \]
\[
u \mapsto \mathcal{P} \nu = \left( \Delta^2 \nu, u \bigg|_{\Omega}, \frac{\partial u}{\partial v} \bigg|_{\partial \Omega} \right),
\]
where \( u \bigg|_{\partial \Omega} \) means the trace to \( \partial \Omega \). By Theorem 7.4 of Lions and Magenes [5], we know that \( \mathcal{P} \) is a bijective map from \( H^4(\Omega) \) onto \( L^2(\Omega) \times H^{7/2}(\partial \Omega) \times H^{5/2}(\partial \Omega) \) and that (A.2) holds for \( s = 4 \). On the other hand, \( \mathcal{P} \) is also a bijective map from \( H^2(\Omega) \) onto \( H^{-2}(\Omega) \times H^{3/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \) and (A.2) holds for \( s = 2 \). In fact, first of all, let us reduce the problem to the case of homogeneous boundary condition by using the function \( w \in H^2(\Omega) \) such that
\[
w \bigg|_{\partial \Omega} = g_1, \quad \frac{\partial w}{\partial v} \bigg|_{\partial \Omega} = g_2 \quad \text{and} \quad \| w \|_2 \leq C \{ \langle g_1 \rangle_{3/2} + \langle g_2 \rangle_{1/2} \}, \tag{A.3}
\]
the existence of such \( w \) being guaranteed by Theorem 8.3 of Lions and Magenes [5]. If we put \( u = v + w \), then the problem is reduced to finding \( v \in H^2(\partial \Omega) \) satisfying the equation \( \Delta^2 v = f - \Delta^2 w \). Therefore, let us consider the variational problem
\[
(\Delta v, \Delta \varphi) = \langle f - \Delta^2 w, \varphi \rangle_2 \quad \forall \varphi \in H^2_{\partial \Omega}, \tag{A.4}
\]
where $\langle \cdot, \cdot \rangle_2$ means the duality between $H^{-2}(\Omega)$ and $H^2_0(\Omega)$. Here we use the fact that $H^{-2}(\Omega)$ is the dual space of $H^2_0(\Omega)$. Moreover, we shall use the fact that the characterization of $H^{-2}(\Omega)$ is that $f \in H^{-2}(\Omega)$ is represented, in non-unique fashion, by

$$f = \sum_{|s| \leq 2} \delta^s f_s, \quad f_s \in L^2(\Omega) \quad \text{and} \quad \|f\|_{-2}^2 = \inf \sum \|f_s\|^2$$

(cf. [5, pp. 70, 71]). In view of (2.10), $(\Delta \varphi, \Delta \psi)$ is a coercive bilinear form on $H^2_0(\Omega) \times H^2_0(\Omega)$ and

$$|\langle f - \Delta^2 w, \varphi \rangle_2| \leq C(\|f\|_{-2} + \|\Delta w\|) \|\varphi\|_2 \quad \forall \varphi \in H^2_0(\Omega). \quad (A.5)$$

Therefore, by the Lax–Milgram theorem there exists a unique $v \in H^2_0(\Omega)$ satisfying (A.4) and by (2.10) and (A.5)

$$\|v\|_2 \leq C\left\{ \|f\|_{-2} + \|\Delta w\| \right\}. \quad (A.6)$$

Putting $u = v + w$, we have $\mathcal{P}u = (f, g_1, g_2)$ and by (A.3) and (A.6) we see that (A.2) holds for $s = 2$. For general $s \in (2, 4)$, the theorem follows from the interpolation theorem, which completes the proof of the theorem.

**Lemma A.2** (cf. [4, pp. 244]). Let $v \in H^3_0(\Omega)$. Then

$$\langle \Delta v \rangle^2_0 \leq 2 \left( \nabla \Delta v, \nabla \frac{\partial}{\partial N} v \right) + C\|v\|^2.$$  

**Proof.** By the divergence theorem, we have

$$\left( \nabla \Delta v, \nabla \frac{\partial}{\partial N} v \right) = \left( \Delta v, \frac{\partial}{\partial v} \frac{\partial}{\partial v} v \right)_{\partial \Omega} - \left( \Delta v, \Delta \frac{\partial}{\partial N} v \right). \quad (A.7)$$

By using the local coordinate systems, we see that

$$\Delta v = \frac{\partial}{\partial v} \frac{\partial}{\partial v} v \quad \forall v \in H^3_0(\Omega). \quad (A.8)$$

Observing that

$$\left( \Delta v, \Delta \frac{\partial}{\partial N} v \right) = \left( \Delta v, \Delta \frac{\partial}{\partial N} v - \frac{\partial}{\partial N} \Delta v \right)$$

$$+ \frac{1}{2}(\Delta v, \Delta v)_{\partial \Omega} - \frac{1}{2} \sum_{j=1}^n ((\partial N_j/\partial x_j) \Delta v, \Delta v),$$

where we have used the fact that $|N(x)|^2 = |v(x)|^2 = 1$ for $x \in \partial \Omega$, by (A.7) and (A.8) our conclusion follows.

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