Transmission problem for hyperbolic thermoelastic systems

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Abstract

In this paper we study a transmission problem in thermoelasticity. We show that the linear system is well posed and that the solution decays exponentially to zero as time goes to infinity. That is, denoting by $E(t)$ the first order energy associated to the thermoelastic system, then there exists positive constants c and γ such that

$$E(t) \leq cE(0)e^{-\gamma t}$$

Introduction

In this paper we consider the asymptotic behaviour of one dimensional bodies which are composed of two different types of materials, one of them is of thermoelastic type, while the other has no thermal effect. That is, we have a material with a localized thermal effect. Since the body is composed of two different types of material the density is not necessarily a continuous function and since the stress strain relation changes from the thermoelastic part to the elastic part, the corresponding model is not a continuous one. The mathematical model which deals with the above situation is called a transmission problem.

Concerning the thermal effect, in the classical linear theory of thermoelasticity the Fourier's law is used to describe the heat conduction in the body. This theory has two principal shortcomings. First, it is unable to account for memory effect which may prevail in some materials, particularly at low temperatures. Secondly, the corresponding parabolic part of the system predicts an unrealistic result, that a thermal disturbance at one point of the body is instantly felt

every where in the body. These observations lead one to believe that for materials with memory, Fourier's law is not a good model and we have to look for another more general constitutive assumption relating the heat flux to the material thermal history.

We overcome the above shortcomings following the Gurtin-Pipkin's model [4] for the temperature. In this case the thermoelastic system is fully hyperbolic, which in particular implies the finite speed of propagation of the thermal disturbances which is physically more realistic. Therefore we consider the elastic - thermoelastic trasmission problem, for which the thermal part does not depend of the present values of the temperature. Hence the constitutive assumptions we use in this paper are

$$\begin{aligned}\sigma(x, t) &= b(x)\varepsilon(x, t) - c(x)\theta(x, t) \\ q(x, t) &= - \int_0^\infty k(s)\theta_x(x, t-s) ds \\ e(x, t) &= c_u\theta(x, t) + \alpha u_x(x, t)\end{aligned}$$

where σ is the axial stress, $\varepsilon = u_x$ is the deformation, q is the heat flux, e is the internal energy and θ the temperature difference from the reference value. The memory kernel k is assumed to be a regular functions decaying to zero as time goes to infinity. The functions b and c are discontinuous function of type

$$b(x) = \begin{cases} \frac{\alpha}{\rho_1} & \text{for } 0 \leq x \leq L_0 \\ \frac{\beta}{\rho_2} & \text{for } L_0 \leq x \leq L. \end{cases} \quad c(x) = \begin{cases} \frac{\alpha}{\rho_1} & \text{for } 0 \leq x \leq L_0 \\ 0 & \text{for } L_0 \leq x \leq L. \end{cases}$$

Let us denote by

$$\mathcal{U}(x, t) = \begin{cases} u(x, t) & \text{for } 0 \leq x \leq L_0 \\ v(x, t) & \text{for } L_0 \leq x \leq L. \end{cases} \quad p(x) = \begin{cases} \rho_1 & \text{for } 0 \leq x \leq L_0 \\ \rho_2 & \text{for } L_0 \leq x \leq L. \end{cases},$$

we see that the couple $(\mathcal{U}, \theta)$ satisfies

$$\begin{aligned}\mathcal{U}_{tt} - (b(x)\mathcal{U}_x)_x + c(x)\theta_x &= 0 \quad \text{a.e. in }]0, L[\times]0, T[\\ \theta_t - \kappa * \theta_{xx} - \alpha\mathcal{U}_{xt} &= 0 \quad \text{a.e. in }]0, L_0[\times]0, T[\end{aligned}$$

which is the thermoelastic system with discontinuous coefficient. By $*$ we are denoting the convolution product, that is $k * g(\cdot, t) = \int_0^t k(t-\tau)g(\cdot, \tau) d\tau$. The transmission condition is given by

$$[p(x)(b(x)\mathcal{U}_x - c(x)\theta)] = 0$$

where by $[v]$ we are denoting the jump in the function v on L_0 . Note that there exists another implicit transmission condition given by the continuity of the function $\mathcal{U}$, this means that $u(L_0, t) = v(L_0, t)$ for any $t \geq 0$.

Note that the dissipation given by the thermal effect is only effective over the interval $[0, L_0]$. Other important point in the above model is that the dissipation does not depend on present values of the temperature. These two above properties makes that the corresponding dissipative effect is weaker than the dissipation given by the thermal effect in classic thermoelasticity when the heat flux follows the Fourier law and also more interesting from the mathematical point of view. To facilitate our calculations we rewrite the above system in terms on u and v and without lost of generality we considered $\rho_1 = \rho_2 = 1$.

Therefore we have

$$u_{tt} - au_{xx} + \alpha\theta_x = 0 \quad \text{in} \quad]0, L_0[\times \mathbb{R}^+ \quad (1)$$

$$\theta_t - k * \theta_{xx} + \alpha u_{xt} = 0 \quad \text{in} \quad]0, L_0[\times \mathbb{R}^+ \quad (2)$$

$$v_{tt} - \beta v_{xx} = 0 \quad \text{in} \quad]L_0, L[\times \mathbb{R}^+ \quad (3)$$

$$\begin{aligned} u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), \\ \theta(x, 0) &= \theta_0(x), \\ v(x, 0) &= v_0(x), & v_t(x, 0) &= v_1(x), \end{aligned}$$

with the following boundary condition

$$u(0, t) = \theta(0, t) = v(L, t) = 0, \quad \theta_x(L_0, t) = 0 \quad \forall t > 0.$$

and transmission condition

$$\begin{aligned} au_x(L_0, t) - \alpha\theta(L_0, t) &= \beta v_x(L_0, t) \quad \forall t > 0 \\ u(L_0, t) &= v(L_0, t) \quad \forall t > 0 \end{aligned}$$

For simplicity we consider that the initial past history is fixed and equals to zero,

$$\theta(x, t) = 0, \quad \forall t < 0^-$$

The main result of this paper is to show that the localized dissipation due to the thermal effect is strong enough to produce exponential decay to zero of the energy. That is we will prove that

the solution decays exponentially to zero as time goes to infinity, provided the kernel is positive definite (see [10]) and decays exponentially to zero. The method we use is based on the energy method, combined with compactness arguments. The fundamental tool to get the exponential stability is to construct a functional $\mathcal{F}$, equivalent to the first order energy times $e^{2\gamma t}$, bounded for γ small enough (this implies the exponential decay of the energy). The main difficulty we had is to deal with boundary terms and also to get pointwise estimates at the transmission point L_0 . We overcome such problems using some observabilities estimates see Lions [8].

The rest of this paper is organized as follows. In the next section 2 we show the existence and regularity of the solution. Finally, in section 3 we show the main result of this paper: The exponential decay of the solution.

Existence and Regularity

In this section we prove the existence and regularity of weak solutions of the hyperbolic thermoelastic equation with memory. We assume that k is a strongly positive definite kernel satisfying

$$k'(t) < 0, \quad k''(t) > 0 \quad \forall t > 0. \quad (4)$$

Our starting point is to establish the following Lemma whose prove can be found in [10]

Lemma 0.1 *Let us suppose that $k \in L^1(\mathbb{R}^+)$ is a strongly positive definite kernel satisfying $k' \in L^1(\mathbb{R}^+)$; then we have*

$$\int_0^t |k * y(\tau)|^2 d\tau \leq \beta_0 K \int_0^t k * y(\tau) y(\tau) d\tau$$

for any $y \in L^1_{\text{loc}}(\mathbb{R}^+)$, where $K = |k|_1^2 + 4|k'|_1^2$ and $\beta_0 > 0$ is such that the function $k(t) - \beta_0^{-1}e^{-t}$ is a positive definite kernel.

Let us introduce the following space

$$\begin{aligned} \mathcal{V} &= \{(\varphi, \psi) \in H^1(0, L_0) \times H^1(L_0, L); \varphi(0) = \psi(L) = 0 \text{ and } \varphi(L_0) = \psi(L_0)\} \\ \mathcal{W} &= \{w \in H^1(0, L_0); w(0) = 0\} \end{aligned}$$

To simplify our analysis let us define the binary operator

$$k \square g(t) = \int_0^t k(t - \tau) \int_0^L |g(x, t) - g(x, \tau)|^2 dx d\tau.$$

Under this notation we have

Lemma 0.2 For any $g \in C([0, T]; H^1(0, L))$ and $\eta \in C^1(\mathbb{R})$

$$\begin{aligned} \int_0^L \int_0^t \eta(t-\tau) g(x, \tau) d\tau g_t(x, t) dx &= -\frac{1}{2} \eta(t) \int_0^L |g|^2 dx + \frac{1}{2} \eta' \square g - \\ &- \frac{1}{2} \frac{d}{dt} \left\{ \eta \square g - \left(\int_0^t \eta d\tau \right) \int_0^L |g|^2 dx \right\}. \end{aligned}$$

Proof: To show the above identity, it is enough to differentiate with respect to the time the expression $\eta \square g$.

The definition of weak solution we use in this work is given as follows

Definition 0.1 We say that (u, v, θ) is a weak solution of the system (1)–(3) when

$$\begin{aligned} (u, v) &\in L^\infty(0, T; \mathcal{V}) \cap W^{1, \infty}(0, T; L^2(0, L_0) \times L^2(L_0, L)) \\ \theta &\in L^\infty(0, T; L^2(0, L_0)) \quad k * \theta_x \in L^2(0, T; L^2(0, L_0)) \end{aligned}$$

and satisfies the following identities

$$\begin{aligned} \int_0^T \int_0^{L_0} \{u \Phi_{tt} + a u_x \Phi_x - \alpha \theta \Phi_x\} dx dt + \int_0^T \int_{L_0}^L \{v \Psi_{tt} + \beta v_x \Psi_x\} dx dt = \\ \int_0^{L_0} u_1 \Phi(0) - u_0 \Phi_t(0) dx + \int_{L_0}^L v_1 \Psi(0) - v_0 \Psi_t(0) dx. \\ - \int_0^T \int_0^{L_0} \{\theta \Gamma_t - k * \theta_x \Gamma_x + \alpha u_x \Gamma_t\} dx dt = \int_0^{L_0} \theta_0 \Gamma(0) dx + \alpha \int_0^{L_0} u_x(0) \Gamma(0) dx \end{aligned}$$

for any $(\Phi, \Psi) \in C^2([0, T]; \mathcal{V})$ and $\Gamma \in C^1([0, T]; \mathcal{W})$ such that $\Phi(T) = \Phi_t(T) = \Psi(T) = \Psi_t(T) = \Gamma(T) = 0$.

Let us introduce the energy functions

$$\begin{aligned} E_1(t; u, \theta) &:= \frac{1}{2} \int_0^{L_0} |u_t|^2 + a |u_x|^2 + |\theta|^2 dx \\ E_2(t; v) &:= \frac{1}{2} \int_{L_0}^L |v_t|^2 + \beta |v_x|^2 dx \\ E(t; u, \theta, v) &:= E_1(t; u, \theta) + E_2(t; v) \end{aligned}$$

$$\begin{aligned} F(t; u, \theta, v) : &= \frac{1}{2} \int_0^{L_0} |u_{tt}|^2 + a |u_{xt}|^2 + |\theta_t|^2 + k(t) |\theta_x|^2 - k' \square \theta_x dx \\ &+ \frac{1}{2} \int_{L_0}^L |v_{tt}|^2 + \beta |v_{xt}|^2 dx. \end{aligned}$$

In these conditions we are able to prove the following Theorem

Theorem 0.1 *Let us suppose that the initial data satisfies $(u_0, v_0) \in \mathcal{V}$, $(u_1, v_1) \in L^2(0, L_0) \times L^2(L_0, L)$ and $\theta_0 \in L^2(0, L_0)$. Then there exists a unique weak solution for (1)–(3). Moreover if we take the initial data $(u_0, v_0) \in [H^2(0, L_0) \times H^2(L_0, L)] \cap \mathcal{V}$, $(u_1, v_1) \in \mathcal{V}$, $\theta_0 \in H^1(0, L_0)$ verifying the compatibility condition given by*

$$au_{0,x}(L_0) - \alpha\theta_0(L_0) = \beta v_{0,x}(L_0)$$

then the solution satisfies,

$$\begin{aligned} (u, v) &\in \bigcap_{k=0}^2 W^{k,\infty}(0, T; H^{2-k}(0, L_0) \times H^{2-k}(L_0, L)) \\ \theta &\in \bigcap_{k=0}^1 W^{k,\infty}(0, T; H^{1-k}(0, L_0)) \\ k * \theta_{xx} &\in L^\infty(0, T; L^2(0, L_0)) \end{aligned}$$

Proof: Let us denote $\{(\varphi_j, \psi_j)\} \times \{w_j\}$ an orthonormal basis of $\mathcal{V} \times \mathcal{W}$ and

$$(u^m(t), v^m(t)) = \sum_{j=1}^m h_{j,m}(t)(\varphi_j, \psi_j), \quad \theta^m(t) = \sum_{j=1}^m g_{j,m}(t)w_j$$

where the functions $h_{j,m}(t)$ and $g_{j,m}(t)$ are given by the solution of the approximated system,

$$\int_0^{L_0} [u_{tt}^m \varphi_j + au_x^m \varphi_{j,x} - \alpha \theta^m \varphi_{j,x}] dx + \int_{L_0}^L [v_{tt}^m \psi_j + \beta v_x^m \psi_{j,x}] dx = 0, \quad (5)$$

$$\int_0^{L_0} [\theta_t^m w_j + k * \theta_x^m w_{j,x} - \alpha u_t^m w_{j,x}] dx = 0, \quad (6)$$

with initial data

$$(u^m(0), v^m(0)) = (u_0^m, v_0^m), \quad (u_t^m(0), v_t^m(0)) = (u_1^m, v_1^m), \quad \theta^m(0) = \theta_0^m.$$

Where

$$(u_0^m, v_0^m) = \sum_{i=1}^m \alpha_i(\varphi_i, \psi_i), \quad (u_1^m, v_1^m) = \sum_{i=1}^m \beta_i(\varphi_i, \psi_i), \quad \theta_0^m = \sum_{i=1}^m \gamma_i w_i$$

are such that

$$(u_0^m, v_0^m) \rightarrow (u_0, v_0), \quad (u_1^m, v_1^m) \rightarrow (u_1, v_1), \quad \theta_0^m \rightarrow \theta_0$$

strongly in their corresponding spaces. The existence of approximated solutions u^m , θ^m and v^m are guaranteed by standard results on Ordinary Differential Equations. Our next step is to show that the approximated solution remains bounded for any $m > 0$. To this end, let us multiply equation (5) by $h'_{j,m}(t)$ and (6) by $g_{j,m}(t)$, summing up the product result in j we arrive at

$$\frac{d}{dt} E(t; u^m, \theta^m, v^m) = - \int_0^{L_0} k * \theta_x^m \theta_x^m dx.$$

Integrating from 0 to t the above relation and using Lemma 0.1 it follows that there exists a positive constant C_0 satisfying

$$E(t; u^m, \theta^m, v^m) + C_0 \int_0^t \int_0^{L_0} |k * \theta_x^m|^2 dx d\tau \leq E(0; u^m, \theta^m, v^m). \quad (7)$$

From our choice of initial data it follows that

$$E(t; u^m, \theta^m, v^m) \quad \text{is bounded for any } m \in \mathbb{N} \quad (8)$$

$$k * \theta_x^m \quad \text{is bounded in } L^2(0, T; L^2(0, L_0)). \quad (9)$$

Differentiating equation (5) and (6) with respect to the time we get

$$\int_0^{L_0} [u_{ttt}^m \varphi_j + a u_{xt}^m \varphi_{j,x} - \alpha \theta_t^m \varphi_{j,x}] dx + \int_{L_0}^L [v_{ttt}^m \psi_j + \beta v_{xt}^m \psi_{j,x}] dx = 0 \quad (10)$$

$$\int_0^{L_0} [\theta_{tt}^m w_j + k(0) \theta_x^m w_{j,x} + k' * \theta_x^m w_{j,x} - \alpha u_{tt}^m w_{j,x}] dx = 0. \quad (11)$$

Multiplying equation (10) by $h_{j,m}''$ and (11) by $g_{j,m}'$ we obtain

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \left[\int_0^{L_0} |u_{tt}^m|^2 + a |u_{xt}^m|^2 + |\theta_t^m|^2 + k(0) |\theta_x^m|^2 dx + \int_{L_0}^L |v_{tt}^m|^2 + \beta |v_{xt}^m|^2 dx \right] = \\ = - \int_0^{L_0} \int_0^t k'(t - \tau) \theta_x^m(x, \tau) d\tau \theta_{xt}^m(x, t) dx. \end{aligned}$$

Using Lemma 0.2 and hypotheses (4) we conclude that

$$\frac{d}{dt} F(t; u^m, \theta^m, v^m) = \frac{k'(t)}{2} \int_0^{L_0} |\theta_x^m|^2 dx - \frac{1}{2} k'' \square_x^m < 0.$$

Integrating from 0 to t the above relation we get

$$\begin{aligned} F(t; u^m, \theta^m, v^m) &\leq \frac{1}{2} \int_0^{L_0} |u_{tt}^m(0)|^2 + a |u_{1,x}^m|^2 + |\theta_t^m(0)|^2 + k(0) |\theta_{0,x}^m|^2 dx \\ &\quad + \frac{1}{2} \int_{L_0}^L |v_{tt}^m(0)|^2 + \beta |v_{1,x}^m|^2 dx. \end{aligned}$$

Now, we do $t \rightarrow 0^+$ in (5) and (6) and multiplying the limit result by $h_{j,m}''(0)$ and $g_{j,m}'(0)$ respectively. Summing up in j and using compatibility condition we get

$$\begin{aligned} \int_0^{L_0} |u_{tt}^m(0)|^2 + |\theta_t^m(0)|^2 dx + \int_{L_0}^L |v_{tt}^m(0)|^2 dx &= \int_{L_0}^L \beta v_{0,xx}^m v_{tt}^m(0) dx \\ &\quad + \int_0^{L_0} \{a u_{0,xx}^m - \alpha \theta_{0,x}^m\} u_{tt}^m(0) - \alpha u_{1,x}^m \theta_t^m(0) dx. \end{aligned}$$

Applying Young's inequality we have

$$(u_{tt}^m(0), v_{tt}^m(0), \theta_t^m(0)) \quad \text{is bounded in} \quad L^2(0, L_0) \times L^2(L_0, L) \times L^2(0, L_0).$$

Therefore

$$F(t; u^m, \theta^m, v^m) \text{ is bounded for any } m \in \mathbb{N}. \quad (12)$$

From (8) and (12) we get that (u, θ, v) obtained taking the limit when $m \rightarrow \infty$ is such that

$$\begin{aligned} (u, v), \quad (u_t, v_t) &\in L^\infty(0, T; \mathcal{V}) \\ (u_{tt}, v_{tt}) &\in L^\infty(L^2(0, L_0) \times L^2(L_0, L)) \\ (\theta, \theta_t) &\in L^\infty(H^1(0, L_0) \times L^2(0, L_0)) \\ k * \theta_x &\in L^2(0, T; L^2(0, L_0)) \end{aligned}$$

and satisfies

$$\begin{aligned} au_{xx} &= u_{tt} + \alpha \theta_x \\ k * \theta_{xx} &= \theta_t + \alpha u_{xt} \\ \beta v_{xx} &= v_{tt} \end{aligned}$$

from this we conclude that

$$\begin{aligned} (u_{xx}, v_{xx}) &\in L^\infty(0, T; L^2(0, L_0) \times L^2(L_0, L)) \\ k * \theta_{xx} &\in L^\infty(0, T; L^2(0, L_0)). \end{aligned}$$

The rest of proof is a matter of routine. □

Uniform rate of decay

In this section we show that the solution of the system (1)–(2) decays exponentially as time goes to infinity. The method we use here is based on the construction of a functional $\mathcal{L}(t)$ equivalent to the first order energy whose derivative is negative proportional to itself. Here we will assume that k and $-k'$ are positive definite kernel satisfying

$$k', k'' \in C^1(\mathbb{R}^+) \quad (13)$$

$$\int_0^T k' * \phi \phi \, dt \leq -C \int_0^T k * \phi \phi \, dt. \quad (14)$$

The above inequality is satisfied for any function k which is sum of exponential functions with varying rates of decay. Since the kernel k decays exponentially, we take $\gamma > 0$ small enough such that there exists $\varepsilon > 0$ satisfying

$$K(t) := e^{\gamma t} k(t) \leq C e^{-\varepsilon t}.$$

Let us denote $U(x, t) = u(x, t) e^{\gamma t}$, $\Theta(x, t) = \theta(x, t) e^{\gamma t}$ and $V(x, t) = v(x, t) e^{\gamma t}$, then (U, Θ, V) satisfies:

$$U_{tt} - aU_{xx} + \alpha\Theta_x = R \quad (15)$$

$$\Theta_t - K * \Theta_{xx} + \alpha U_{xt} = S, \quad (16)$$

$$V_{tt} - \beta V_{xx} = Q \quad (17)$$

where R , S and Q are given by:

$$R := 2\gamma U_t - \gamma^2 U, \quad S := \gamma\Theta + \alpha\gamma U_x, \quad Q := 2\gamma V_t - \gamma^2 V.$$

with U, Θ and V satisfying the same boundary conditions and initial values as in system (1)–(3).

Let us introduce the energy function

$$\begin{aligned} E_1(t; U, \Theta) &:= \frac{1}{2} \int_0^{L_0} |U_t|^2 + a|U_x|^2 + |\Theta|^2 dx \\ E_2(t; V) &:= \frac{1}{2} \int_{L_0}^L |V_t|^2 + \beta|V_x|^2 dx \\ E(t; U, \Theta, V) &:= E_1(t; U, \Theta) + E_2(t; V). \end{aligned}$$

To show the exponential decay of the solution (u, θ, v) it is enough to prove that $E(t, U, \Theta, V)$ is bounded. To this end, we start with the following Lemma.

Lemma 0.3 *Let us suppose that the initial data satisfy*

$$(u_0, v_0) \in [H^2(0, L_0) \times H^2(L_0, L)] \cap \mathcal{V}, \quad (u_1, v_1) \in \mathcal{V}, \quad \theta_0 \in H^1(0, L_0)$$

verifying the compatibility condition given by

$$au_{0,x}(L_0) - \alpha\theta_0(L_0) = \beta v_{0,x}(L_0)$$

then we have

$$\frac{d}{dt} E(t; U, \Theta, V) \leq - \int_0^{L_0} K * \Theta_x \Theta_x dx + c\gamma E(t; U, \Theta, V).$$

Proof: Multiplying equation (15) by U_t and equation (16) by Θ and integrating from 0 to L_0 we get

$$\begin{aligned} \frac{d}{dt} E_1(t; U, \Theta) &= - \int_0^{L_0} K * \Theta_x \Theta_x dx + \int_0^{L_0} (RU_t + S\Theta) dx \\ &\quad + aU_x(L_0, t) U_t(L_0, t) - \alpha\Theta(L_0, t) V_t(L_0, t). \end{aligned} \quad (18)$$

On the other hand, multiplying equation (17) by V_t and integrating from L_0 to L we get

$$\frac{d}{dt} E_2(t; V) := \int_{L_0}^L V_t Q dx - \beta V_x(L_0, t) V_t(L_0, t) \quad (19)$$

After applying Holder's inequality and Young's inequality we have

$$\begin{aligned} \left| \int_0^{L_0} RU_t \, dx \right| &\leq c\gamma \left(\int_0^{L_0} |U_t|^2 \, dx + \int_0^{L_0} |U_x|^2 \, dx \right) \\ \left| \int_0^{L_0} S\Theta \, dx \right| &\leq c\gamma \left(\int_0^{L_0} |\Theta|^2 \, dx + \int_0^{L_0} |U_x|^2 \, dx \right) \\ \left| \int_{L_0}^L V_t Q \, dx \right| &\leq c\gamma \left(\int_{L_0}^L |V_t|^2 \, dx + \int_{L_0}^L |V_x|^2 \, dx \right) \end{aligned}$$

From now on c will denote a positive constant, easy to estimate, which can be different in different places. Thereby, from transmission's conditions our conclusion follows $\square$

Lemma 0.4 *With the same hypotheses as Lemma 0.3 we have*

$$\begin{aligned} -\frac{d}{dt} \int_0^{L_0} K * \Theta \Theta \, dx &= -K(0) \int_0^{L_0} |\Theta|^2 \, dx + \int_0^{L_0} |K * \Theta_x|^2 \, dx \\ &\quad + \alpha U_t(L_0, t) K * \Theta(L_0, t) - \int_0^{L_0} \Theta K' * \Theta \, dx \\ &\quad - \alpha \int_0^{L_0} U_t K * \Theta_x \, dx - \int_0^{L_0} S(K * \Theta) \, dx \end{aligned} \quad (20)$$

$$\begin{aligned} \frac{d}{dt} \int_0^{L_0} UU_t \, dx &\leq -\frac{a}{2} \int_0^{L_0} |U_x|^2 \, dx + \int_0^{L_0} |U_t|^2 \, dx \\ &\quad + \frac{\alpha^2}{2a} \int_0^{L_0} |\Theta|^2 \, dx + \left\{ \frac{24a}{L_0} + \frac{24\alpha^2}{L_0} \right\} |U(L_0, t)|^2 \\ &\quad + \frac{L_0}{96} \left\{ a |U_x(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} + c\gamma E_1(t; U, \Theta). \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{d}{dt} \int_{L_0}^L (x - L) V_x V_t \, dx &\leq \frac{\lambda_0}{2} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} \\ &\quad - E_2(t; V) + c\gamma E_2(t; V) \end{aligned} \quad (22)$$

where $\lambda_0 = (L - L_0) \max\{1, \frac{\bar{c}}{\beta}\}$ and $\bar{c} = \max\{\frac{a}{2}, \frac{\alpha^2}{2}\}$.

Proof: We have

$$\begin{aligned} \frac{d}{dt} \int_0^{L_0} \Theta(K * \Theta) \, dx &= \int_0^{L_0} \Theta_t K * \Theta \, dx + K(0) \int_0^{L_0} |\Theta|^2 \, dx + \int_0^{L_0} K' * \Theta \Theta \, dx \\ &= K(0) \int_0^{L_0} |\Theta|^2 \, dx + \int_0^{L_0} K * \Theta_{xx} (K * \Theta) \, dx - \alpha \int_0^{L_0} U_{xt} (K * \Theta) \, dx \\ &\quad + \int_0^{L_0} S(K * \Theta) \, dx + \int_0^{L_0} K' * \Theta \Theta \, dx \\ &= K(0) \int_0^{L_0} |\Theta|^2 \, dx - \int_0^{L_0} |K * \Theta_x|^2 \, dx - \alpha U_t(L_0, t) K * \Theta(L_0, t) \\ &\quad + \int_0^{L_0} S(K * \Theta) \, dx + \int_0^{L_0} K' * \Theta \Theta \, dx + \alpha \int_0^{L_0} U_t K * \Theta_x \, dx. \end{aligned}$$

On the other hand from (15) we have

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} UU_t \, dx &= \int_0^{L_0} |U_t|^2 \, dx + \int_0^{L_0} UU_{tt} \, dx \\
&= \int_0^{L_0} |U_t|^2 \, dx + a \int_0^{L_0} UU_{xx} \, dx - \alpha \int_0^{L_0} U \Theta_x \, dx + \int_0^{L_0} RU \, dx \\
&= \int_0^{L_0} |U_t|^2 \, dx + aU(L_0, t) U_x(L_0, t) - \alpha U(L_0, t) \Theta(L_0, t) \\
&\quad - a \int_0^{L_0} |U_x|^2 \, dx + \alpha \int_0^{L_0} U_x \Theta \, dx + \int_0^{L_0} RU \, dx \\
&\leq -\frac{a}{2} \int_0^{L_0} |U_x|^2 \, dx + \int_0^{L_0} |U_t|^2 \, dx + \frac{\alpha^2}{2a} \int_0^{L_0} |\Theta|^2 \, dx \\
&\quad + aU(L_0, t) U_x(L_0, t) - \alpha U(L_0, t) \Theta(L_0, t) + \int_0^{L_0} RU \, dx.
\end{aligned}$$

Since

$$\begin{aligned}
|aU(L_0, t) U_x(L_0, t)| &\leq \frac{24a}{L_0} |U(L_0, t)|^2 + \frac{aL_0}{96} |U_x(L_0, t)|^2 \\
|\alpha U(L_0, t) \Theta(L_0, t)| &\leq \frac{24\alpha^2}{L_0} |U(L_0, t)|^2 + \frac{L_0}{96} |\Theta(L_0, t)|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} UU_t \, dx &\leq -\frac{a}{2} \int_0^{L_0} |U_x|^2 \, dx + \int_0^{L_0} |U_t|^2 \, dx + \\
&\quad + \frac{\alpha^2}{2a} \int_0^{L_0} |\Theta|^2 \, dx + \left\{ \frac{24a}{L_0} + \frac{24\alpha^2}{L_0} \right\} |U(L_0, t)|^2 \\
&\quad + \frac{L_0}{96} \left\{ a |U_x(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} + c\gamma E_1(t; U, \Theta).
\end{aligned} \tag{23}$$

Now we proof the last estimate,

$$\begin{aligned}
\frac{d}{dt} \int_{L_0}^L \varphi(x) V_t V_x \, dx &= \frac{1}{2} \int_{L_0}^L \varphi(x) \frac{d}{dx} |V_t|^2 \, dx + \frac{\beta}{2} \int_{L_0}^L \varphi(x) \frac{d}{dx} |V_x|^2 \, dx \\
&\quad + \int_{L_0}^L Q \varphi(x) V_x \, dx \\
&= \frac{\varphi(L_0)}{2} \left\{ \beta |V_x(L_0, t)|^2 + |V_t(L_0, t)|^2 \right\} \\
&\quad - \frac{1}{2} \int_{L_0}^L \varphi'(x) \left\{ |V_t|^2 + \beta |V_x|^2 \right\} \, dx + \int_{L_0}^L Q \varphi(x) V_x \, dx.
\end{aligned}$$

Taking $\varphi(x) = (x - L)$ we arrive to our conclusion. $\square$

Let us introduce the multipliers q and w satisfying

$$\begin{aligned}
-q_{xx} &= U_t && \text{in } (0, L_0) \times \mathbb{R}^+ \\
q(0, t) &= q(L_0, t) = 0 && \text{in } \mathbb{R}^+
\end{aligned}$$

$$\begin{aligned}
-w_{xx} &= \Theta - \bar{\Theta} && \text{in } (0, L_0) \times \mathbb{R}^+ \\
w_x(0, t) &= w_x(L_0, t) = 0 && \text{in } \mathbb{R}^+
\end{aligned}$$

where $\bar{\Theta} = \frac{1}{L_0} \int_0^{L_0} \Theta \, dx$. We easily obtain

$$\int_0^{L_0} |w_x|^2 \, dx \leq c \int_0^{L_0} |\Theta|^2 \, dx,$$

Lemma 0.5 *With the same hypotheses as Lemma 0.3 we have*

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} q_x \Theta \, dx &\leq -\frac{\alpha}{2} \int_0^{L_0} |U_t|^2 \, dx + \frac{\alpha a}{32} \int_0^{L_0} |U_x|^2 \, dx + \left[\frac{16c_p^2 \delta_1 a}{\alpha L_0} + \frac{16a}{\alpha} + \alpha \right] \int_0^{L_0} |\Theta|^2 \, dx \\
&+ \left(\frac{64\alpha \lambda_1^2}{L_0} + \frac{16\lambda_1^2 K(0)}{\alpha L_0} \right) \|q_x(\cdot, t)\|_{H^{3/4}}^2 + \frac{\alpha L_0}{64K(0)} |K * \Theta_x(0, t)|^2 \\
&+ \frac{\alpha L_0}{256} |U_t(L_0, t)|^2 + \frac{1}{2\alpha} \int_0^{L_0} |K * \Theta_x|^2 \, dx + c\gamma E_1(t; U, \Theta)
\end{aligned}$$

where λ_1 and δ_1 are the embedding constants of $H^{3/4}(0, L_0) \hookrightarrow L^\infty(0, L_0)$ and $L^2(0, L_0) \hookrightarrow L^1(0, L_0)$, respectively, and c_p is a constant satisfying

$$|f(L_0)|^2 \leq c_p^2 \int_0^L |f_x|^2 \, dx, \quad \int_0^L |f|^2 \, dx \leq c_p^2 \int_0^L |f_x|^2 \, dx, \quad \forall f \in \mathcal{W}$$

Proof. Note that

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} q_x \Theta \, dx &= \int_0^{L_0} q_{xt} \Theta \, dx + \int_0^{L_0} q_x \Theta_t \, dx \\
&= \underbrace{\int_0^{L_0} q_{xt} \Theta \, dx}_{I_1} + \underbrace{\int_0^{L_0} q_x K * \Theta_{xx} \, dx}_{I_2} - \underbrace{\alpha \int_0^{L_0} q_x U_{xt} \, dx}_{I_3} + \underbrace{\int_0^{L_0} q_x S \, dx}_{I_4}.
\end{aligned}$$

Recalling the definitions of q_x and w we have

$$\begin{aligned}
I_1 &= \int_0^{L_0} q_{xt} (\Theta - \bar{\Theta}) \, dx = - \int_0^{L_0} q_{xt} w_{xx} \, dx = - \int_0^{L_0} U_{tt} w_x \, dx \\
&= -a \int_0^{L_0} U_{xx} w_x \, dx + \alpha \int_0^{L_0} \Theta_x w_x \, dx - \int_0^{L_0} R w_x \, dx \\
&= a \int_0^{L_0} U_x w_{xx} \, dx - \alpha \int_0^{L_0} \Theta w_{xx} \, dx - \int_0^{L_0} R w_x \, dx \\
&= -a \int_0^{L_0} U_x (\Theta - \bar{\Theta}) \, dx + \alpha \int_0^{L_0} |\Theta|^2 \, dx - \frac{\alpha}{L_0} \left(\int_0^{L_0} \Theta \, dx \right)^2 - \int_0^{L_0} R w_x \, dx \\
&= -a \int_0^{L_0} U_x \Theta \, dx + a \bar{\Theta} U(L_0, t) + \alpha \int_0^{L_0} |\Theta|^2 \, dx - \frac{\alpha}{L_0} \left(\int_0^{L_0} \Theta \, dx \right)^2 \\
&\quad - \int_0^{L_0} R w_x \, dx.
\end{aligned} \tag{24}$$

Since

$$\begin{aligned}
\left| a \int_0^{L_0} U_x \Theta \, dx \right| &\leq \frac{\alpha a}{64} \int_0^{L_0} |U_x|^2 \, dx + \frac{16a}{\alpha} \int_0^{L_0} |\Theta|^2 \, dx \\
|U(L_0, t)|^2 &\leq c_p^2 \int_0^{L_0} |U_x|^2 \, dx \\
\int_0^{L_0} R w_x \, dx &\leq c \gamma E_1(t; U, \Theta) \\
|a \bar{\Theta} U(L_0, t)| &\leq \frac{16c_p^2 a}{\alpha} |\bar{\Theta}|^2 + \frac{\alpha a}{64c_p^2} |U(L_0, t)|^2 \\
&\leq \frac{16c_p^2 a}{\alpha} \left| \frac{1}{L_0} \int_0^{L_0} \Theta \, dx \right|^2 + \frac{\alpha a}{64c_p^2} |U(L_0, t)|^2 \\
&\leq \frac{16c_p^2 \delta_1 a}{\alpha L_0^2} \int_0^{L_0} |\Theta|^2 \, dx + \frac{\alpha a}{64} \int_0^{L_0} |U_x|^2 \, dx
\end{aligned}$$

Using theses estimates in (24) we get

$$I_1 \leq \frac{\alpha a}{32} \int_0^{L_0} |U_x|^2 \, dx + \left(\frac{16a}{\alpha} + \frac{16c_p^2 \delta_1 a}{\alpha L_0^2} + \alpha \right) \int_0^{L_0} |\Theta|^2 \, dx + c \gamma E_1(t; U, \Theta)$$

Now we consider the term

$$\begin{aligned}
I_2 &= q_x K * \Theta_x|_0^{L_0} - \int_0^{L_0} q_{xx} K * \Theta_x \, dx \\
&\leq \|q_x(\cdot, t)\|_{L^\infty} |K * \Theta_x(0, t)| + \int_0^{L_0} U_t K * \Theta_x \, dx \\
&\leq \lambda_1 \|q_x(\cdot, t)\|_{H^{3/4}} |K * \Theta_x(0, t)| + \int_0^{L_0} U_t K * \Theta_x \, dx.
\end{aligned}$$

Applying Holder's inequality and Young's inequality in the last expression we have

$$I_2 \leq \frac{\alpha}{2} \int_0^{L_0} |U_t|^2 \, dx + \frac{16K(0)\lambda_1^2}{\alpha L_0} \|q_x(\cdot, t)\|_{H^{3/4}}^2 + \frac{\alpha L_0}{64K(0)} |K * \Theta_x(0, t)|^2 + \frac{1}{2\alpha} \int_0^{L_0} |K * \Theta_x|^2 \, dx.$$

Analogously note that I_3 satisfies

$$\begin{aligned}
I_3 &= -\alpha \int_0^{L_0} |U_t|^2 \, dx + \alpha q_x U_t|_0^{L_0} \\
&\leq -\alpha \int_0^{L_0} |U_t|^2 \, dx + \frac{\alpha L_0}{256} |U_t(L_0, t)|^2 + \frac{64\alpha\lambda_1^2}{L_0} \|q_x(\cdot, t)\|_{H^{3/4}}^2.
\end{aligned}$$

Finally, we have that

$$I_4 = \int_0^{L_0} q_x (\gamma \Theta + \alpha \gamma U_x) \, dx \leq c \gamma E_1(t; U, \Theta).$$

Taking the summatory of the functions I_i our conclusion follows. $\square$

Lemma 0.6 *With the same hypotheses as Lemma 0.3 we have*

$$\begin{aligned}
\frac{d}{dt} \mathcal{N}_1(t) \leq & -\frac{K(0)L_0}{4} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} \\
& -\frac{L_0}{4} |K * \Theta_x(0, t)|^2 + \frac{K(0)}{2} \int_0^{L_0} |U_t|^2 dx + aK(0) \int_0^{L_0} |U_x|^2 dx \\
& + K(0) \int_0^{L_0} |\Theta|^2 dx + \frac{1}{2} \int_0^{L_0} |K * \Theta_x|^2 dx \\
& + \left(\frac{L_0^2}{8K(0)} + \frac{\alpha^2 L_0^2}{8aK(0)} \right) \int_0^{L_0} |K' * \Theta_x|^2 dx \\
& + c\gamma \left\{ E_1(t; U, \Theta) + \int_0^{L_0} |K * \Theta_x|^2 dx \right\}
\end{aligned}$$

where $\mathcal{N}_1(t) = \int_0^{L_0} (\frac{L_0}{2} - x) \{ K(0)U_t U_x + \Theta K * \Theta_x + \alpha U_x K * \Theta_x \} dx$

Proof: Multiplying (15) by $\varphi(x) U_x$ and integrating from 0 to L_0 we get

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} \varphi(x) U_t U_x dx &= \frac{1}{2} \int_0^{L_0} \varphi(x) \frac{d}{dx} |U_t|^2 dx + \frac{a}{2} \int_0^{L_0} \varphi(x) \frac{d}{dx} |U_x|^2 dx \\
& - \alpha \int_0^{L_0} \Theta_x \varphi(x) U_x dx + \int_0^{L_0} R \varphi(x) U_x dx \\
&= \frac{\varphi(L_0)}{2} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 \right\} \\
& - \frac{a\varphi(0)}{2} |U_x(0, t)|^2 - \frac{1}{2} \int_0^{L_0} \varphi'(x) \left\{ |U_t|^2 + a |U_x|^2 \right\} dx \\
& - \alpha \int_0^{L_0} \Theta_x \varphi(x) U_x dx + \int_0^{L_0} R \varphi(x) U_x dx.
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\frac{d}{dt} \int_0^{L_0} \varphi(x) K * \Theta_x \Theta dx &= \frac{K(0)}{2} \int_0^{L_0} \varphi(x) \frac{d}{dx} |\Theta|^2 dx + \int_0^{L_0} \varphi(x) K' * \Theta_x \Theta dx \\
& + \frac{1}{2} \int_0^{L_0} \varphi(x) \frac{d}{dx} |K * \Theta_x|^2 dx - \alpha \int_0^{L_0} \varphi(x) K * \Theta_x U_{xt} dx \\
& + \int_0^{L_0} S \varphi(x) K * \Theta_x dx. \\
&= \frac{K(0)\varphi(L_0)}{2} |\Theta(L_0, t)|^2 - \frac{K(0)}{2} \int_0^{L_0} \varphi'(x) |\Theta|^2 dx \\
& + \int_0^{L_0} \varphi(x) K' * \Theta_x \Theta dx - \frac{\varphi(0)}{2} |K * \Theta_x(0, t)|^2 \\
& - \frac{1}{2} \int_0^{L_0} \varphi'(x) |K * \Theta_x|^2 dx - \alpha \frac{d}{dt} \int_0^{L_0} \varphi(x) K * \Theta_x U_x dx \\
& + \alpha \int_0^{L_0} \varphi(x) K(0) \Theta_x U_x dx + \alpha \int_0^{L_0} \varphi(x) K' * \Theta_x U_x dx \\
& + \int_0^{L_0} S \varphi(x) K * \Theta_x dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \frac{d}{dt} \int_0^{L_0} \varphi(x) K * \Theta_x [\Theta + \alpha U_x] dx \\
= & \frac{K(0)\varphi(L_0)}{2} |\Theta(L_0, t)|^2 - \frac{K(0)}{2} \int_0^{L_0} \varphi'(x) |\Theta|^2 dx \\
& + \int_0^{L_0} \varphi(x) K' * \Theta_x \Theta dx - \frac{\varphi(0)}{2} |K * \Theta_x(0, t)|^2 \\
& - \frac{1}{2} \int_0^{L_0} \varphi'(x) |K * \Theta_x|^2 dx + \alpha K(0) \int_0^{L_0} \varphi(x) \Theta_x U_x dx \\
& + \alpha \int_0^{L_0} \varphi(x) K' * \Theta_x U_x dx + \int_0^{L_0} S\varphi(x) K * \Theta_x dx.
\end{aligned}$$

Taking $\varphi(x) = (\frac{L_0}{2} - x)$ in above identities and using the estimate

$$\begin{aligned}
\left| \alpha \int_0^{L_0} (\frac{L_0}{2} - x) U_x K' * \Theta_x dx \right| & \leq \frac{K(0)a}{2} \int_0^{L_0} |U_x|^2 dx + \frac{L_0^2 \alpha^2}{8aK(0)} \int_0^{L_0} |K' * \Theta_x|^2 dx, \\
\left| \int_0^{L_0} (\frac{L_0}{2} - x) \Theta K' * \Theta_x dx \right| & \leq \frac{K(0)}{2} \int_0^{L_0} |\Theta|^2 dx + \frac{L_0^2}{8K(0)} \int_0^{L_0} |K' * \Theta_x|^2 dx.
\end{aligned}$$

our conclusion follows. $\square$

To show the uniform rate decay we will use a result due to Kim [6]

Lemma 0.7 *Let us denote $\{z^k\}$ a sequence of function satisfying*

$$\begin{aligned}
z^k & \xrightarrow{*} z & \text{in} & L^\infty(0, T; H^p(0, L_0)) \\
z_t^k & \xrightarrow{*} z_t & \text{in} & L^\infty(0, T; H^q(0, L_0))
\end{aligned}$$

when $k \rightarrow \infty$, for $p < q$. Then we have for any $r < q$ that

$$z^k \rightarrow z \quad \text{in} \quad C(0, T; H^r(0, L_0))$$

Proof. See [6].

Lemma 0.8 *With the same hypotheses as Lemma 0.3 we have, for any $\delta > 0$ exist $C_\delta > 0$ independent of initial data such that to T large enough*

$$\int_0^T \left[|U(L_0, t)|^2 + \|q_x(., t)\|_{H^{\frac{3}{4}}}^2 \right] dt \leq \delta \int_0^T E(t; U, \Theta, V) dt + C_\delta \int_0^T \int_0^L |K * \Theta_x|^2 dx dt$$

for any solution (U, Θ, V) of system (15) – (17)

Proof: We reason by contradiction. Let us suppose that there exists a sequence of initial data $(U_0^\nu, \Theta_0^\nu, V_0^\nu)$, (U_1^ν, V_1^ν) and positive constant $\delta_0 > 0$ such that the corresponding solution of system

$$U_{tt}^\nu - aU_{xx}^\nu + \alpha\Theta_x^\nu = 2\gamma U_t^\nu - \gamma^2 U^\nu \quad (25)$$

$$\Theta_t^\nu - K * \Theta_{xx}^\nu + \alpha U_{xt}^\nu = \gamma\Theta^\nu + \alpha\gamma U_x^\nu, \quad (26)$$

$$V_{tt}^\nu - \beta V_{xx}^\nu = 2\gamma V_t^\nu - \gamma^2 V^\nu \quad (27)$$

$$\begin{aligned} U^\nu(x, 0) &= U_0^\nu(x), & U_t^\nu(x, 0) &= U_1^\nu(x), & \Theta^\nu(x, 0) &= \Theta_0^\nu(x), \\ V^\nu(x, 0) &= V_0^\nu(x), & V_t^\nu(x, 0) &= V_1^\nu(x), \end{aligned}$$

with the following boundary condition

$$U^\nu(0, t) = \Theta^\nu(0, t) = V^\nu(L, t) = \Theta_x^\nu(L, t) = 0 \quad \forall t > 0.$$

and transmission's condition

$$\begin{aligned} \beta V_x^\nu(L_0, t) &= aU_x^\nu(L_0, t) - \alpha\Theta^\nu(L_0, t) \quad \forall t > 0 \\ U^\nu(L_0, t) &= V^\nu(L_0, t) \quad \forall t > 0 \end{aligned}$$

satisfying

$$\int_0^T \left[|U^\nu(L_0, t)|^2 + \|q_x^\nu(\cdot, \tau)\|_{H^{\frac{3}{4}}}^2 \right] dt = 1 \quad (28)$$

where q^ν is solution of

$$\begin{aligned} -q_{xx}^\nu &= U_t^\nu & \text{in } (0, L_0) \times \mathbb{R}^+ \\ q^\nu(0, t) &= q^\nu(L_0, t) = 0 & \text{in } \mathbb{R}^+ \end{aligned}$$

and verifying the following inequality

$$1 \geq \delta_0 \int_0^T E(t; U^\nu, \Theta^\nu, V^\nu) dt + \nu \int_0^T \int_0^L |K * \Theta_x^\nu|^2 dx dt,$$

It follows that

$$\int_0^T E(t; U^\nu, \Theta^\nu, V^\nu) dt \text{ is bounded for any } \nu \in \mathbb{N} \quad (29)$$

and also that

$$K * \Theta_x^\nu \rightarrow 0 \quad \text{in } L^2(0, T; L^2(0, L)). \quad (30)$$

From (29) we have there exist a subsequence of $(U^\nu, \Theta^\nu, V^\nu)_{\nu \in \mathbb{N}}$, which we denote in the same way, such that

$$\begin{cases} U_t^\nu \xrightarrow{*} U_t & \text{in } L^\infty(0, T; L^2(0, L_0)) \\ \Theta^\nu \xrightarrow{*} \Theta & \text{in } L^\infty(0, T; L^2(0, L_0)) \\ V_t^\nu \xrightarrow{*} V_t & \text{in } L^\infty(0, T; L^2(L_0, L)) \\ U_x^\nu \xrightarrow{*} U_x & \text{in } L^\infty(0, T; L^2(0, L_0)) \\ V_x^\nu \xrightarrow{*} V_x & \text{in } L^\infty(0, T; L^2(L_0, L)) \end{cases} \quad (31)$$

Since $K * \Theta^\nu(0, t) = 0$ for any ν and any t we get that $K * \Theta(0, t) = 0$ and (30) implies $K * \Theta_x = 0$. So we get that $K * \Theta = 0$. Therefore, using the Volterra's resolvent we conclude $\Theta = 0$. From $-q_{xx}^\nu = U_t^\nu$ and (31) it follows that

$$(q_x^\nu, q_t^\nu) \quad \text{is bounded in} \quad [L^2(0, T; H^1(0, L_0))]^2.$$

Applying Lions-Aubin's Theorem (See [7]) we conclude that

$$q_x^\nu \rightarrow q_x \quad \text{in} \quad L^2\left(0, T; H^{\frac{3}{4}}(0, L_0)\right). \quad (32)$$

On the other side, from (31) and Lemma 0.7 it follows that

$$U^\nu \rightarrow U \quad \text{in} \quad C(0, T; H^r(0, L_0)) \quad (33)$$

for $r < 1$. Using (32) and (33) in (28) we have

$$\int_0^T \left[|U(L_0, t)|^2 + \|q_x(\cdot, t)\|_{H^{\frac{3}{4}}}^2 \right] dt = 1. \quad (34)$$

Substituing $\Theta = 0$ in (26) we have

$$U_{xt}^\nu = \gamma U_x^\nu.$$

Since $U(0, t) = 0$ we conclude that

$$U(x, t) = U(x, 0)e^{\gamma t}.$$

Using this fact in (25) we have that

$$-aU_{xx} = 0.$$

Since $U(0, t) = 0$ then we can write $U(x, t) = cxe^{\gamma t}$. On the other hand V satisfies

$$\begin{aligned} V_{tt} - \beta V_{xx} &= 2\gamma V_t - \gamma^2 V \\ V(L_0, t) &= cL_0 e^{\gamma t}, \quad \beta V_x(L_0, t) = ace^{\gamma t}, \quad V(L, t) = 0 \end{aligned}$$

Let us denote by $v(x, t) = e^{-\gamma t}V$ we have

$$v_{tt} - \beta v_{xx} = 0$$

$$v(L_0, t) = cL_0, \quad \beta v_x(L_0, t) = ac, \quad v(L, t) = 0.$$

Differentiating with respect to t and denoting by $W = v_t$ we have that

$$W_{tt} - \beta W_{xx} = 0$$

$$W(L_0, t) = 0, \quad W_x(L_0, t) = 0, \quad W(L, t) = 0.$$

Therefore we can extend the solution in the following way

$$\bar{W} = \begin{cases} 0 & x \in [0, L_0], \quad t > 0 \\ W & x \in [L_0, L], \quad t > 0 \end{cases}.$$

Since $\bar{W} \in H^1(0, L)$ then $\bar{W}$ is a weak solution of the wave equation vanishing over $[0, L_0]$. Therefore from Holgrem's Theorem (See [5]) we conclude that $\bar{W} = 0$, which implies that $v_t = 0$, therefore $V(x, t) = \varphi(x)e^{\gamma t}$. Then we get the following Elliptic transmission problem

$$-aU_{xx} = 0 \tag{35}$$

$$-\beta V_{xx} = 0 \tag{36}$$

$$U(L_0, t) = V(L_0, t), \quad aU_x(L_0, t) = \beta V_x(L_0, t) \quad U(0, t) = 0, \quad V(L, t) = 0.$$

Multiplying equation (35) by U and equation (36) by V we get

$$\begin{aligned} a \int_0^{L_0} |U_x|^2 dx - aU_x U|_0^{L_0} &= 0 \\ \beta \int_{L_0}^L |V_x|^2 dx - \beta V_x V|_{L_0}^L &= 0 \end{aligned}$$

Using the transmission conditions, we conclude that

$$U = V = 0$$

Therefore the convergence in (32) and (33) implies that

$$q_x = U = 0 \quad \text{a.e.} \quad \text{in} \quad (0, T) \times (0, L_0).$$

Hence, we arrive to

$$\int_0^T \left[|U(L_0, t)|^2 + \|q_x(\cdot, \tau)\|_{H^{\frac{3}{4}}}^2 \right] dt = 0$$

But this is contradictory with (34). The proof is now complete.

The exponential decay is summarized in the following Theorem.

Theorem 0.2 *Let us suppose that the initial data satisfy*

$$(u_0, v_0) \in [H^2(0, L_0) \times H^2(L_0, L)] \cap \mathcal{V}, \quad (u_1, v_1) \in \mathcal{V}, \quad \theta_0 \in H^1(0, L_0)$$

verifying the compatibility condition given by

$$au_{0,x}(L_0) - \alpha\theta_0(L_0) = \beta v_{0,x}(L_0)$$

there exist positive constant C_0 and γ such that

$$E(t; u, \theta, v) \leq C_0 E(0; u, \theta, v) e^{-\gamma t}.$$

Proof: From Lemmas 0.5 and 0.6 we have

$$\begin{aligned} \frac{d}{dt} \left[\int_0^{L_0} q_x \Theta \, dx + \frac{\alpha}{8K(0)} \mathcal{N}_1(t) \right] &\leq -\frac{7\alpha L_0}{256} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} \\ &\quad - \frac{7\alpha}{16} \int_0^{L_0} |U_t|^2 \, dx + \frac{5\alpha a}{32} \int_0^{L_0} |U_x|^2 \, dx \\ &\quad + \left(\frac{9\alpha}{8} + \frac{16ac_p^2 \delta_1}{\alpha L_0} + \frac{16a}{\alpha} \right) \int_0^{L_0} |\Theta|^2 \, dx \\ &\quad + \left(\frac{1}{2\alpha} + \frac{\alpha}{16K(0)} \right) \int_0^{L_0} |K * \Theta_x|^2 \, dx \\ &\quad + c_0 \int_0^{L_0} |K' * \Theta_x|^2 \, dx \\ &\quad + \left(\frac{64\alpha \lambda_1^2}{L_0} + \frac{16\lambda_1^2 K(0)}{\alpha L_0} \right) \|q_x(., t)\|_{H^{3/4}}^2 \\ &\quad + c\gamma \left\{ E_1(t; U, \Theta) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right\} \end{aligned}$$

where $c_0 = \left(\frac{L_0^2 \alpha}{64K(0)^2} + \frac{\alpha^3 L_0^2}{64aK(0)^2} \right).$

Let us introduce the functional

$$\mathcal{N}_2(t) = \int_0^{L_0} q_x \Theta \, dx + \frac{\alpha}{8K(0)} \mathcal{N}_1(t) + \frac{3\alpha}{8} \int_0^{L_0} U U_t \, dx.$$

From last inequality and (21) of the Lemma 0.4 we get

$$\begin{aligned} \frac{d}{dt} \mathcal{N}_2(t) &\leq -\frac{3\alpha L_0}{128} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} \\ &\quad - \frac{\alpha a}{32} \int_0^{L_0} |U_x|^2 \, dx - \frac{\alpha}{16} \int_0^{L_0} |U_t|^2 \, dx + c_0 \int_0^{L_0} |K' * \Theta_x|^2 \, dx \\ &\quad + c_1 |U(L_0, t)|^2 + \left(\frac{64\alpha \lambda_1^2}{L_0} + \frac{16\lambda_1^2 K(0)}{\alpha L_0} \right) \|q_x(., t)\|_{H^{3/4}}^2 \\ &\quad + c_2 \int_0^{L_0} |\Theta|^2 \, dx + c_3 \int_0^{L_0} |K * \Theta_x|^2 \, dx \\ &\quad + c\gamma \left\{ E_1(t; U, \Theta) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right\} \end{aligned} \tag{37}$$

where

$$c_1 = \frac{3\alpha}{8} \left(\frac{24a}{L_0} + \frac{24\alpha^2}{L_0} \right), \quad c_2 = \left(\frac{9\alpha}{8} + \frac{16ac_p^2\delta_1}{\alpha L_0} + \frac{16a}{\alpha} + \frac{3\alpha^3}{16a} \right),$$

$$c_3 = \left(\frac{1}{2\alpha} + \frac{\alpha}{16K(0)} \right).$$

Note that

$$|K * \Theta(L_0, t)|^2 \leq c_p^2 \int_0^{L_0} |K * \Theta_x|^2 dx.$$

$$\left| \alpha \int_0^{L_0} U_t K * \Theta_x dx \right| \leq \frac{\alpha K(0)}{96c_2} \int_0^{L_0} |U_t|^2 dx + \frac{24c_2}{K(0)} \int_0^{L_0} |K * \Theta_x|^2 dx,$$

$$|\alpha U_t(L_0, t) K * \Theta(L_0, t)| \leq \frac{\alpha L_0 K(0)}{768c_2} |U_t(L_0, t)|^2 + \frac{192\alpha c_2 c_p^2}{L_0 K(0)} \int_0^{L_0} |K * \Theta_x|^2 dx,$$

$$\left| \int_0^{L_0} \Theta K' * \Theta dx \right| \leq \frac{K(0)}{2} \int_0^{L_0} |\Theta|^2 dx + \frac{c_p^2}{2K(0)} \int_0^{L_0} |K' * \Theta_x|^2 dx.$$

Using the above estimate in (20) from Lemma 0.4 it follows

$$\begin{aligned} -\frac{d}{dt} \int_0^{L_0} K * \Theta \Theta dx &\leq -\frac{K(0)}{2} \int_0^{L_0} |\Theta|^2 dx + \frac{\alpha L_0 K(0)}{768c_2} |U_t(L_0, t)|^2 \\ &\quad + \frac{\alpha K(0)}{96c_2} \int_0^{L_0} |U_t|^2 dx + \frac{c_p^2}{2K(0)} \int_0^{L_0} |K' * \Theta_x|^2 dx \\ &\quad + \left(1 + \frac{192\alpha c_2 c_p^2}{L_0 K(0)} + \frac{24c_2}{K(0)} \right) \int_0^{L_0} |K * \Theta_x|^2 dx \\ &\quad + c\gamma \left\{ E_1(t; U, \Theta) + \int_0^{L_0} |K * \Theta_x|^2 dx \right\}. \end{aligned} \quad (38)$$

Let us write

$$\mathcal{N}_3(t) = \mathcal{N}_2(t) - \frac{3c_2}{K(0)} \int_0^{L_0} K * \Theta \Theta dx.$$

From (37) and (38) we get

$$\begin{aligned} \frac{d}{dt} \mathcal{N}_3(t) &\leq -\frac{5\alpha L_0}{256} \left\{ a |U_x(L_0, t)|^2 + |U_t(L_0, t)|^2 + |\Theta(L_0, t)|^2 \right\} \\ &\quad - \frac{\alpha a}{32} \int_0^{L_0} |U_x|^2 dx - \frac{\alpha}{32} \int_0^{L_0} |U_t|^2 dx - \frac{c_2}{2} \int_0^{L_0} |\Theta|^2 dx \\ &\quad + c_1 |U(L_0, t)|^2 + c_4 \int_0^{L_0} |K * \Theta_x|^2 dx + c_5 \int_0^{L_0} |K' * \Theta_x|^2 dx \\ &\quad + c_6 \|q_x(., t)\|_{H^{3/4}}^2 + c\gamma \left\{ E_1(t; U, \Theta) + \int_0^{L_0} |K * \Theta_x|^2 dx \right\} \end{aligned} \quad (39)$$

where

$$c_4 = \left[c_3 + \frac{3c_2}{K(0)} \left(1 + \frac{192\alpha c_2 c_p^2}{L_0 K(0)} + \frac{24c_2}{K(0)} \right) \right], \quad c_5 = \left(c_0 + \frac{3c_2 c_p^2}{2K(0)^2} \right)$$

$$c_6 = \left(\frac{64\alpha\lambda_1^2}{L_0} + \frac{16\lambda_1^2 K(0)}{\alpha L_0} \right).$$

Let us introduce the following functional:

$$\mathcal{N}_4(t) = \mathcal{N}_3(t) + \frac{\alpha L_0}{64\lambda_0} \int_{L_0}^L (x - L) V_x V_t \, dx.$$

From (22) of the Lemma 0.4 and last inequality

$$\begin{aligned} \frac{d}{dt} \mathcal{N}_4(t) \leq & -\mu_0 E(t; U, \Theta, V) \\ & + \mu_1 \left(|U(L_0, t)|^2 + \|q_x(\cdot, t)\|_{H^{3/4}}^2 \right) \\ & + c_4 \int_0^{L_0} |K * \Theta_x|^2 \, dx + c_5 \int_0^{L_0} |K' * \Theta_x|^2 \, dx \\ & + c\gamma \left\{ E(t; U, \Theta, V) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right\} \end{aligned} \quad (40)$$

where $\mu_0 = C(\alpha, a, L_0, \beta, \lambda_0, c_p, \delta_1)$ and $\mu_1 = \max\{c_1 + \frac{\alpha L_0 \bar{c}}{256\lambda_0^2}, c_6\}$. Now we define

$$\begin{aligned} \mathcal{L}(t) &= \mathcal{N}_4(t) + NE(t; U, \Theta, V) \\ \mathcal{F}(t) &= \mathcal{L}(t) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \end{aligned}$$

Observe that

$$\begin{aligned} \frac{d}{dt} \int_0^{L_0} |K * \Theta_x|^2 \, dx &= 2 \int_0^{L_0} K' * \Theta_x K * \Theta_x \, dx + 2K(0) \int_0^{L_0} K * \Theta_x \Theta_x \, dx \\ &\leq \int_0^{L_0} |K' * \Theta_x|^2 \, dx + \int_0^{L_0} |K * \Theta_x|^2 \, dx + 2K(0) \int_0^{L_0} K * \Theta_x \Theta_x \, dx \end{aligned}$$

From this, Lemma 0.3 and (40) we get

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t) \leq & -\mu_0 E(t; U, \Theta, V) + \mu_1 \left(|U(L_0, t)|^2 + \|q_x(\cdot, t)\|_{H^{3/4}}^2 \right) \\ & + (c_4 + 1) \int_0^{L_0} |K * \Theta_x|^2 \, dx + (c_5 + 1) \int_0^{L_0} |K' * \Theta_x|^2 \, dx \\ & - (N - 2K(0)) \int_0^{L_0} K * \Theta_x \Theta_x \, dx + Nc\gamma E(t; U, \Theta, V) \\ & + c\gamma \left\{ E(t; U, \Theta, V) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right\}. \end{aligned}$$

Integrating from 0 to t the above relation, using Lemma 0.8 we have

$$\begin{aligned} \mathcal{F}(t) - \mathcal{F}(0) \leq & -\mu_0 \int_0^t E(\tau; U, \Theta, V) \, d\tau + \mu_1 \delta \int_0^t E(\tau; U, \Theta, V) \, d\tau \\ & - (N - 2K(0)) \int_0^t \int_0^{L_0} K * \Theta_x \Theta_x \, dx \, d\tau \end{aligned}$$

$$\begin{aligned}
& +(\mu_1 C_\delta + c_4 + 1) \int_0^t \int_0^{L_0} |K * \Theta_x|^2 \, dx \, d\tau \\
& +(c_5 + 1) \int_0^t \int_0^{L_0} |K' * \Theta_x|^2 \, dx \, d\tau + Nc\gamma \int_0^t E(\tau; U, \Theta, V) d\tau \\
& +c\gamma \int_0^t \left[E(\tau; U, \Theta, V) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right] d\tau.
\end{aligned}$$

Observe that, using Lemma 0.1, hypotheses on k and k' we get

$$\begin{aligned}
\int_0^t \int_0^{L_0} K * \Theta_x \Theta_x \, dx d\tau & \geq C \int_0^t \int_0^{L_0} |K * \Theta_x|^2 \, dx d\tau \\
\int_0^t \int_0^{L_0} |K' * \Theta_x|^2 \, dx d\tau & \leq C \int_0^t \int_0^{L_0} |K * \Theta_x|^2 \, dx d\tau
\end{aligned}$$

From this and choosing $N > 4K(0)$ we conclude that

$$\begin{aligned}
\mathcal{F}(t) - \mathcal{F}(0) & \leq -\mu_0 \int_0^t E(\tau; U, \Theta, V) d\tau + \mu_1 \delta \int_0^t E(\tau; U, \Theta, V) d\tau \\
& -\left(\frac{NC}{2} - \mu_1 C_\delta - c_4 - 1 - c_5 C - C + c\gamma\right) \int_0^t \int_0^{L_0} |K * \Theta_x|^2 \, dx d\tau \\
& +(N+1)c\gamma \int_0^t E(\tau; U, \Theta, V) d\tau
\end{aligned}$$

Choose $\delta > 0$ such that $\delta\mu_1 < \frac{\mu_0}{4}$. Taking N large enough and γ sufficiently small we conclude that

$$\mathcal{F}(t) - \mathcal{F}(0) \leq -\frac{\mu_0}{2} \int_0^t \left[E(\tau; U, \Theta, V) + \int_0^{L_0} |K * \Theta_x|^2 \, dx \right] d\tau < 0$$

and

$$\begin{aligned}
\mathcal{L}(t) & \leq k_1 [E(t; U, \Theta, V) + \int_0^{L_0} |K * \Theta_x|^2 \, dx] \\
\mathcal{L}(t) & \geq k_2 [E(t; U, \Theta, V) - \int_0^{L_0} |K * \Theta_x|^2 \, dx]
\end{aligned}$$

where k_1 and k_2 are positive constants.

Since

$$\begin{aligned}
E(t; U, \Theta, V) & \leq \frac{1}{k_2} [\mathcal{L}(t) + \int_0^{L_0} |K * \Theta_x|^2 \, dx] \\
& \leq \frac{1}{k_2} \mathcal{F}(t) \leq \frac{1}{k_2} \mathcal{F}(0) \\
& \leq \frac{1}{k_2} \mathcal{L}(0) \leq \frac{k_1}{k_2} E(0; U, \Theta, V).
\end{aligned}$$

Our conclusion follows. The proof is now complete. $\square$

References

- [1] B.D. Coleman, M.E. Gurtin, *Equipresence and constitutive equations for rigid heat conductors*, Z. Angew. Math. Phys **18**, 199-208 (1967).
- [2] M. Fabrizio and B. Lazzari, *On the existence and asymptotic stability of solutions for linearly viscoelastic solids*, Arch. Rat. Mech. Anal. **116**, 139-152 (1991).
- [3] C. Giorgi, M. G. Naso; *On the exponential stability of linear non-Fourier thermoviscoelastic bar*, quaderni del Seminario Matematico di Brescia **2/97**, (1997).
- [4] M.E. Gurtin, A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rational Mech. Anal. **31**, 113-126 (1968).
- [5] L.Hörmander, *Linear partial differential operators*, Springer Verlag, New York, (1976).
- [6] J.U.Kim, *A boundary thin obstacle problem for a wave equation*. Commun. in Partial Differential Equations, 14(8&9), 1011-1026 (1989).
- [7] , J. L. Lions, *Quelques methodes de résolution des problénes aux limites non linéaires*. Dunod, Paris 1969.
- [8] J.L. Lions, *Contrôlabilité exacte perturbations et stabilisation de systèmes distribués*. Collection RMA, Masson Paris 1988 (tome 1).
- [9] J.E. Muñoz Rivera; *Asymptotic Behaviour in Linear Viscoelasticity*. Quarterly of Applied Mathematics, **III-4**, 629-648 (1994).
- [10] O. J. Staffans; *On a nonlinear hyperbolic Volterra Equation*. Siam J. Math, Anal. **11**, 5, 793-812 (1980).