EXISTENCE AND EXPONENTIAL DECAY IN NONLINEAR THERMOELASTICITY

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1. INTRODUCTION

It is well known that in the absence of dissipation, smooth solution of nonlinear elastic materials develop singularities in finite time, while for thermoelastic materials the conduction of the heat equation provides dissipation that competes with the destabilizing effect of nonlinearity in the elastic response. The level of subtlety of this dissipation depends on the boundary condition that the displacement and the thermal difference are supporting. Slemrod [1] showed the global existence of smooth solution for small data when the boundary is either traction-free and at a constant temperature or rigidly clamped and thermally insulated. A similar result was obtained by Zheng [2]. These boundary conditions get a simpler damping mechanism because they imply additional boundary conditions for u and the thermal difference θ, that is, if an end is clamped then the displacement u and the thermal difference θ satisfy $u_{xx} = 0$ and $θ_{xxx} = 0$ there respectively. So we can make additional partial integrations which led to the desire a priori $L^2$-estimate.

In case of Dirichlet boundary condition for which the boundary is rigidly clamped and held at a constant temperature we lost the value of $u_{xx}$ in that point and instead of it we get $u_{xx} + αθ_x = 0$. So this case leads ill behaved boundary terms and it is not possible to apply directly the multiplicative techniques to secure global estimate. Recently Racke and Shibata [3] proved Global existence of a smooth solution for these boundary conditions. To do this the authors showed the algebraic decay of the energy for the linear equation by studying the spectral properties of the stationary linearized problem. The rate of decay depends on higher regularity of the initial data and therefore the global existence result depends on the initial data to be small in $H^m(0, L)$ with $m$ large. One of the authors of this paper proved in [4] (see also the work of Kim [5]) that the solution of the linearized thermoelastic system decays exponentially as time goes to infinity. This fact allows us to get simpler existence result for the corresponding nonlinear equation as was shown in [6] for small data $(u_0, u_1)$ in $H^3(0, L) \times H^2(0, L)$.

The system in question is written as follows

$$u_{tt} - [W, u]_t = 0, \quad \text{in } [0, L] \times [0, \infty[ \quad (1.1)$$

$$(θ + τ_0)[N(u_x, θ)]_t - [Q(u_x, θ_x, θ)]_x = 0, \quad \text{in } [0, L] \times [0, \infty[ \quad (1.2)$$

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With the initial data given by

\[ u(x, 0) = u_0(x); \quad u_t(x, 0) = u_1(x); \quad \theta(x, 0) = \theta_0(x) \]  

(1.3)

and boundary conditions

\[ u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0. \]  

(1.4)

We are denoting by \( u \) the displacement, by \( \theta = T - \tau \) the thermal difference, where \( T \) is the absolute temperature and \( \tau \) is the reference temperature which we will assume to be constant. Finally, by \( S \) we will denote the stress tensor of Piola-Kirchoff, \( N \) stands for the specific entropy and by \( Q \) the heat flux.

We would like to remark that the dissipation given by the thermal difference is not strong enough to prevent development of singularities. The work of Hrusa and Messauodi showed that for a special class of nonlinear thermoelastic materials which occupy the whole line, there are smooth initial data for which the solution will develop singularities in finite time.

The main result of this paper is to improve the work in [6] by taking initial data \((u_0, u_1)\) small in \( H^1(0, L) \times H^1(0, L) \)-norm. This fact allows us to choose large data \((u_0, u_1)\) in the \( H^3(0, L) \times H^2(0, L) \)-norm. The approach we use here is different from others, we explore the dissipative properties to construct a Liapunov functional whose derivative is negative proportional to itself and we look for estimates of the nonlinear terms in functions of the dissipative terms associated to the thermoelastic system. The fact together with the local existence result (see [1]) give the estimate we need to get the global existence of smooth solutions.

2. EXISTENCE AND ASYMPTOTIC BEHAVIOUR

In this section we will assume that the functions \( S, N, Q \) are in \( C^4 \) satisfying the following hypotheses

\[
\begin{align*}
\frac{\partial S}{\partial u_x}(0, 0) &= 1; \quad \frac{\partial S}{\partial \theta}(0, 0) = 0; \quad \frac{\partial N}{\partial u_x}(0, 0) = 0; \\
\frac{\partial N}{\partial \theta}(0, 0) &= 0; \quad \frac{\partial Q}{\partial u_x}(0, 0, 0) = 0; \quad \frac{\partial Q}{\partial \theta}(0, 0, 0) = 0.
\end{align*}
\]  

(2.1)

To simplify notations we will introduce

\[
-\frac{\partial S}{\partial \theta}(0, 0) =: \alpha; \quad \frac{\partial Q}{\partial \theta}(0, 0, 0) = k > 0; \quad \frac{\partial N/\partial u_x}{\partial N/\partial \theta}(0, 0, 0) = \beta.
\]

Where the product \( \alpha \beta > 0 \). For the initial data we will impose

\[ u_0 \in H^3(0, L); \quad u_1 \in H^2(0, L); \quad u_2 \in H^1(0, L); \quad \theta_1 \in H^2(0, L); \quad \theta_0 \in H^3(0, L). \]  

(2.2)

By \( u_2 \) and \( \theta_1 \) we are denoting

\[
u_2 =: [S(u_x, \theta)]_{x=0} \quad \theta_1 =: \left[ \frac{\partial N}{\partial u_x} u_x + \frac{Q(u_x, \theta_x, \theta)}{(\theta + \tau_0)(\partial N/\partial \theta)} \right]_{x=0}
\]  

(2.3)

(2.4)

satisfying the compatibility conditions

\[ u_0 = u_1 = u_2 = \theta_1 = \theta_0 = 0 \quad \text{at } x = L, \ x = 0. \]  

(2.5)
With this hypotheses we can show that there exists only one local solution for system (1.1)-(1.4) (see [7]), defined in the maximal interval \([0, T_m] \). So, to get global smooth solution we will show that

\[
\|\theta(t, \cdot)\|_{H^2(0, L)} + \|u(t, \cdot)\|_{H^2(0, L)} \leq c; \quad \forall t \geq 0.
\]

To do this we will regard system (1.1)-(1.4) as

\[
\begin{align*}
    u_{tt} - u_{xx} + \alpha \theta_x &= F, \quad \text{in } ]0, T_m[ \times ]0, T_m[; \\
    \theta_t - k \theta_{xx} + \beta u_{xt} &= G, \quad \text{in } ]0, T_m[ \times ]0, T_m[. \\
\end{align*}
\]

Where

\[
F = \left\{ \frac{\partial S}{\partial u_x}(u_x, \theta) - 1 \right\} u_{xx} + \left\{ \frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha \right\} \theta_x
\]

\[
G = \left\{ \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)} - k \right\} \theta_{xx} - \left\{ \frac{\partial N/\partial u_x}{\partial N/\partial \theta} - \beta \right\} u_{xt}
\]

\[
+ \frac{\partial Q/\partial u_x}{(\theta + \tau_0)(\partial N/\partial \theta)} u_{xx} + \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)} \theta_x.
\]

For simplicity we will put

\[
\eta_1 = \frac{\partial S}{\partial u_x}(u_x, \theta) - 1; \quad \eta_2 = \frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha;
\]

\[
W_1 = \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)} - k; \quad W_2 = \frac{\partial N/\partial u_x}{\partial N/\partial \theta} - \beta;
\]

\[
W_3 = \frac{\partial Q/\partial u_x}{(\theta + \tau_0)(\partial N/\partial \theta)}; \quad W_4 = \frac{\partial Q/\partial \theta}{(\theta + \tau_0)(\partial N/\partial \theta)}.
\]

To facilitate our analysis let us introduce the linear system

\[
\begin{align*}
    U_{tt} - U_{xx} + \alpha \psi_x &= \Phi, \quad \text{in } ]0, L[ \times ]0, T_m[; \\
    \psi_t - k \psi_{xx} + \beta U_{xt} &= \Gamma, \quad \text{in } ]0, L[ \times ]0, T_m[. \\
\end{align*}
\]

\[
U(x, 0) = U_0, \quad U_t(x, 0) = U_1, \quad \psi(x, 0) = \psi_0,
\]

\[
U(0, t) = U(L, t) = \psi(0, t) = \psi(L, t) = 0.
\]

From now on and without loss of generality we will assume that \(\alpha\) and \(\beta\) are positive real numbers. First we will study the asymptotic behaviour of the linearized equation (2.8)-(2.9). To do this we define the following functionals

\[
E_1(t; U, \psi) = \frac{1}{2} \int_0^L \left\{ |U_t|^2 + |U_x|^2 + \frac{\alpha}{\beta} |\psi|^2 \right\} \, dx;
\]

\[
E_2(t; U, \psi) = \frac{1}{2} \int_0^L \left\{ |U_{xt}|^2 + |U_{xx}|^2 + \frac{\alpha}{\beta} |\psi_{x1}|^2 \right\} \, dx
\]

\[
E_3(t; U, \psi) = \frac{1}{2} \int_0^L \left\{ |U_{xt}|^2 + |U_{xx}|^2 + \frac{\alpha}{\beta} |\psi_{x1}|^2 \right\} \, dx.
\]
Let us multiply equation (2.8) by $U$, and (2.9) by $(\alpha/\beta)\psi$ and summing the product result we have
\[
\frac{d}{dt} E_1(t; U, \psi) = -k \frac{\alpha}{\beta} \int_0^L \left( \sum_{i=0}^{L} |\psi_{xi}|^2 \right) dx + \int_0^L \left( \sum_{i=0}^{L} \left( \frac{\alpha}{\beta} U_{xi} + \frac{\alpha}{\beta} \psi_{xi} \right) \right) dx.
\]

Assuming regular data, and since $U_t$ and $\psi_t$ have the same boundary conditions, we get
\[
\frac{d}{dt} E_2(t; U, \psi) = -k \frac{\alpha}{\beta} \int_0^L \left( \sum_{i=0}^{L} |\psi_{xi}|^2 \right) dx + \int_0^L \left( \sum_{i=0}^{L} \left( \frac{\alpha}{\beta} U_{xi} + \frac{\alpha}{\beta} \psi_{xi} \right) \right) dx.
\]

To get the above identities we use essentially the fact that $U_t$ and $\psi_t$ have the same boundary conditions as $U$ and $\psi$. But this is not the case for $U_x$ and $\psi_x$. It is in this point that the typical difficulty for boundary conditions of type Dirichlet-Dirichlet appears. Let us see in detail this fact. Multiplying equation (2.8) by $-U_{xxt}$ and (2.9) by $-(\alpha/\beta)\psi_{xx}$ and summing up the product result we have
\[
\frac{d}{dt} E_3(t; U, \psi) = -k \frac{\alpha}{\beta} \int_0^L \left( \sum_{i=0}^{L} |\psi_{xx}|^2 \right) dx + \int_0^L \left( \sum_{i=0}^{L} \left( \frac{\alpha}{\beta} U_{xx} + \frac{\alpha}{\beta} \psi_{xx} \right) \right) dx.
\]

The derivative of $E_3$ has a pointwise term involving second order derivatives. Which is not possible to bound using directly the Sobolev inequalities. To overcome this fact we will use the following lemma.

**Lemma 2.1.** Let us take $(\nu_0, \nu_1, f) \in H^1_0(0, L) \cap H^2(0, L) \times H^1_0(0, L) \times H^1(0, T; L^2(0, L))$ and let $\nu$ be the solution of
\[
\begin{align*}
\nu_{xx} - \nu_{tt} = f(x, t) & \text{ in } ]0, L[ \times ]0, T[ \\
\nu(0, t) = \nu(L, t) = 0 & \text{ on } ]0, L[ \times ]0, T[
\end{align*}
\]

then the following identity holds
\[
\frac{L}{4} [\nu_x^2(L, t) + \nu_x^2(0, t)] = \frac{d}{dt} \int_0^L \left( x - \frac{L}{2} \right) \nu_x \psi_x dx + \frac{1}{2} \int_0^L \left( \nu_x^2 + \nu_t^2 \right) dx - \int_0^L \left( x - \frac{L}{2} \right) f \psi_x dx.
\]

**Proof.** Multiplying (2.12) by $(x - L/2)\nu_x$ and integrating over $[0, L]$ we have
\[
\int_0^L \left( x - \frac{L}{2} \right) \nu_{xx} \psi_x dx = \int_0^L \left( x - \frac{L}{2} \right) \nu_x \psi_x dx = \int_0^L \left( x - \frac{L}{2} \right) f \psi_x dx.
\]
Since \( v_i(0, t) = v_i(L, t) = 0 \), direct calculations yields

\[
\int_0^L \left( x - \frac{L}{2} \right) v_{1, x} v_x \, dx = \frac{d}{dt} \int_0^L \left( x - \frac{L}{2} \right) v_{1, x} v_x \, dx - \int_0^L \left( x - \frac{L}{2} \right) v_{1, x x} \, dx
\]

\[
= \frac{d}{dt} \int_0^L \left( x - \frac{L}{2} \right) v_{1, x} v_x \, dx + \frac{1}{2} \int_0^L v_t^2 \, dx. \tag{2.14}
\]

On the other hand

\[
\int_0^L \left( x - \frac{L}{2} \right) v_{1, x x} v_x \, dx = \frac{1}{2} \int_0^L \left( x - \frac{L}{2} \right) (v_t^2)_{x} \, dx
\]

\[
= \frac{L}{4} [v_t^2(L, t) + v_t^2(0, t)] - \frac{1}{2} \int_0^L v_t^2 \, dx. \tag{2.15}
\]

From (2.13) to (2.15) our result follows. ■

Motivated in Lemma 2.1 we introduce the following functional

\[
E_4(t) = -\int_0^L \left( x - \frac{L}{2} \right) U_{xx} U_{tt} \, dx.
\]

Using equation (2.8) and Lemma 2.1 we easily get

\[
\frac{d}{dt} E_4(t) = -\frac{L}{4} \left( |U_{xx}(0, t)|^2 + |U_{xx}(L, t)|^2 \right) + \frac{1}{2} \int_0^L \left( |U_{xx}|^2 + |U_{tt}|^2 \right) \, dx
\]

\[
+ \alpha \int_0^L \left( x - \frac{L}{2} \right) \psi_{xx} U_{xx} \, dx - \int_0^L \left( x - \frac{L}{2} \right) \Phi_t U_{xx} \, dx. \tag{2.16}
\]

Finally, we define the following functions

\[
E_5(t) = \int_0^L U_{xx} \psi \, dx; \quad E_6(t) = \int_0^L U_{xx} U_x \, dx,
\]

\[
\Phi(t; U, \psi) = \int_0^L \left( |U_t|^2 + |U_{xx}|^2 + |U_{xx}|^2 + |\psi|^2 + |\psi_t|^2 + |\psi_{xx}|^2 \right) \, dx
\]

\[
\Psi(t; U, \psi) = \int_0^L \left( |U_{xx}|^2 + |U_{xx}|^2 + |\psi_{xx}|^2 + |\psi_{xx}|^2 \right) \, dx.
\]

We will prove in the following lemma that there exists a linear combination of the functions \( E_i \) (\( i = 1, \ldots, 6 \)), we will denote by \( K \), that is

\[
K(t; u, \psi) = \kappa_1 E_1 + \kappa_2 E_2 + \frac{10k}{\alpha} E_3 + \frac{\beta}{6} E_4 + E_5 + \frac{\beta}{2} E_6,
\]

which is a Liapunov functional. This is shown more precisely in the following lemma.
Lemma 2.2. There exist positive constants $\kappa_i$ ($i = 1, 2$) and $c_0$, $c_1$ such that the derivative of $K(t; U, \psi)$ defined above satisfies

$$
\frac{d}{dt} K(t; U, \psi) \leq -\frac{k^2}{\beta} \int_0^L \left( |\psi_x|^2 + |\psi_{xx}|^2 + |\psi_{xxx}|^2 \right) dx - \beta \frac{L}{8} \int_0^L \left( |U_{xx}|^2 + |U_{xxx}|^2 \right) dx
$$

$$
- \frac{\beta L}{48} \left( |U_{xx}(0, t)|^2 + |U_{xx}(L, t)|^2 \right) + R(t; U, \psi), \quad (2.17)
$$

where

$$
R(t; U, \psi) = \kappa_1 \int_0^L \left\{ \mathcal{F} U_x + \frac{\alpha}{\beta} \mathcal{S} \psi \right\} dx + \kappa_2 \int_0^L \left\{ \mathcal{S} \psi_{xx} + \frac{\alpha}{\beta} \mathcal{G} \psi_{xx} \right\} dx
$$

$$
- \frac{10k}{\alpha} \int_0^L \left\{ \mathcal{S} \psi_{xx} + \frac{\alpha}{\beta} \mathcal{G} \psi_{xx} \right\} dx - \frac{\beta}{6} \int_0^L \left( x - \frac{L}{2} \right) \mathcal{S} \psi U_x dx
$$

$$
- \frac{10}{\alpha} \int_0^L \left\{ \mathcal{S} \psi_x - \mathcal{G} H_{xx} \right\} dx - \frac{\beta}{2} \int_0^L \mathcal{F} \psi_{xx} dx + \frac{\beta}{4} \int_0^L |\mathcal{F}|^2 dx,
$$

and

$$
c_0 \mathcal{H}(t; U, \psi) \leq K(t, U, \psi) \leq c_1 \mathcal{H}(t; U, \psi). \quad (2.18)
$$

Proof. Using (2.8)–(2.9) we obtain that

$$
\frac{d}{dt} E_5(t) = -\beta \int_0^L |U_{xx}|^2 dx + \alpha \int_0^L |\psi_x|^2 dx + k \int_0^L \psi_x U_x dx
$$

$$
- \int_0^L \psi_x U_{xx} dx - \int_0^L \left( \mathcal{F} \psi_x - \mathcal{G} U_{xx} \right) dx, \quad (2.19)
$$

similarly, using (2.8) we get

$$
\frac{d}{dt} E_6(t) = \int_0^L |U_{xx}|^2 dx - \int_0^L |U_{xx}|^2 dx + \alpha \int_0^L U_{xx} \psi_x dx - \int_0^L \mathcal{F} U_{xx} dx. \quad (2.20)
$$

From (2.19) and (2.20) we easily obtain

$$
\frac{d}{dt} \left\{ E_5(t) + \frac{\beta}{2} E_6(t) \right\} = -\frac{\beta}{2} \int_0^L |U_{xx}|^2 dx - \frac{\beta}{2} \int_0^L |U_{xx}|^2 dx + \alpha \int_0^L |\psi_x|^2 dx + k \int_0^L \psi_x U_x dx - \int_0^L U_{xx} \psi_x dx
$$

$$
+ \alpha \int_0^L \psi_x^2 dx + k \int_0^L \psi_x U_x dx - \int_0^L U_{xx} \psi_x dx
$$

$$
- \frac{\beta}{2} \int_0^L \mathcal{F} U_{xx} dx - \int_0^L \left( \mathcal{F} \psi_x - \mathcal{G} U_{xx} \right) dx. \quad (2.21)
$$

On the other hand using (2.8) it is not difficult to see that

$$
\int_0^L |U_{xx}|^2 dx \leq 3 \int_0^L \left[ |U_{xx}|^2 + \alpha^2 |\psi_x|^2 + |\mathcal{F}|^2 \right] dx.
$$
From (2.16) we get

\[
\frac{d}{dt} E_4(t) \leq - \frac{L}{4} \left( |U_{xx}(0, t)|^2 + |U_{xx}(L, t)|^2 \right) + \frac{3}{2} \int_0^L \left( |U_{xx}|^2 + |U_{xxr}|^2 \right) dx
+ \frac{3\alpha^2}{2} \int_0^L |\psi_x|^2 dx + \alpha \int_0^L (x - \frac{L}{2}) \psi_{xx} U_{xxr} dx
- \int_0^L \left( x - \frac{L}{2} \right) \mathcal{F}_r U_{xx} dx + \frac{3}{2} \int_0^L |\mathcal{F}|^2 dx.
\]  

Relation (2.21) together with (2.22) yields

\[
\frac{d}{dt} \left( \beta E_4(t) + E_5(t) + \frac{\beta}{2} E_6(t) \right)
\leq - \frac{\beta L}{24} \left( |U_{xx}(0, t)|^2 + |U_{xx}(L, t)|^2 \right) - \frac{\beta}{4} \int_0^L \left( |U_{xxr}|^2 + |U_{xxx}|^2 \right) dx
+ \frac{\alpha^2 \beta}{4} \int_0^L |\psi_x|^2 dx + \frac{L\alpha \beta}{12} \int_0^L \psi_{xx} U_{xxr} dx + \left( \frac{\alpha \beta}{2} - 1 \right) \int_0^L U_{xx} \psi_x dx
+ \alpha \int_0^L |\psi_x|^2 dx + k \int_0^L \psi_{xx} U_{xxr} dx - \int_0^L \left( \mathcal{F} \psi_x - \mathcal{G} U_{xxr} \right) dx
- \frac{\beta}{6} \int_0^L \left( x - \frac{L}{2} \right) \mathcal{F}_r U_{xxr} dx + \frac{\beta}{4} \int_0^L |\mathcal{F}|^2 dx - \frac{\beta}{2} \int_0^L |\mathcal{F}|^2 dx.
\]  

Note that

\[
\frac{L\alpha \beta}{12} \int_0^L \psi_{xx} U_{xxr} dx + k \int_0^L \psi_{xx} U_{xxr} dx
\leq \frac{\beta}{8} \int_0^L |U_{xxr}|^2 dx + \frac{L^2 \alpha^2 \beta}{36} \int_0^L |\psi_{xxr}|^2 dx + \frac{4k^2}{\beta} \int_0^L |\psi_{xxr}|^2 dx
\]  

and

\[
\left( \frac{\alpha \beta}{2} - 1 \right) \int_0^L U_{xxr} \psi_x dx \leq \frac{\beta}{8} \int_0^L |U_{xxr}|^2 dx + \frac{2}{\beta} \left( \frac{\alpha \beta}{2} - 1 \right)^2 \int_0^L |\psi_x|^2 dx.
\]  

Substitution of (2.24) and (2.25) into (2.23) yields

\[
\frac{d}{dt} \left( \beta E_4(t) + E_5(t) + \frac{\beta}{2} E_6(t) \right)
\leq - \frac{\beta L}{24} \left( |U_{xx}(0, t)|^2 + |U_{xxr}(L, t)|^2 \right) - \frac{\beta}{8} \int_0^L \left( |U_{xxr}|^2 + |U_{xxx}|^2 \right) dx + c_2 \int_0^L |\psi_x|^2 dx
+ \frac{4k^2}{\beta} \int_0^L |\psi_{xxr}|^2 dx + \frac{L^2 \alpha^2 \beta}{36} \int_0^L |\psi_{xxr}|^2 dx - \int_0^L \left( \mathcal{F} \psi_x - \mathcal{G} U_{xxr} \right) dx
- \frac{\beta}{6} \int_0^L \left( x - \frac{L}{2} \right) \mathcal{F}_r U_{xxr} dx + \frac{\beta}{4} \int_0^L |\mathcal{F}|^2 dx - \frac{\beta}{2} \int_0^L |\mathcal{F}|^2 dx.
\]
From the Gagliardo–Nirenberg inequality we get
\[ \alpha \psi_x U_t \bigg|_{x=0}^{x=L} = 2\alpha c_0 \left( \int_0^L \psi_x^2 \, dx \right)^{1/4} \left( \int_0^L (|\psi_x|^2 + |\psi_{xx}|^2) \, dx \right)^{1/4} \left( \int_0^L (|U_t(0,t)|^2 + |U_t(L,t)|^2) \right)^{1/2} \]
\[ \leq c_3 \int_0^L |\psi_x|^2 \, dx + \frac{k\alpha}{2\beta} \int_0^L |\psi_{xx}|^2 \, dx + \frac{\alpha\beta L}{480k} \left( |U_t(0,t)|^2 + |U_t(L,t)|^2 \right), \]
for a positive constant \( c_3 \). Using (2.11) we get
\[ \frac{d}{dt} E_3(t) \leq -\frac{k\alpha}{2\beta} \int_0^L |\psi_{xx}|^2 \, dx + c_3 \int_0^L |\psi_x|^2 \, dx \]
\[ \quad - \int_0^L \left( \mathcal{F} U_{xx} + \frac{\alpha}{\beta} \mathcal{G} \psi_x \right) \, dx + \frac{\alpha\beta L}{480k} \left( |U_t(0,t)|^2 + |U_t(L,t)|^2 \right). \]
Our result follows from (2.26) and the last inequality for \( \kappa_1 \) and \( \kappa_2 \) satisfying
\[ \kappa_1 \geq \frac{\beta}{\kappa\alpha} \left( \frac{10kc_3}{\alpha} + c_2 \right) + \frac{k_2}{\alpha} \quad \text{and} \quad \kappa_2 \geq \frac{L^2\alpha\beta^2}{36k} + \frac{k_2}{\alpha}. \]
Finally, for \( \kappa_1 \) and \( \kappa_2 \) big enough inequality (2.18) holds. The proof is now complete.

To get global solution we will suppose that the initial data satisfy
\[ \|u_0\|_{L^2(0,L)}^2 + \|u_1\|_{H^1(0,L)}^2 + \|\theta_0\|_{L^2(0,L)}^2 + \|\theta_1\|_{H^1(0,L)}^2 + \|u_2\|_{L^2(0,L)}^2 < \epsilon^2. \quad \text{(2.27)} \]
Since \( u_1, u_2 \) and \( \theta_1 \) satisfy condition (2.2), then there exist a positive constant \( \mu \) such that
\[ \|u_1, u_2\|_{L^2(0,L)}^2 + \|\theta_1, u_2\|_{L^2(0,L)}^2 + \|u_2, u_2\|_{L^2(0,L)}^2 < \mu^2, \quad \text{(2.28)} \]
where \( \epsilon \) is small (\( < 1 \)) and \( \mu \) is large (\( > 1 \)). From (2.27) and (2.28) we have
\[ \mathcal{M}(0, u, \theta) < \epsilon^2; \quad \mathcal{M}(0, u_1, \theta_1) < \epsilon^2. \quad \text{(2.29)} \]
Using the continuity of the solution, it follows
\[ \mathcal{M}(t, u, \theta) + \frac{2\lambda}{c_0} \int_0^L \int_0^L \left( |\theta_x|^2 + |\theta_{xx}|^2 \right) \, dx \, dt \leq ce^2 \quad \forall t \in [0, t_0], \quad \text{(2.30)} \]
\[ \mathcal{N}(t, u_1, \theta_1) + \frac{2\lambda}{c_0} \int_0^L \int_0^L \left( |\theta_{x1}|^2 + |\theta_{x1}|^2 \right) \, dx \, ds \leq c\mu^2 \quad \forall t \in [0, t_0], \quad \text{(2.31)} \]
for some \( t_0 > 0, t_0 \leq T_m, c = 6c/c_0 \geq 1 \), where \( c_0, c_1 \) are defined by inequality (2.18), \( \lambda = \min(\kappa^2/2\beta, \beta/16) \). Let us define the functions
\[ g_1(t) = \mathcal{M}(t; u, \theta) + \frac{2\lambda}{c_0} \int_0^L \int_0^L \left( |\theta_x|^2 + |\theta_{xx}|^2 \right) \, dx \, dt; \]
\[ g_2(t) = \mathcal{M}(t; u_1, \theta_1) + \frac{2\lambda}{c_0} \int_0^L \int_0^L \left( |\theta_{x1}|^2 + |\theta_{x1}|^2 \right) \, dx \, ds. \]
In this conditions there exists $t_0$ for which we have

$$\begin{align*}
g_1(t) &\leq \frac{6c_1}{c_0} \varepsilon^2, \\
g_2(t) &\leq \frac{6c_1}{c_0} \mu^2 \\
&\forall t \in [0, t_0[.
\end{align*} \tag{2.32}$$

Denoting by

$$\begin{align*}
t_1 &= \sup \left\{ \tau_1 > 0; g_1(t) \leq \frac{6c_1}{c_0} \varepsilon^2 \text{ in } [0, \tau_1[ \right\}, \\
t_2 &= \sup \left\{ \tau_2 > 0; g_2(t) \leq \frac{6c_1}{c_0} \mu^2 \text{ in } [0, \tau_2[ \right\},
\end{align*}$$

and by $t_3 =: \min\{t_1, t_2\}$. We only have two cases: (i) $t_3 = T_m$, (ii) $t_3 < T_m$. The first one implies that the solution is bounded and therefore $T_m = +\infty$. It remains only to consider the case (ii), which will be studied in our final theorem. From Sobolev’s embedding theorem and inequality (2.32) we get

$$\begin{align*}
|u_x(x, t)| &\leq c_0 \varepsilon, \\
|\theta(x, t)| &\leq c_0 \varepsilon \\
&\forall (x, t) \in [0, L] \times [0, t_3[.
\end{align*} \tag{2.33}$$

It is not difficult to see that there exists a positive constant $c_1$ such that

$$\begin{align*}
|\theta_x(x, t)| &\leq c_1 \sqrt{\varepsilon}, \\
&\forall (x, t) \in [0, L] \times [0, t_3[.
\end{align*} \tag{2.34}$$

In fact, we have

$$\begin{align*}
\int_0^L |\theta_{xx}|^2 \, dx &= \int_0^L |\theta_{0,xx}|^2 \, dx + 2 \int_0^t \int_0^L \theta_{xx} \theta_{xx} \, dx \, d\tau \\
&\leq \varepsilon^2 + 2 \left[ \int_0^t \int_0^L |\theta_{xx}|^2 \, dx \, d\tau \right]^{1/2} \left\{ \int_0^t \int_0^L |\theta_{xx}|^2 \, dx \, d\tau \right\}^{1/2} \\
&\leq c_0 \mu \varepsilon. \tag{2.35}
\end{align*}$$

For $\varepsilon < 1$ and $\mu > 1$. From the Gagliardo–Nirenberg inequality and (2.35) we get (2.34). Therefore, for $\delta > 0$ there exists $\varepsilon > 0$ for which (2.32), (2.33) implies

$$\begin{align*}
|n_i| < \delta, \quad i = 1, 2; \\
|W_j| < \delta, \quad j = 1, \ldots, 4. \tag{2.36}
\end{align*}$$

From the Gagliardo–Nirenberg inequality and (2.32) we easily deduce that

$$\begin{align*}
|u_{xx}(x, t)| &\leq c_2 \sqrt{\varepsilon}, \\
|u_{tt}(x, t)| &\leq c_2 \sqrt{\varepsilon}, \\
|\theta_t(x, t)| &\leq c_2 \sqrt{\varepsilon}, \\
&\forall (x, t) \in [0, L] \times [0, t_3[.
\end{align*} \tag{2.37}$$

for some $c_2 > 0$. Finally, using equation (2.6), (2.7) and inequalities (2.36), (2.37), we conclude that there exists $c_3 > 0$ satisfying

$$\begin{align*}
|\theta_{xx}(x, t)| &\leq c_3 \sqrt{\varepsilon}; \\
|u_{xx}(x, t)| &\leq c_2 \sqrt{\varepsilon} \quad \text{in } [0, L] \times [0, t_3[. \tag{2.38}
\end{align*}$$
Let us denote by

\[ v = \sup_{|x| \leq \varepsilon} \{ |\partial^\rho \eta_i(x)|, |\partial^\rho W_j(x)|; i = 1, 2; j = 1, \ldots, 4, 1 \leq \rho \leq 4 \}, \]

where \( \partial^\rho \) stands for the partial derivative of order \( |\rho| \). From (2.6) it follows

\[ u_{xxx} = u_{txx} - \alpha \theta_{xx} - \nabla \eta_1 \cdot (u_{xx}, \theta_x)u_{xx} - \eta_1 u_{xxx} - \nabla \eta_2 \cdot (u_{xx}, \theta_x)\theta_x - \eta_2 \theta_{xx}, \]

from this identity we find that

\[ \|u_{xxx}\|_{L^2(0, L)}^2 \leq c_4 \mu^2 \] in \([0, t_3]\).

Similarly we have

\[ \|\theta_{xxx}\|_{L^2(0, L)}^2 \leq c_4 \mu^2; \quad \|\theta_{xx}\|_{L^2(0, L)}^2 \leq c_4 \mu^2 \] in \([0, t_3]\).

Finally, we get

\[ \|\theta_{xx}\|_{L^2(0, L)}^2 \leq 3 \varepsilon \mu \] in \([0, t_3] \}; \quad |\theta_{xx}(x, t)|^2 \leq c_5 \sqrt{\varepsilon} \] in \([0, L] \times [0, t_3]\). (2.39)

**Lemma 2.3.** Let us suppose that the initial data satisfies conditions (2.1), (2.2), (2.3), (2.4), (2.5), (2.27) and (2.28), then there exist positive constants \( C_i \) for which the following inequalities hold

\[ \int_0^L \mathcal{F}_1 U_{tx} \, dx \leq \frac{1}{2} \frac{d}{dt} \int_0^L \eta_1 |U_{tx}|^2 \, dx + C_1(\sqrt{\varepsilon} + \delta) \mathcal{M}(t; U, \psi). \] (2.40)

\[ \left| \int_0^L \left( \frac{x - L}{2} \right) \mathcal{F}_1 U_{tx} \, dx \right| \leq C_2(\sqrt{\varepsilon} + \delta) \mathcal{M}(t; U, \psi) + \frac{L\delta}{4} \left( |U_{tx}(0, t)|^2 + |U_{tx}(L, t)|^2 \right) \] (2.41)

\[ \int_0^L \mathcal{G}_1 \psi, U \, dx \leq C_3(\delta + 4\varepsilon) \mathcal{M}(t; U, \psi). \] (2.42)

**Proof.** First we consider the case \((U, \psi) = (u, \theta)\) and \( \mathcal{F} = F \). From (2.8) it follows that

\[ \int_0^L \mathcal{F}_1 U_{xx} \, dx = \int_0^L [\eta_1 u_{xx}, u_{xx}] \, dx + \int_0^L [\eta_2 \theta_x, u_{xx}] \, dx. \]

On the other hand

\[ \int_0^L [\eta_1 u_{xx}, u_{xx}] \, dx = \frac{1}{2} \int_0^L \nabla \eta_1 \cdot (u_{xx}, \theta_x) |u_{xx}|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_0^L \eta_1 |u_{xx}|^2 \, dx. \]

\[ \int_0^L [\eta_2 \theta_x, u_{xx}] \, dx = \int_0^L \nabla \eta_2 \cdot (u_{xx}, \theta_x) \theta_x u_{xx} \, dx + \int_0^L \eta_2 \theta_x u_{xx} \, dx. \]
We then have from (2.33), (2.36) and (2.38)

\[
\int_0^L [\eta_1, u_{xx}], u_{xx} \, dx \leq c_1 \sqrt{\varepsilon} \mathcal{M}(t; u, \theta) + \frac{1}{2} \frac{d}{dt} \int_0^L \eta_1 |u_{xx}|^2 \, dx.
\]

\[
\int_0^L [\eta_2, \theta_x], u_{xx} \, dx \leq c_1 \sqrt{\varepsilon + \delta} \mathcal{M}(t; u, \theta),
\]

so from the last three inequalities relation (2.40) follows. Let us consider the case \((U, \psi) = (u_t, \theta_t)\) and \(\mathcal{F} = F_t\). So we have

\[
\int_0^L \mathcal{F}_t U_{xx} \, dx = \int_0^L F_t u_{xx} \, dx.
\]

Using the identities

\[
F_t u_{xx} = \eta_{1,tt} u_{xx} + 2 \eta_{1,t} u_{xxx} + \eta_{1} u_{xxt} + \eta_{2,tt} \theta_x + 2 \eta_{2,t} \theta_{xx} + \eta_{2} \theta_{xxt} \quad (2.43)
\]

\[
\eta_{1,tt} = (u_{xt}, \theta) \mathcal{G}_{\eta_1}(u_{xt}, i) + \nabla \eta_1 \cdot (u_{xxt}, \theta_t), \quad \eta_{1,t} = \nabla \eta_1 \cdot (u_{xx}, \theta_t), \quad (2.44)
\]

we get that inequality (2.40) also holds in this case for an appropriate constant \(C_1\). To prove (2.41) we only consider the case \((U, \psi, \mathcal{F}) = (u_t, \theta_t, F_t)\). The other is simpler. So

\[
-\int_0^L \left( x - \frac{L}{2} \right) \mathcal{F}_t U_{xx} \, dx = \int_0^L \left( x - \frac{L}{2} \right) F_t u_{xx} \, dx,
\]

using (2.34), (2.36), (2.37) we then have from (2.43) and (2.44) that

\[
-\int_0^L \left( x - \frac{L}{2} \right) F_t u_{xx} \, dx \leq C_2 (\delta + \sqrt{\varepsilon})\mathcal{M}(t; u_t, \theta_t) - \int_0^L \left( x - \frac{L}{2} \right) \eta_1 u_{xxt} u_{xxt} \, dx.
\]

Integrating by parts and using (2.36) we obtain

\[
-\int_0^L \left( x - \frac{L}{2} \right) \eta_1 u_{xxt} u_{xxt} \, dx \leq \int_0^L \left( x - \frac{L}{2} \right) \nabla \eta_1 \cdot (u_{xx}, \theta_x) |u_{xxt}|^2 \, dx
\]

\[
+ \frac{L \delta}{4} (|U_{xt}(0, t)|^2 + |U_{xt}(L, t)|^2),
\]

so (2.41) follows. Finally, to prove (2.42) we only prove for \((U, \psi) = (u_t, \theta_t)\) and \(\mathcal{F} = F_t\), \(G = G_t\). Then we have

\[
\int_0^L G_t \theta_{xx} \, dx = \int_0^L G_t \theta_{tt} \, dx,
\]

where

\[
G = W_1 \theta_{xx} - W_2 u_{xx} + W_3 u_{xx} + W_4 \theta_x.
\]
We will prove that
\[ \int_0^L \{ W_1 \theta_{xx} \theta_t \} dx \leq C_2 (\delta + \sqrt{\varepsilon}) M(t; u_t, \theta_t). \]
The other terms are proved in a similar way.
\[ \int_0^L W_1 \theta_{xxt} \theta_t dx = \int_0^L (u_{xt}, \theta_{xt}, \theta_t) \mathcal{K}_{W_1}(u_{xt}, \theta_{xt}, \theta_t) \theta_{xx} \theta_t dx 
+ \int_0^L \nabla W_1 \cdot (u_{xt}, \theta_{xt}, \theta_t) \theta_{xx} \theta_t dx 
+ \int_0^L W_1 \theta_{xxt} \theta_t dx 
+ 2 \int_0^L \nabla W_1 \cdot (u_{xt}, \theta_{xt}, \theta_t) \theta_{xx} \theta_t dx \]
\[ \leq c \varepsilon M(t; u_t, \theta_t) + \int_0^L W_1 \theta_{xxt} \theta_t dx. \]
Since
\[ \int_0^L W_1 \theta_{xxt} \theta_t dx = -\int_0^L \nabla W_1 \cdot (u_{xx}, \theta_{xx}, \theta_t) \theta_{xt} \theta_t dx - \int_0^L W_1 \theta_{xx}^2 dx \]
\[ \leq C \varepsilon (\varepsilon \delta + \delta) M(t; u_t, \theta_t). \]
Therefore our result follows.

**Lemma 2.4.** Under the same hypothesis as in Lemma 2.3, there exists positive constants $C$ for which the following inequality holds
\[ R(t; U, \psi) \leq \frac{d}{dt} S(t; U, \psi) + \frac{\beta L \delta}{24} |U_x(0, t)|^2 + |U_x(L, t)|^2 
+ C(\delta + \sqrt{\varepsilon}) M(t; U, \psi), \]
where
\[ S(t; U, \psi) = \frac{1}{2} \left( \kappa_2 + \frac{10k}{\alpha} \right) \int_0^L \eta_1 |U_x|^2 dx - \frac{10k}{\alpha} \int_0^L \mathcal{F} U_x dx
\[ - \alpha \kappa_2 \int_0^L \mathcal{F} \psi_x dx + \frac{K_2}{2} \int_0^L |\mathcal{F}|^2 dx. \]

**Proof.** Using Lemma 2.3 and recalling the definition of $R$ our conclusion follows.

**Theorem 2.1.** Let us take $S, N, Q$ in $C^3$ satisfying (2.1)–(2.3) and with the same hypotheses as in Lemma 2.3, then there exists only one global solution of system (1.1)–(1.4) which decays exponentially as time goes to infinity.
Proof. We will suppose that $S$, $N$, $Q$ are in $C^4$ and the initial data belongs to $H^4(0, L)$ satisfying the compatibility condition as in Theorem 5.1 of [1]. Our result will follow using the well known density arguments. From Lemmas 2.3 and 2.4 we get
\[
\frac{d}{dt} K(t; U, \psi) \leq \frac{d}{dt} S(t; U, \psi) - \min\left(\frac{k^2}{2\beta}, \frac{\beta}{16}\right) M(t; U, \psi),
\]
provided $\epsilon$ and $\delta$ are small enough. Let us denote by
\[
\mathcal{L}(t; U, \psi) = K(t; U, \psi) - S(t; U, \psi).
\]
From (2.18) we get
\[
\frac{c_0}{2} M(t; U, \psi) \leq \mathcal{L}(t; U, \psi) \leq \frac{3}{2} c_1 M(t; U, \psi),
\]
provided $\epsilon$ and $\delta$ are small enough. Inequalities (2.45) and (2.46) imply
\[
\frac{d}{dt} \mathcal{L}(t; U, \psi) \leq -c_4 M(t; U, \psi)
\]
for some positive constant $c_4$ which together with (2.46) yields
\[
\frac{c_0}{2} M(t, U, \psi) \leq \mathcal{L}(t; U, \psi) \leq \mathcal{L}(0; U, \psi) e^{-\gamma t} = \frac{3c_1}{2} \mathcal{M}(0; U, \psi) e^{-\gamma t},
\]
for a positive constant $\gamma$. On the other hand, for (2.45) and recalling the definition of $\mathcal{L}(t; U, \psi)$ we obtain
\[
\mathcal{L}(t; U, \psi) + \lambda \int_0^t M(\tau; U, \psi) d\tau \leq \mathcal{L}(0; U, \psi).
\]
Where $\lambda = \min\{k^2/2\beta, \beta/16\}$. From (2.48), we get
\[
\frac{2}{c_0} \lambda \int_0^t \int_0^L |\psi_{xx}|^2 + |\psi_{xx}|^2 dx d\tau < \frac{2}{c_0} \mathcal{L}(0; U, \psi) < \frac{3c_1}{c_0} \mathcal{M}(0; U, \psi).
\]
From (2.29), (2.46), (2.48) and (2.49) we easily obtain
\[
g_1(t) < \frac{3c_1}{c_0} e^{2(e^{-\gamma t} + 1)}, \quad g_2(t) < \frac{3c_1}{c_0} \mu^2(e^{-\gamma t} + 1),
\]
so, letting $t \to t_3$ we get
\[
g_1(t_3) \leq \frac{3c_1}{c_0} e^{2(e^{-\gamma t_3} + 1)} < \frac{6c_1}{c_0} e^2, \quad (2.50)
\]
\[
g_2(t_3) \leq \frac{3c_1}{c_0} \mu^2(e^{-\gamma t_3} + 1) < \frac{6c_1}{c_0} \mu^2. \quad (2.51)
\]
Since $t_3 = t_1$ or $t_3 = t_2$ inequality (2.50) or (2.51) is contradictory to the maximility of $t_1$ or $t_2$ respectively. Then it follows that $t_2 = T_m = \infty$ therefore the solution must be global and inequality (2.47) holds for any $t > 0$, thus the exponential decay follows. The proof is now complete. \[\square\]
REFERENCES


