Global existence and exponential stability for a contact problem between two thermoelastic beams

Giovanna Bonfanti\textsuperscript{a}, Jaime E. Muñoz Rivera\textsuperscript{b,c}, Maria Grazia Naso\textsuperscript{a,\ast}

\textsuperscript{a} Dipartimento di Matematica, Facoltà di Ingegneria, Università di Brescia Via Valotti 9, 25133 Brescia, Italy
\textsuperscript{b} National Laboratory for Scientific Computation, Rua Getulio Vargas 333, Quitandinha-Petrópolis 25651-070, Rio de Janeiro, RJ, Brazil
\textsuperscript{c} Instituto de Matemática – UFRJ, Av. Horácio Macedo, Centro de Tecnologia, Cidade Universitária – Ilha do Fundão 21941-972, Rio de Janeiro, RJ, Brazil

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A B S T R A C T

In this paper we analyze a dynamic unilateral contact problem between two thermoelastic beams. We establish the existence of a weak global-in-time solution, by a penalization method. Moreover, we study the asymptotic behavior of such a solution proving that the energy associated to the system decays exponentially to zero, as time goes to infinity.

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1. Introduction

In this paper, we investigate the mechanical behavior of two thermoelastic beams that are in unilateral contact across a joint. We suppose that the area-centers of gravity of beams in their (stress free and isothermal) reference configurations are given by the intervals \([0, l_0]\) and \([l_0, l]\), respectively. For \(T > 0\), denoting by \(u = u(x, t) : (0, l_0) \times (0, T) \rightarrow \mathbb{R}\) and \(v = v(x, t) : (l_0, l) \times (0, T) \rightarrow \mathbb{R}\) the vertical displacements, and by \(\theta = \theta(x, t) : (0, l_0) \times (0, T) \rightarrow \mathbb{R}\) and \(\varphi = \varphi(x, t) : (l_0, l) \times (0, T) \rightarrow \mathbb{R}\) the thermal moments of the beams, we describe the evolution of the system by the following equations:

\[
\begin{align*}
  u_{tt}(x, t) &+ k_1 u_{xxxx}(x, t) + m_1 \theta_{xx}(x, t) = 0 & \text{in} & \ (0, l_0) \times (0, T), \\
  \theta_t(x, t) &+ \tau_1 \theta_{xx}(x, t) - m_1 u_{xx}(x, t) = 0 & \text{in} & \ (0, l_0) \times (0, T), \\
  v_{tt}(x, t) &+ k_2 v_{xxxx}(x, t) + m_2 \varphi_{xx}(x, t) = 0 & \text{in} & \ (l_0, l) \times (0, T), \\
  \varphi_t(x, t) &+ \tau_2 \varphi_{xx}(x, t) - m_2 v_{xx}(x, t) = 0 & \text{in} & \ (l_0, l) \times (0, T),
\end{align*}
\]

(1.1)

where \(k_i, \tau_i, m_i (i = 1, 2)\) are coefficients satisfying \(k_i, \tau_i > 0\) and \(m_i \neq 0\). Without loss of generality we may assume that \(m_i > 0\). We supplement (1.1) with the initial conditions

\[
\begin{align*}
  u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & \theta(x, 0) &= \theta_0(x) & \text{in} & \ (0, l_0), \\
  v(x, 0) &= v_0(x), & v_t(x, 0) &= v_1(x), & \varphi(x, 0) &= \varphi_0(x) & \text{in} & \ (l_0, l),
\end{align*}
\]

(1.2)

\ast Corresponding author.

E-mail addresses: bonfanti@ing.unibs.it (G. Bonfanti), rivera@lncc.br (J.E. Muñoz Rivera), naso@ing.unibs.it (M.G. Naso).
For a detailed derivation of the modeling of thermoelastic beams, we refer, e.g., to [12]. Here and in what follows, the subscripts \( x \) and \( t \) indicate partial derivatives.

Concerning the joint at \( x = l_0 \), we model it with the classical Signorini non-penetration condition (see, e.g., [5,8,9]) and we allow the joint with gap \( g \) to be asymmetrical so that \( g = g_1 + g_2 \), where \( g_1 > 0 \) and \( g_2 > 0 \) are, respectively, the upper and lower clearance, when the system is at rest (see Fig. 1). Then, the right end of the left beam is assumed to be within the clearance of the left end of the right beam, namely

\[
v(l_0, t) - g_2 \leq u(l_0, t) \leq v(l_0, t) + g_1 \quad \text{in} \ (0, T).
\]

(1.4)

In addition to (1.4), we assume that the stresses at the joint are equal, namely

\[
\sigma(t) := \sigma_1(l_0, t) = \sigma_2(l_0, t) \quad \text{in} \ (0, T),
\]

(1.5)

where

\[
\sigma_1(l_0, t) = -k_1 u_{xx}(l_0, t) - m_1 \theta_x(l_0, t),
\]

\[
\sigma_2(l_0, t) = -k_2 v_{xx}(l_0, t) - m_2 \phi_x(l_0, t).
\]

Moreover, we prescribe

\[
-\sigma(t) \in \partial \chi_{\nu(l_0, t)}(u(l_0, t)) \quad \text{in} \ (0, T),
\]

(1.6)

where \( \partial \chi_{\nu} \) denotes the subdifferential of the indicator function \( \chi_{\nu} \).

\[
\chi_{\nu}(\phi) = \begin{cases} 0 & \text{if } v - g_2 \leq \phi \leq v + g_1, \\ +\infty & \text{otherwise,} \end{cases}
\]

namely

\[
\partial \chi_{\nu}(\phi) = \begin{cases} (\infty, 0] & \text{if } \phi = v - g_2, \\ 0 & \text{if } v - g_2 < \phi < v + g_1, \\ [0, +\infty) & \text{if } \phi = v + g_1. \end{cases}
\]

Let us spend a few words on the condition expressed by (1.6). When \( v(l_0, t) - g_2 < u(l_0, t) < v(l_0, t) + g_1 \) is verified, there is no contact, the ends at \( x = l_0 \) are free, and \( \sigma(t) = 0 \). On the other hand, when \( v(l_0, t) - g_2 = u(l_0, t) \) or \( u(l_0, t) = v(l_0, t) + g_1 \), the ends at \( x = l_0 \) are in contact. More precisely, when the contact occurs at the lower end, relations \( v(l_0, t) - g_2 = u(l_0, t) \) and \( \sigma(t) \geq 0 \) hold; when the contact takes place at the upper end, relations \( u(l_0, t) = v(l_0, t) + g_1 \) and \( \sigma(t) \leq 0 \) are verified.

Finally, we suppose that

\[
u_{xx}(l_0, t) = 0, \quad \theta(l_0, t) = 0, \quad v_{xx}(l_0, t) = 0, \quad \phi(l_0, t) = 0 \quad \text{in} \ (0, T).
\]

(1.7)

This implies that the ends, evaluated at \( x = l_0 \), do not exert moments on each other.

The problem specified by (1.1)–(1.7) can be regarded as an extension to the thermoelastic case of the problem studied in [9]. Let us outline that it turns out to be interesting to investigate non-isothermal situations and take into account thermal effects. In fact, dynamic models for vibrations transmission across joints are of considerable interest in various industrial settings and in many applications, as, e.g., the satellite dynamics where the temperature plays a significant role and the contribution of the heat flux to bending is very important.

The first goal of the present paper is to obtain a global in time existence result for problem (1.1)–(1.7). The main analytical difficulties arise from the ill-behaved boundary terms induced by the constraint (1.6) and from the low regularity of the
weak solution. This regularity ceiling is related to the possible velocity discontinuity upon impact. Therefore, we consider an approximate version of the problem (1.1)-(1.7) by introducing a normal compliance condition (Remark 3.1 below) as regularization of the Signorini condition (1.6). Then, we prove a well-posedness result for the approximate problem by means of a Faedo–Galerkin scheme (Proposition 3.2), we derive suitable a priori estimates and we pass to the limit in the regularization parameter obtaining the existence of a solution to the original problem (Theorem 2.2). As far as the uniqueness of the solution to the limit problem is concerned, we recall that it remains an open question. The relevant part of our paper is to prove the exponential stability of a solution to the problem (1.1)-(1.7) as time goes to infinity (see Theorem 2.3). First, we work in the approximate framework: we find the exponential decay for the approximate solution by introducing a suitable Lyapunov functional and by using the multiplier method. Then, by weak lower semicontinuity arguments, we achieve the exponential decay for a solution to the original problem.

Before proceeding, let us recall some related results in the literature. The dynamics of contact problems, involving only a single displacement and/or a single variation of temperature, have been studied extensively by several authors (see, e.g., [1,2,5,6,11,16,17]). For instance, mathematical models describing the dynamic evolution of a thermoviscoelastic rod which may contact or impact a rigid or reactive obstacle are proposed in [4,10].

A second way of research is related to the study of the asymptotic behavior of the solutions. The exponential energy decay rate for weak solutions of a thermoelastic rod, contacting a rigid obstacle, is analyzed in [13].

A semilinear system of energy–elasticity equations that model the dynamic longitudinal deformations of a thermoelastic rod, fixed at one end and constrained at the other, is considered in [7]. At the contact end the obstacle is assumed to be deformable and friction is taken into account, in the interaction between the rod and the obstacle. Existence and exponential decay of weak solutions are obtained.

In [15] the authors study a model for dynamic contact between a thermoviscoelastic rod and a rigid obstacle. Contact is modeled by the Signorini unilateral condition, which also contributes a strong non-linearity to the problem. The existence of a weak solution and a power law in time energy decay rate for the problem are established. Since the modulus of elasticity is allowed to vanish, exponential decay cannot be expected.

Concerning the contact problems between two bodies, we recall the already quoted paper [9] where the authors analyze the dynamic unilateral contact between two elastic or viscoelastic beams. An existence result is established and the vibration transmission across the joint between the beams is numerically investigated.

Finally, let us recall the contribution [14] where the thermoelastic and viscoelastic contact problems of two rods are considered and the existence of a weak solution is shown. Here, the authors prove that the weak solution to the thermoelastic contact problem converges to zero exponentially as time goes to infinity, while the weak solution of the viscoelastic contact problem decays to zero with the same rates as the relaxation functions do. A numerical approximation of the problem of quasistatic contact between two thermoelastic rods is studied in [3] as well.

Now, we briefly sketch the plan of the present paper. In Section 2 we introduce a variational formulation of (1.1)-(1.7) and we state our results. Section 3 is devoted to the proof of the existence of a weak solution to (1.1)-(1.7) and in Section 4 we prove the main result of the paper about the exponential stability.

2. Main results

To provide a variational formulation of the problem (1.1)-(1.7), we introduce the following spaces:

\[
\mathcal{V}_1 = \{ w \in H^2(0,l_0) : w(0) = 0 , \ w_x(0) = 0 \},
\]

\[
\mathcal{V}_2 = \{ w \in H^2(l_0,l) : w(0) = 0 , \ w_x(l) = 0 \},
\]

\[
\mathcal{H}_1 = \{ w \in H^1(0,l_0) : w(l_0) = 0 \},
\]

\[
\mathcal{H}_2 = \{ w \in H^1(l_0,l) : w(l_0) = 0 \},
\]

\[
\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2.
\]

Next, to incorporate the constraint specified by (1.4), we set

\[
\mathcal{K} = \{ (w_1 , w_2) \in \mathcal{V}_1 \times \mathcal{V}_2 : w_2(l_0) - g_2 \leq w_1(l_0) \leq w_2(l_0) + g_1 \},
\]

representing the convex set of admissible pairs of displacements \((u, v)\).

Concerning the initial data, we assume that

\[
(u_0 , v_0) \in \mathcal{K}, \quad (u_1 , v_1) \in L^2(0,l_0) \times L^2(l_0,l), \quad (\theta_0 , \phi_0) \in L^2(0,l_0) \times L^2(l_0,l).
\]

We may now specify the variational problem we are dealing with by introducing the following definition of weak solution to the problem (1.1)-(1.7).
**Definition 2.1.** Let \( u_0, v_0, u_1, v_1, \theta_0, \) and \( \varphi_0 \) be given as in (2.1)-(2.3). A quadruple \((u, \theta, v, \varphi)\) is a weak solution to the problem (1.1)-(1.7) when

\[
(u, v) \in \mathcal{W}^{1, \infty}(0, T; L^2(0, I_0) \times L^2(I_0, I)) \cap L^\infty(0, T; \mathcal{K}),
\]

\[
(\theta, \varphi) \in L^\infty(0, T; L^2(0, I_0) \times L^2(I_0, I)) \cap L^2(0, T; \mathcal{H}),
\]

\[
u(x, 0) = u_0(x) \text{ in } (0, I_0),
\]

\[
\nu(x, 0) = v_0(x) \text{ in } (I_0, I),
\]

and satisfies the relations

\[
\int_0^T \int_0^l \left\{-u_1(x, t) [w(x, t) - u(x, t)]_x + k_1 u_{xx}(x, t) [w(x, t) - u(x, t)]_x - m_1 \theta(x, t) [w(x, t) - u(x, t)]_x \right\} dx dt
\]

\[
+ \int_0^T \int_0^l \left\{-v_1(x, t) [z(x, t) - v(x, t)]_x + k_2 v_{xx}(x, t) [z(x, t) - v(x, t)]_x - m_2 \varphi(x, t) [z(x, t) - v(x, t)]_x \right\} dx dt
\]

\[
\geq \int_0^l u_1(x) [w(x, 0) - u_0(x)] dx + \int_0^l v_1(x) [z(x, 0) - v_0(x)] dx
\]

(2.6)

for all \((w, z) \in \mathcal{W}^{1, 1}(0, T; L^2(0, I_0) \times L^2(I_0, I)) \cap L^2(0, T; \mathcal{K})\) such that \(w(\cdot, T) = u(\cdot, T)\) and \(z(\cdot, T) = v(\cdot, T)\), and

\[
\int_0^T \int_0^l \left[-\theta(x, t) \psi_1(x, t) + \tau_1 \theta_x(x, t) \psi_1(x, t) + m_1 u_{xx}(x, t) \psi_1(x, t) \right] dx dt
\]

\[
= \int_0^l [\theta_0(x) - m_1 u_{0xx}(x)] \psi_1(x, 0) dx,
\]

(2.7)

\[
\int_0^T \int_0^l \left[-\varphi(x, t) \eta_1(x, t) + \tau_2 \varphi_x(x, t) \eta_1(x, t) + m_2 v_{xx}(x, t) \eta_1(x, t) \right] dx dt
\]

\[
= \int_0^l [\varphi_0(x) - m_2 v_{0xx}(x)] \eta_1(x, 0) dx.
\]

(2.8)

for all \((\psi, \eta) \in \mathcal{W}^{1, 1}(0, T; L^2(0, I_0) \times L^2(I_0, I)) \cap L^2(0, T; \mathcal{H})\) such that \(\psi(\cdot, T) = 0\) and \(\eta(\cdot, T) = 0\).

Here are the main results of the paper.

**Theorem 2.2.** Under assumptions (2.1)–(2.3), there exists a weak solution to problem (1.1)–(1.7).

The proof of this result will be carried out in Section 3, by a regularization, a priori estimates, and passage to the limit procedure.

Next, in Section 4, we investigate the asymptotic behavior of the weak solutions provided by Theorem 2.2. Denoting by

\[
E(t, u, \theta, v, \varphi) := \frac{1}{2} \int_0^l \left[|u_t(x, t)|^2 + k_1 |u_{xx}(x, t)|^2 + |\theta(x, t)|^2 \right] dx + \frac{1}{2} \int_0^l \left[|v_t(x, t)|^2 + k_2 |v_{xx}(x, t)|^2 + |\varphi(x, t)|^2 \right] dx
\]

the energy associated with the system, we establish in the following theorem that it decays exponentially to zero, as \(t \to +\infty\).

**Theorem 2.3.** Let \((u, \theta, v, \varphi)\) be a weak solution to the problem (1.1)–(1.7) provided by Theorem 2.2. Then, there exist two positive constants \(M, \gamma\), independent of \(t\), such that

\[
E(t, u, \theta, v, \varphi) \leq ME(0, u, \theta, v, \varphi)e^{-\gamma t}, \quad t \geq 0.
\]

(2.10)

Before proceeding, let us collect here some properties which will be useful in the sequel. We recall that, by the Sobolev embedding theorem, the continuous injections hold

\[
H^1(0, I_0) \hookrightarrow C^{0,1/2}([0, I_0]), \quad H^1(I_0, I) \hookrightarrow C^{0,1/2}([I_0, I]),
\]

and, in particular, there exists a positive constant \(C_5\) such that

\[
\|w\|_{C^0(0, I_0)} \leq C_5 \|w\|_{H^1(0, I_0)}, \quad \forall w \in H^1(0, I_0),
\]

\[
\|w\|_{C^0(I_0, I)} \leq C_5 \|w\|_{H^1(I_0, I)}, \quad \forall w \in H^1(I_0, I).
\]

(2.11)
Moreover, we will use the Young inequality
\[ ab \leq (\delta/2)a^2 + (2\delta)^{-1}b^2, \quad \forall a, b \in \mathbb{R}, \quad \delta > 0. \]
Finally, for the sake of simplicity, we will employ the same symbols \( C \) for different constants, even in the same formula. In particular, we will denote by the same symbol \( C_p \) different constants due to the use of the Poincaré inequality on the intervals \([0, l_0]\) or \([l_0, l]\).

3. Global existence result

Let us briefly sketch the proof of Theorem 2.2. Firstly, in Section 3.1, we approximate the problem (1.1)–(1.7) by a penalization procedure and we prove a well-posedness result for the regularized problem (Proposition 3.2 below). Then, in Section 3.2, we show that a sequence of approximate solutions converges to a solution of the original problem.

3.1. Approximating problems

We introduce the families of initial data \([u_0^\varepsilon, v_0^\varepsilon]_{\varepsilon > 0}, [u_1^\varepsilon, v_1^\varepsilon]_{\varepsilon > 0}, [v_0^\varepsilon, \theta_0^\varepsilon]_{\varepsilon > 0}, [\theta_1^\varepsilon, \phi_0^\varepsilon]_{\varepsilon > 0}\) satisfying
\[
(u_0^\varepsilon, v_0^\varepsilon) \in \left[H^4(0, l_0) \times H^4(l_0, l)\right] \cap \mathcal{K},
\]
\[
(u_1^\varepsilon, v_1^\varepsilon) \in V_1 \times V_2,
\]
\[
(\theta_0^\varepsilon, \phi_0^\varepsilon) \in H^2(l_0, l) \times H^2(l_0, l).
\]
Now, we introduce a penalized version of the problem (1.1)–(1.7) by regularizing the Signorini contact condition with a normal compliance condition (see Remark 3.1 below). For any \( \varepsilon > 0 \) let us consider the following system:
\[
\begin{align*}
u_1^{\varepsilon}(x, t) + k_1 u_{xxx}^\varepsilon(x, t) + m_1 \theta_x^\varepsilon(x, t) &= 0 \quad \text{in} \ (0, l_0) \times (0, T), \\
\theta_x^\varepsilon(x, t) - \tau_1 \phi_x^\varepsilon(x, t) - m_1 u_{xxt}^\varepsilon(x, t) &= 0 \quad \text{in} \ (0, l_0) \times (0, T), \\
v_1^{\varepsilon}(x, t) + k_2 v_{xxx}^\varepsilon(x, t) + m_2 \phi_x^\varepsilon(x, t) &= 0 \quad \text{in} \ (l_0, l) \times (0, T), \\
\phi_x^\varepsilon(x, t) - \tau_2 \phi_{xx}^\varepsilon(x, t) - m_2 v_{xxt}^\varepsilon(x, t) &= 0 \quad \text{in} \ (l_0, l) \times (0, T),
\end{align*}
\]
(3.4)

Together with the initial conditions
\[
\begin{align*}
u_1^\varepsilon(x, 0) &= u_0^\varepsilon(x), & u_1^\varepsilon(x, 0) &= u_1^\varepsilon(x), & \theta^\varepsilon(x, 0) &= \theta_0^\varepsilon(x) \quad \text{in} \ [0, l_0], \\
v_1^\varepsilon(x, 0) &= v_0^\varepsilon(x), & v_1^\varepsilon(x, 0) &= v_1^\varepsilon(x), & \phi^\varepsilon(x, 0) &= \phi_0^\varepsilon(x) \quad \text{in} \ [l_0, l],
\end{align*}
\]
(3.5)

the boundary conditions at \( x = 0 \) and \( x = l \),
\[
\begin{align*}
u_1^\varepsilon(0, t) &= 0, & u_1^\varepsilon(0, t) &= 0, & \theta_1^\varepsilon(0, t) &= 0 \quad \text{in} \ [0, T], \\
v_1^\varepsilon(l, t) &= 0, & v_1^\varepsilon(l, t) &= 0, & \phi_1^\varepsilon(l, t) &= 0 \quad \text{in} \ [0, T].
\end{align*}
\]
(3.6)

And the boundary conditions at the joint \( x = l_0 \),
\[
\begin{align*}
u_{xx}^\varepsilon(l_0, t) &= 0, & \theta_x^\varepsilon(l_0, t) &= 0, & v_{xx}^\varepsilon(l_0, t) &= 0, & \phi_x^\varepsilon(l_0, t) &= 0 \quad \text{in} \ [0, T], \\
\sigma_1^\varepsilon(l_0, t) &= \sigma_2^\varepsilon(l_0, t) := \sigma^\varepsilon(t) \quad \text{in} \ [0, T],
\end{align*}
\]
(3.7)

where
\[
\begin{align*}
\sigma_1^\varepsilon(l_0, t) &= -k_1 u_{xxx}^\varepsilon(l_0, t) - m_1 \theta_x^\varepsilon(l_0, t), \\
\sigma_2^\varepsilon(l_0, t) &= -k_2 v_{xxx}^\varepsilon(l_0, t) - m_2 \phi_x^\varepsilon(l_0, t),
\end{align*}
\]
(3.8)

And
\[
\sigma^\varepsilon(t) = -\frac{1}{\varepsilon} \left[\left[u^\varepsilon(l_0, t) - v^\varepsilon(l_0, t) - g_1^\varepsilon\right]^+ + \left[v^\varepsilon(l_0, t) - u^\varepsilon(l_0, t) - g_2^\varepsilon\right]^+\right] - \varepsilon u_{xx}^\varepsilon(l_0, t) + \varepsilon v_{xx}^\varepsilon(l_0, t).
\]
(3.9)

Here and in the sequel, \([f]^+ := \max\{f, 0\}\) denotes the positive part of \( f \).

Remark 3.1. Assuming (3.11) we are considering a normal compliance condition (see, e.g., [9, 11, 14]) as a regularization of the Signorini contact condition (1.6). Actually, we relax the non-penetration condition by assuming for instance that the stops at the left end of the right beam are flexible. As \( \varepsilon \to 0 \), we recover formally the constraint (1.4) and the condition (1.6). Moreover, let us stress that the term \( -\varepsilon \left[u_{xx}^\varepsilon(l_0, t) - v_{xx}^\varepsilon(l_0, t)\right] \) (introduced in [14]) will play a crucial role in the proof of the uniqueness of the approximating solution (see (3.33) below).
The following result establishes the well-posedness of the above problem.

**Proposition 3.2.** Let \( u^\varepsilon_0, v^\varepsilon_0, u^\varepsilon_1, v^\varepsilon_1, \theta^\varepsilon_0, \phi^\varepsilon_0 \) be given as in (3.1)–(3.3) and compatible with the boundary conditions (3.6)–(3.11) for \( t = 0 \). Then, there exists a unique quadruple \((u^\varepsilon, v^\varepsilon, \theta^\varepsilon, \phi^\varepsilon)\) such that

\[
(u^\varepsilon, v^\varepsilon) \in W^{2,\infty}(0, T; L^2(0, l_0) \times L^2(0, l_0)) \cap W^{1,\infty}(0, T; H^2(0, l_0) \times H^2(0, l_0) \times L^\infty(0, T; H^4(0, l_0) \times H^4(0, l_0)), \tag{3.12}
\]

\[
(\theta^\varepsilon, \phi^\varepsilon) \in W^{1,\infty}(0, T; L^2(0, l_0) \times L^2(0, l_0)) \cap L^\infty(0, T; H^2(0, l_0) \times H^2(0, l_0)), \tag{3.13}
\]

fulfilling (3.4)–(3.11).

**Proof.** Existence. The proof proceeds by using the Faedo–Galerkin method.

**Construction of Galerkin approximations.** We choose bases \( \{w_i\}_{i \in \mathbb{N}}, \{\psi_i\}_{i \in \mathbb{N}}, \{z_i\}_{i \in \mathbb{N}}, \{\eta_i\}_{i \in \mathbb{N}} \) of the spaces \( \mathcal{V}_1, \mathcal{H}_1, \mathcal{V}_2, \mathcal{H}_2 \), respectively, such that \( u^\varepsilon_0, u^\varepsilon_1 \in \text{span}\{w_1, w_2\}, v^\varepsilon_0, v^\varepsilon_1 \in \text{span}\{z_1, z_2\}, \theta^\varepsilon_0 = \psi_1, \phi^\varepsilon_0 = \eta_1 \). For any \( n \in \mathbb{N} \), let us denote by

\[
u^n(u, t) = \sum_{i=1}^{n} h^n_i(t)w_i(x), \quad \theta^n(u, t) = \sum_{i=1}^{n} p^n_i(t)\psi_i(x), \tag{3.14}
\]

\[
u^n(v, t) = \sum_{i=1}^{n} g^n_i(t)z_i(x), \quad \phi^n(v, t) = \sum_{i=1}^{n} q^n_i(t)\eta_i(x),
\]

the solutions of the following system of \( 4n \) ordinary differential equations

\[
\begin{align*}
\int_0^{l_0} u^n_{t1}(x, t)w_j(x)dx + k_1 \int_0^{l_0} u^n_{xx}(x, t)w_{jxx}(x)dx &= m_1 \int_0^{l_0} \theta^n(x, t)w_j(x)dx - \sigma^n(t)w_j(l_0) = 0, \\
\int_0^{l_0} \theta^n(x, t)\psi_j(x)dx + \tau_1 \int_0^{l_0} \theta^n(x, t)\psi_{jx}(x)dx &= m_1 \int_0^{l_0} \nu^n(x, t)\psi_j(x)dx = 0, \\
\int_0^{l_0} v^n_{t1}(x, t)z_j(x)dx + k_2 \int_0^{l_0} v^n_{xx}(x, t)z_{jxx}(x)dx &= m_2 \int_0^{l_0} \phi^n(x, t)z_j(x)dx + \sigma^n(t)z_j(l_0) = 0, \\
\int_0^{l_0} \phi^n(x, t)\eta_j(x)dx + \tau_2 \int_0^{l_0} \phi^n(x, t)\eta_{jx}(x)dx &= m_2 \int_0^{l_0} \nu^n(x, t)\eta_j(x)dx = 0,
\end{align*}
\]

for \( j = 1, \ldots, n \), with

\[
\sigma^n(t) = -\frac{1}{\varepsilon}\left\{ \left[ u^n(l_0, t) - v^n(l_0, t) - g_1 \right]^+ - \left[ v^n(l_0, t) - u^n(l_0, t) - g_2 \right]^+ \right\} - \varepsilon u^n_t(l_0, t) + \varepsilon v^n_t(l_0, t),
\]

and

\[
u^n(u, 0) = u^n_0(x), \quad u^n_t(0, 0) = u^n_1(x), \quad \theta^n(u, 0) = \theta^n_0(x), \tag{3.15}
\]

\[
u^n(v, 0) = v^n_0(x), \quad v^n_t(0, 0) = v^n_1(x), \quad \phi^n(v, 0) = \phi^n_0(x). \tag{3.16}
\]

**Existence of Galerkin approximations.** System (3.14) appended by initial conditions (3.15)–(3.16) admits a local solution, and the a priori estimates derived below show that this solution can be extended to \((0, T)\), for any \( T > 0 \).

**A priori estimates.** First we differentiate equations in (3.14) with respect to \( t \), namely

\[
\begin{align*}
\int_0^{l_0} u^n_{ttt}(x, t)w_j(x)dx + k_1 \int_0^{l_0} u^n_{xxx}(x, t)w_{jxxx}(x)dx &= m_1 \int_0^{l_0} \theta^n_{xx}(x, t)w_{jx}(x)dx \\
&+ \left\{ \frac{1}{\varepsilon} \sigma^n(t) + \varepsilon [u^n_{tt}(l_0, t) - v^n_{tt}(l_0, t)] \right\}w_j(l_0) = 0, \\
\int_0^{l_0} \theta^n_{tt}(x, t)\psi_j(x)dx + \tau_1 \int_0^{l_0} \theta^n_{xx}(x, t)\psi_{jxx}(x)dx &= m_1 \int_0^{l_0} \nu^n_{xx}(x, t)\psi_{jx}(x)dx + \varepsilon \phi^n_{xx}(x, t)\eta_j(x)dx = 0,
\end{align*}
\]
\[
\int_0^l v_{txx}^n(x, t) z_j(x) \, dx + k_2 \int_0^l v_{txx}^n(x, t) z_j(x) \, dx - m_2 \int_0^l \phi_{xx}^n(x, t) z_j(x) \, dx - \left\{ \frac{1}{\varepsilon} B^n(t) + \varepsilon \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] \right\} z_j(l_0) = 0,
\]

\[
\int_0^l \phi_{xt}^n(x, t) \eta_j(x) \, dx + \tau_2 \int_0^l \phi_{xx}^n(x, t) \eta_j(x) \, dx + m_2 \int_0^l v_{xx}^n(x, t) \eta_j(x) \, dx = 0.
\]

(3.17)

for \( j = 1, \ldots, n \), where

\[
B^n(t) := \frac{d}{dt} \left[ \left[ u^n(l_0, t) - v^n(l_0, t) - g_1 \right] - \left[ v^n(l_0, t) - u^n(l_0, t) - g_2 \right] \right].
\]

Multiplying (3.17)1 by \( h_{1tt}^n(t) \), (3.17)2 by \( p_{2tt}^n(t) \), (3.17)3 by \( g_{3tt}^n(t) \), (3.17)4 by \( q_{4tt}^n(t) \), respectively, summing over \( j \), adding up, we get

\[
\frac{d}{dt} E^n(t) + \tau_1 \int_0^l \left| \phi_{xt}^n(x, t) \right|^2 \, dx + \tau_2 \int_0^l \left| \phi_{xx}^n(x, t) \right|^2 \, dx + \varepsilon \int_0^l \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] \, dx = \frac{1}{\varepsilon} B^n(t) \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right],
\]

(3.18)

where \( E^n(t) := E(t, u^n, \sigma^n, v^n, \phi^n) \) according to (2.9). Now we estimate the right-hand side of (3.18), recalling that \( |f(t)| \leq |f_1| \). Using the Young and Poincaré inequalities and the Sobolev embedding theorem (cf. (2.11)), we find

\[
\frac{1}{\varepsilon} \left| B^n(t) \right| \left| u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right| \leq E \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] + \frac{1}{2\varepsilon^2} \left| B^n(t) \right|^2
\]

\[
\leq E \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] + C \left[ \left| u_{tt}^n(l_0, t) \right|^2 + \left| v_{tt}^n(l_0, t) \right|^2 \right]
\]

\[
\leq E \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] + C \int_0^l \left| u_{xx}^n(x, t) \right|^2 \, dx + C \int_0^l \left| v_{xx}^n(x, t) \right|^2 \, dx.
\]

(3.19)

for some positive constant \( C \) independent of \( n \). After an integration of (3.18) over \( (0, t) \), on account of (3.19), we have

\[
E^n(t) + \tau_1 \int_0^t \int_0^l \left| \phi_{xt}^n(x, \tau) \right|^2 \, dx \, d\tau + \tau_2 \int_0^t \int_0^l \left| \phi_{xx}^n(x, \tau) \right|^2 \, dx \, d\tau
\]

\[
\leq E^n(0) + C \int_0^t \int_0^l \left[ u_{tt}^n(l_0, t) - v_{tt}^n(l_0, t) \right] \, dx + C \int_0^t \int_0^l \left| u_{xx}^n(x, t) \right|^2 \, dx \, d\tau.
\]

(3.20)

Now we prove that the second-order energy is bounded initially, i.e. that

\[
E^n(0) = \int_0^l \left[ \left| u_{tt}^n(x, 0) \right|^2 + k_1 \left| u_{1xx}^n(x, 0) \right|^2 + \left| \phi_{tt}^n(x, 0) \right|^2 \right] \, dx + \int_0^l \left| v_{tt}^n(x, 0) \right|^2 \, dx + k_2 \left| v_{xx}^n(x, 0) \right|^2 + \left| \phi_{tt}^n(x, 0) \right|^2 \right] \, dx
\]

is bounded independently of \( n \). To this aim, we will take advantage of the special bases chosen above, containing the initial data. In fact, let us multiply Eq. (3.14)1 by \( h_{1tt}^n(t) \), summing up over \( j = 1, \ldots, n \), let \( t \to 0 \). Taking into account (3.15), we find that

\[
\int_0^l \left[ u_{tt}^n(x, 0) \right]^2 \, dx + k_1 \int_0^l u_{1xx}^n(x, 0) u_{1xx}^n(x, 0) \, dx - m_1 \int_0^l \phi_{xx}^n(x) u_{1xx}^n(x, 0) \, dx - \sigma^n(0) u_{tt}^n(l_0, 0) = 0.
\]

Integrating by parts and owing to the compatibility conditions (3.6)-(3.11) for \( t = 0 \), we have

\[
\int_0^l \left[ u_{tt}^n(x, 0) \right]^2 \, dx + k_1 \int_0^l u_{1xx}^n(x, 0) u_{1xx}^n(x, 0) \, dx + m_1 \int_0^l \phi_{xx}^n(x) u_{tt}^n(x, 0) \, dx - \left[ k_1 u_{1xx}^n(l_0) + m_1 \phi_{xx}^n(l_0) + \sigma^n(0) \right] u_{tt}^n(l_0, 0) = 0,
\]

and then, by the Young inequality, there exists a positive constant \( C \) independent of \( n \), such that

\[
\int_0^l \left| u_{tt}^n(x, 0) \right|^2 \, dx \leq C \int_0^l \left| u_{1xx}^n(x) \right|^2 \, dx + \int_0^l \left| \phi_{xx}^n(x) \right|^2 \, dx.
\]
Analogously, let us multiply Eq. (3.14) by $\rho^n_j(t)$, summing up over $j = 1, \ldots, n$, letting $t \to 0$ and accounting for (3.15), we obtain
\[
\int_0^l |\theta^n_j(x,0)|^2 \, dx + \tau_1 \int_0^l \theta^n_{0,xx}(x)\theta_{j,xx}^n(x,0) \, dx + m_1 \int_0^l u^n_{1,xx}(x)\theta_{j,xx}^n(x,0) \, dx = 0.
\]
Integrating by parts and accounting for the compatibility conditions verified by $\theta^n_0$ and $u^n_1$ (cf. (3.2) and (3.6)), we find
\[
\int_0^l |\theta^n_j(x,0)|^2 \, dx = \tau_1 \int_0^l \theta^n_{0,xx}(x)\theta_{j,xx}^n(x,0) \, dx + m_1 \int_0^l u^n_{1,xx}(x)\theta_{j,xx}^n(x,0) \, dx \\
\leq C \left[ \int_0^l |u^n_{1,xx}(x)|^2 \, dx + \int_0^l |\theta^n_{0,xx}(x)|^2 \, dx \right] + \frac{1}{2} \int_0^l |\theta^n_j(x,0)|^2 \, dx.
\]
By similar procedure we can estimate the terms
\[
\int_0^l \left| \nabla_{tt} u^n_j(x,0) \right|^2 \, dx \quad \text{and} \quad \int_0^l \left| \nabla_{tt} \phi^n_j(x,0) \right|^2 \, dx,
\]
and conclude that $E^n(0)$ is bounded independently of $n$. Thus, from (3.20), by the Gronwall lemma we deduce that
\[
\begin{align*}
 u^n & \text{ is bounded in } W^{2,\infty}(0, T; L^2(0, l_0)) \cap W^{1,\infty}(0, T; H^2(0, l_0)), \\
 \theta^n & \text{ is bounded in } W^{1,\infty}(0, T; L^2(0, l_0)) \cap H^1(0, T; H^1(0, l_0)), \\
 v^n & \text{ is bounded in } W^{2,\infty}(0, T; L^2(0, l_0)) \cap W^{1,\infty}(0, T; H^2(0, l_0)), \\
 \psi^n & \text{ is bounded in } W^{1,\infty}(0, T; L^2(0, l_0)) \cap H^1(0, T; H^1(0, l_0)).
\end{align*}
\]
Convergence of the Galerkin approximations. In accordance with classical compactness results, we can extract subsequences denoted by the same symbols such that, as $n \to \infty$, there exist $u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \psi^\varepsilon$ with
\[
\begin{align*}
 u^n & \rightharpoonup^* u^\varepsilon \quad \text{in } W^{2,\infty}(0, T; L^2(0, l_0)) \cap W^{1,\infty}(0, T; H^2(0, l_0)), \\
 \theta^n & \rightharpoonup^* \theta^\varepsilon \quad \text{in } W^{1,\infty}(0, T; L^2(0, l_0)) \cap H^1(0, T; H^1(0, l_0)), \\
 v^n & \rightharpoonup^* v^\varepsilon \quad \text{in } W^{2,\infty}(0, T; L^2(0, l_0)) \cap W^{1,\infty}(0, T; H^2(0, l_0)), \\
 \psi^n & \rightharpoonup^* \psi^\varepsilon \quad \text{in } W^{1,\infty}(0, T; L^2(0, l_0)) \cap H^1(0, T; H^1(0, l_0)).
\end{align*}
\]
Moreover, using a generalized version of the Ascoli theorem (see, e.g., [18, Corollary 4]), we can deduce the following strong convergences:
\[
\begin{align*}
 u^n & \to u^\varepsilon \quad \text{in } C^1([0, T]; H^{2-s}(0, l_0)), \quad s > 0, \\
 \theta^n & \to \theta^\varepsilon \quad \text{in } C^0([0, T]; H^{1-s}(0, l_0)), \quad s > 0, \\
 v^n & \to v^\varepsilon \quad \text{in } C^1([0, T]; H^{2-s}(0, l_0)), \quad s > 0, \\
 \psi^n & \to \psi^\varepsilon \quad \text{in } C^0([0, T]; H^{1-s}(0, l_0)), \quad s > 0.
\end{align*}
\]
By standard procedure, thanks to (3.21)–(3.22) we pass to the limit in (3.14) as $n \to +\infty$ and we recover (3.4) as well as the initial and the boundary conditions (3.5)–(3.11). In particular, $\theta^n_0 \in L^\infty(0, T; L^2(0, l_0))$ is deduced by comparison in (3.4)\_2, taking (3.21)\_1 and (3.21)\_2 into account, then a comparison in (3.4)\_1 gives $\theta^n_{1,xx} \in L^\infty(0, T; L^2(0, l_0))$. Analogously, by (3.4)\_4 and then (3.4)\_3, we read the additional regularity for $\psi^\varepsilon$ and $v^\varepsilon$ specified by (3.12) and (3.13).

(Uniqueness) Let $(u^n, \theta^n, v^n, \psi^n)$ and $(w^n, \phi^n, z^n, \eta^n)$ be two solutions of (3.4)–(3.11) whose regularity is specified by (3.12)–(3.13). Then
\[
(\bar{u}^\varepsilon, \bar{\theta}^\varepsilon, \bar{v}^\varepsilon, \bar{\psi}^\varepsilon) := (u^\varepsilon - w^\varepsilon, \theta^\varepsilon - \psi^\varepsilon, v^\varepsilon - z^\varepsilon, \psi^\varepsilon - \eta^\varepsilon)
\]
satisfies
\[ \ddot{u}^e_{tt}(x,t) + k_1 \dddot{u}^e_{xxx}(x,t) + m_1 \dddot{\phi}^e_{xx}(x,t) = 0 \quad \text{in} \ (0, l_0) \times (0, T), \]
\[ \dot{\theta}^e_t(x,t) - \tau_1 \dot{\theta}^e_t(x,t) - m_1 \ddot{u}^e_{xx}(x,t) = 0 \quad \text{in} \ (0, l_0) \times (0, T), \]
\[ \ddot{v}^e_{tt}(x,t) + k_2 \dddot{v}^e_{xxx}(x,t) + m_2 \dddot{\phi}^e_{xx}(x,t) = 0 \quad \text{in} \ (l_0, l) \times (0, T), \]
\[ \dot{\phi}^e_t(x,t) - \tau_2 \dot{\phi}^e_t(x,t) - m_2 \ddot{v}^e_{xx}(x,t) = 0 \quad \text{in} \ (l_0, l) \times (0, T), \]

(3.23)

together with
\[ \ddot{u}^e(x,0) = 0, \quad \ddot{u}^e_t(x,0) = 0, \quad \dddot{\phi}^e(x,0) = 0 \quad \text{in} \ [0, l_0], \]
\[ \ddot{v}^e(x,0) = 0, \quad \ddot{v}^e_t(x,0) = 0, \quad \dddot{\phi}^e(x,0) = 0 \quad \text{in} \ [l_0, l], \]
\[ \ddot{u}^e(0,t) = 0, \quad \ddot{u}^e_x(0,t) = 0, \quad \dddot{\phi}^e(0,t) = 0 \quad \text{in} \ [0, T], \]
\[ \ddot{v}^e(l,t) = 0, \quad \ddot{v}^e_x(l,t) = 0, \quad \dddot{\phi}^e(l,t) = 0 \quad \text{in} \ [0, T], \]
\[ \ddot{u}^e_{xx}(l_0,t) = 0, \quad \ddot{v}^e_{xx}(l_0,t) = 0, \quad \dddot{\phi}^e_{xx}(l_0,t) = 0 \quad \text{in} \ [0, T]. \]

(3.24)

(3.25)

(3.26)

(3.27)

where
\[ \ddot{\sigma}^e_1(l_0,t) = -k_1 \ddot{u}^e_{xx}(l_0,t) - m_1 \dddot{\phi}^e_x(l_0,t), \]
\[ \ddot{\sigma}^e_2(l_0,t) = -k_2 \ddot{v}^e_{xx}(l_0,t) - m_2 \dddot{\phi}^e_x(l_0,t). \]

(3.28)

(3.29)

and
\[ \dddot{\sigma}^e(t) = -\frac{1}{\rho} \left[ \left[ u^e_t(l_0,t) - v^e_t(l_0,t) - g_1 \right]^+ - \left[ v^e_t(l_0,t) - u^e_t(l_0,t) - g_2 \right]^+ \right] \]
\[ + \frac{1}{g} \left[ \left[ w^e_t(l_0,t) - z^e_t(l_0,t) - g_1 \right]^+ - \left[ z^e_t(l_0,t) - w^e_t(l_0,t) - g_2 \right]^+ \right] - \epsilon \left[ \ddot{u}^e_t(l_0,t) - \ddot{v}^e_t(l_0,t) \right]. \]

(3.30)

Multiplying (3.23)\_1 by \( \dddot{u}^e_t \) in \( L^2(0,l_0) \), (3.23)\_2 by \( \dddot{v}^e_t \) in \( L^2(0,l_0) \), (3.23)\_3 by \( \dddot{\phi}^e_x \) in \( L^2(l_0,l) \), (3.23)\_4 by \( \dddot{\phi}^e_x \) in \( L^2(l_0,l) \), respectively, and summing up, we find
\[ \frac{d}{dt} \dddot{E}^e(t) = -\tau_1 \int_0^{l_0} \left| \dddot{u}^e_x(x,t) \right|^2 \, dx - \tau_2 \int_0^l \left| \dddot{\phi}^e_x(x,t) \right|^2 \, dx + \dddot{\sigma}^e(t) \left[ \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) \right] \]
\[ = -\tau_1 \int_0^{l_0} \left| \dddot{u}^e_x(x,t) \right|^2 \, dx - \tau_2 \int_0^l \left| \dddot{\phi}^e_x(x,t) \right|^2 \, dx - \epsilon \left[ \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) \right]^2 \]
\[ - \frac{1}{\rho} \left[ \left[ u^e_t(l_0,t) - v^e_t(l_0,t) - g_1 \right]^+ - \left[ v^e_t(l_0,t) - u^e_t(l_0,t) - g_2 \right]^+ \right] \]
\[ + \frac{1}{g} \left[ \left[ w^e_t(l_0,t) - z^e_t(l_0,t) - g_1 \right]^+ - \left[ z^e_t(l_0,t) - w^e_t(l_0,t) - g_2 \right]^+ \right] \]
\[ - \left[ \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) - \dddot{\phi}^e_x(l_0,t) \right] \left[ \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) \right]. \]

(3.31)

where, according to (2.9), \( \dddot{E}^e(t) := E(t, \dddot{u}^e, \dddot{v}^e, \dddot{\phi}^e) \). Now we estimate the last term on the right-hand side of (3.31). First, since \( |f^+ - g^+| \leq |f - g| \), we can write
\[ \left| \left[ u^e_t(l_0,t) - v^e_t(l_0,t) - g_1 \right]^+ - \left[ v^e_t(l_0,t) - u^e_t(l_0,t) - g_2 \right]^+ - \left[ w^e_t(l_0,t) - z^e_t(l_0,t) - g_1 \right]^+ \right| \]
\[ \leq \left| \left[ u^e_t(l_0,t) - v^e_t(l_0,t) - g_1 \right]^+ - \left[ w^e_t(l_0,t) - z^e_t(l_0,t) - g_1 \right]^+ \right| \]
\[ + \left| \left[ z^e_t(l_0,t) - w^e_t(l_0,t) - g_2 \right]^+ - \left[ z^e_t(l_0,t) - w^e_t(l_0,t) - g_2 \right]^+ \right| \]
\[ \leq \left| u^e_t(l_0,t) - v^e_t(l_0,t) - w^e_t(l_0,t) \right| + \left| w^e_t(l_0,t) - z^e_t(l_0,t) \right| \]
\[ \leq 2 \left| u^e_t(l_0,t) - v^e_t(l_0,t) \right| + \left| w^e_t(l_0,t) - z^e_t(l_0,t) \right| \]
\[ \leq 2 \left[ u^e_t(l_0,t) - v^e_t(l_0,t) \right] + \left| w^e_t(l_0,t) - z^e_t(l_0,t) \right|. \]

(3.32)

Then, applying the Young and Poincaré inequalities and the Sobolev embedding theorem (cf. (2.11)), (3.31) becomes
\[ \frac{d}{dt} \dddot{E}^e(t) \leq -\tau_1 \int_0^{l_0} \left| \dddot{u}^e_x(x,t) \right|^2 \, dx - \tau_2 \int_0^l \left| \dddot{\phi}^e_x(x,t) \right|^2 \, dx - \epsilon \left[ \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) \right]^2 \]
\[ + \frac{2}{\rho} \left| \dddot{u}^e_t(l_0,t) \right| + \left| \dddot{v}^e_t(l_0,t) \right| \left| \dddot{u}^e_t(l_0,t) - \dddot{v}^e_t(l_0,t) \right|. \]
hence, in particular

As a first step, we derive a priori estimates for (suitable norms of) the sequence

Let us introduce the following energy functions:

\[
E(t) := E(t, u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon),
\]

\[
J^\varepsilon(t) := \frac{1}{2\varepsilon} \left( \left[ u^\varepsilon(t, 0) - v^\varepsilon(t, 0, l) - g_1 \right]^2 + \left[ v^\varepsilon(t, 0) - u^\varepsilon(t, 0, l) - g_2 \right]^2 \right).
\]

\[
E^\varepsilon(t) := E^\varepsilon(t) + J^\varepsilon(t).
\]

By the Gronwall lemma and recalling that \( E^\varepsilon(0) = 0 \), we find that \( E^\varepsilon(t) = 0 \) on \((0, T)\). This implies that \((u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon) = (w^\varepsilon, \psi^\varepsilon, z^\varepsilon, \eta^\varepsilon)\), and the proof of Proposition 3.2 is complete. □

### 3.2. Proof of Theorem 2.2

The idea is to consider a sequence of approximate solutions (provided by Proposition 3.2) and to show their convergence (as \( \varepsilon \to 0 \)) to a weak solution of the problem \((1.1)-(1.7)\).

From now on, we let \( \varepsilon \) vary, say, in \((0, 1)\). Concerning the approximating initial data, we assume that

\[
(u_0^\varepsilon, v_0^\varepsilon) \to (u_0, v_0) \quad \text{in} \quad H^2(0, l_0) \times H^2(l_0, l),
\]

\[
(u_1^\varepsilon, v_1^\varepsilon) \to (u_1, v_1) \quad \text{in} \quad L^2(0, l_0) \times L^2(l_0, l),
\]

\[
(\theta_0^\varepsilon, \varphi_0^\varepsilon) \to (\theta_0, \varphi_0) \quad \text{in} \quad L^2(0, l_0) \times L^2(l_0, l).
\]

Let us introduce the following energy functions:

\[
E^\varepsilon(t) := E(t, u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon).
\]

As a first step, we derive a priori estimates for (suitable norms of) the sequence \((u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)\) provided by Proposition 3.2. To this aim, we multiply every equation of \((3.4)\) respectively by \(u_1^\varepsilon, \theta_1^\varepsilon, v_1^\varepsilon, \varphi_1^\varepsilon\), we integrate on the related domains \([0, l_0]\) or \([l_0, l]\), and we sum up the resulting relations. Thus, on account of \((3.6)-(3.11)\), we find

\[
\frac{d}{dt} E^\varepsilon(t) = -\tau_1 \int_0^{l_0} |\theta_0^\varepsilon(x, t)|^2 \, dx - \tau_2 \int_0^t |\psi_0^\varepsilon(x, t)|^2 \, dx + \frac{4}{\varepsilon^3} \left[ |\tilde{u}^\varepsilon(0, t)|^2 + |\tilde{v}^\varepsilon(0, t)|^2 \right]
- \tau_1 \int_0^{l_0} |\theta_1^\varepsilon(x, t)|^2 \, dx - \tau_2 \int_0^t |\psi_1^\varepsilon(x, t)|^2 \, dx - \varepsilon |u_2^\varepsilon(0, t) - v_1^\varepsilon(0, t)|^2
- \frac{1}{2\varepsilon} \int_0^t \left( \left[ u^\varepsilon(t, 0) - v^\varepsilon(t, 0, l) - g_1 \right]^2 + \left[ v^\varepsilon(t, 0) - u^\varepsilon(t, 0, l) - g_2 \right]^2 \right).\]

Then, there holds

\[
\frac{d}{dt} E^\varepsilon(t) = -\tau_1 \int_0^{l_0} |\theta_0^\varepsilon(x, t)|^2 \, dx - \tau_2 \int_0^t |\psi_0^\varepsilon(x, t)|^2 \, dx - \varepsilon |u_2^\varepsilon(0, t) - v_1^\varepsilon(0, t)|^2.
\]

Now, we integrate in time \((3.40)\). On account of \((3.1)\) (which yields \( J^\varepsilon(0) = 0 \)) and \((3.34)-(3.36)\), we derive the following upper bound:

\[
E^\varepsilon(t) + \varepsilon \int_0^t |u_1^\varepsilon(t, \tau) - v_1^\varepsilon(t, \tau)|^2 \, d\tau + \tau_1 \int_0^{l_0} |\theta_0^\varepsilon(x, \tau)|^2 \, dx + \tau_2 \int_0^t |\psi_0^\varepsilon(x, \tau)|^2 \, dx \leq E^\varepsilon(0) \leq C,
\]

hence, in particular

\[
\frac{1}{\varepsilon} \left[ |u^\varepsilon(0, t) - v^\varepsilon(0, t) - g_1|^2 + |v^\varepsilon(0, t) - u^\varepsilon(0, t) - g_2|^2 \right] \leq C.
\]
for some positive constant $C$ depending on $\|u_1\|_{L^2(0,l_0)}^2$, $\|u_0\|_{H^1(0,l_0)}^2$, $\|\theta_0\|_{L^2(0,l_0)}^2$, $\|v_1\|_{H^2(0,l_0)}^2$, $\|v_0\|_{H^2(0,l_0)}^2$, $\|\varphi_0\|_{L^2(0,l_0)}^2$, but independent of $\varepsilon$. Thus, well-known weak and weak* compactness results ensure that there exits a subsequence of $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ still denoted by $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ such that the following convergences hold

\[
\begin{align*}
  u^\varepsilon & \rightharpoonup u \quad \text{in} \quad W^{1,\infty}(0,T;L^2(0,l_0)) \cap L^\infty(0,T;H^2(0,l_0)), \\
  \theta^\varepsilon & \rightharpoonup \theta \quad \text{in} \quad L^\infty(0,T;L^2(0,l_0)) \cap L^2(0,T;H^1(0,l_0)), \\
  v^\varepsilon & \rightharpoonup v \quad \text{in} \quad W^{1,\infty}(0,T;L^2(0,l_0)) \cap L^\infty(0,T;H^2(0,l_0)), \\
  \varphi^\varepsilon & \rightharpoonup \varphi \quad \text{in} \quad L^\infty(0,T;L^2(0,l_0)) \cap L^2(0,T;H^1(0,l_0)),
\end{align*}
\]

as $\varepsilon \to 0$. Moreover, estimate (3.41) implies that

\[
\varepsilon [u^\varepsilon(0,\cdot) - v^\varepsilon(0,\cdot)] \to 0 \quad \text{in} \quad L^2(0,T).
\]

Next, using the generalized Ascoli theorem (see, e.g., [18, Corollary 4]) we deduce the following strong convergences:

\[
\begin{align*}
  u^\varepsilon & \to u \quad \text{in} \quad C^0([0,T];H^{2-\delta}(0,l_0)), \quad s > 0, \\
  v^\varepsilon & \to v \quad \text{in} \quad C^0([0,T];H^{2-\delta}(0,l_0)), \quad s > 0.
\end{align*}
\]

Now, we prove that the quadruple $(u, \theta, v, \varphi)$ is a weak solution to the problem (1.1)–(1.7) in the sense of Definition 2.1. Thanks to the above convergences the quadruple $(u, \theta, v, \varphi)$ verifies the regularity specified in (2.4) and (2.5). Furthermore, it follows from (3.42) that $(u(t), v(t)) \in \mathcal{K}$ for all $t \in (0,T)$. Now, let $(w, z) \in W^{1,\infty}(0,T;L^2(0,l_0) \times L^2(0,l_0)) \cap L^2(0,T;\mathcal{K})$ such that $w(\cdot, T) = u(\cdot, T)$ and $z(\cdot, T) = v(\cdot, T)$. Then, it follows from (3.41), (3.43), (3.5)–(3.11) that

\[
\begin{align*}
\int_0^T \int_0^l & \left\{ -u^\varepsilon(t,x) [w(x,t) - u^\varepsilon(x,t)]_t + k_1 u^\varepsilon(x,t) [w(x,t) - u^\varepsilon(x,t)]_x - m_1 \theta^\varepsilon(x,t) [w(x,t) - u^\varepsilon(x,t)]_x \\
& + k_2 v^\varepsilon(x,t) [z(x,t) - v^\varepsilon(x,t)]_t - m_2 \varphi^\varepsilon(x,t) [z(x,t) - v^\varepsilon(x,t)]_x \right\} dx \, dt \\
& \quad + \varepsilon \left[ u^\varepsilon(0,l_0,t) - v^\varepsilon(0,l_0,t) \right] \left[ w(0,l_0,t) - z(0,l_0,t) + v^\varepsilon(0,l_0,t) - u^\varepsilon(0,l_0,t) \right] dt \\
& \geq \int_0^l \left[ u^\varepsilon(0,\cdot) - u^\varepsilon_0 \right] dx + \int_0^l \left[ v^\varepsilon(0,\cdot) - v^\varepsilon_0 \right] dx.
\end{align*}
\]

Now, the main difficulty in the passage to the limit procedure is to show the convergence of the quadratic terms in (3.50). To this aim we will use the following lemma whose proof can be achieved arguing as in [11, Lemma 4.3].

**Lemma 3.3.** Let $(u^\varepsilon, \theta^\varepsilon, v^\varepsilon, \varphi^\varepsilon)$ a solution to (3.4)–(3.11) provided by Proposition 3.2. Then,

\[
\lim_{\varepsilon \to 0} \sup \int_0^T \int_0^l \left[ u^\varepsilon(x,t)^2 - u^\varepsilon_{xx}(x,t)^2 \right] dx \, dt \
\leq \int_0^T \int_0^l \left[ u_1(x,t)^2 - u_{1xx}(x,t)^2 \right] dx \, dt,
\]

and

\[
\lim_{\varepsilon \to 0} \sup \int_0^T \int_0^l \left[ v^\varepsilon(x,t)^2 - v^\varepsilon_{xx}(x,t)^2 \right] dx \, dt \
\leq \int_0^T \int_0^l \left[ v_1(x,t)^2 - v_{1xx}(x,t)^2 \right] dx \, dt.
\]

Now, on account of (3.34)–(3.35), (3.43)–(3.49), and (3.51)–(3.52), we pass to the $\limsup_{\varepsilon \to 0}$ in (3.50) and we recover (2.6). Finally, using (3.43)–(3.46) along with (3.34) and (3.36) it is easy to show that $(\theta, \varphi)$ solves (2.7)–(2.8) and the proof of Theorem 2.2 is complete.
4. Exponential decay

First, we show that the energy associated with the approximate system decays exponentially. To this end, let us introduce the following functionals:

\[ I_1^{u,\theta}(t) := \int_{0}^{l_0} u^\varepsilon(x, t)u_{t1}^\varepsilon(x, t) \, dx, \quad I_1^{\psi,\varphi}(t) := \int_{0}^{l_0} \psi^\varepsilon(x, t)\varphi_t^\varepsilon(x, t) \, dx, \]

\[ I_2^{u,\theta}(t) := -\int_{0}^{l_0} q(x)u^\varepsilon(x, t)u_{t1}^\varepsilon(x, t) \, dx, \quad I_2^{\psi,\varphi}(t) := -\int_{0}^{l_0} q(x)\psi^\varepsilon(x, t)\varphi_t^\varepsilon(x, t) \, dx, \]

\[ I_3^{u,\theta}(t) := \int_{0}^{l_0} \left[ \int_{0}^{l} q(x)u_{x1}^\varepsilon(x, t) \, dx \right] \psi^\varepsilon(x, t) \, dx, \quad I_3^{\psi,\varphi}(t) := -\int_{0}^{l_0} \left[ \int_{0}^{l} \theta^\varepsilon(x, t) \, dx \right] q(x)\psi^\varepsilon(x, t) \, dx, \]

where \( q(x) = x - l_0 \), with \( x \in [0, l] \). Using the Young and Poincaré inequalities it is easy to see

\[ |I_1^{u,\theta}(t)| + |I_1^{\psi,\varphi}(t)| \leq C E^\varepsilon(t), \quad i = 1, 2, 3, \]

for a positive constant \( C \). Here and in what follows, when it is not necessary to write it explicitly, we will employ the same symbol \( C \) for different constants, depending on \( k_i, m_i, \tau_i, i = 1, 2, l, l_0, C_p \) and \( C_S \), but independent of \( \varepsilon \). Inequalities used subsequently are proved in the following lemmas.

**Lemma 4.1.** Let \((u^\varepsilon, \theta^\varepsilon, \psi^\varepsilon, \varphi^\varepsilon)\) be the solution provided by Proposition 3.2. Then, there holds

\[
\frac{d}{dt} \left[ I_1^{u,\theta}(t) + I_1^{\psi,\varphi}(t) + \frac{\varepsilon}{2} |u^\varepsilon(l_0, t) - \psi^\varepsilon(l_0, t)|^2 \right] \leq -k_1 \int_{0}^{l_0} |u^\varepsilon_{xx}(x, t)|^2 \, dx - \frac{k_2}{2} \int_{0}^{l} |\psi^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |\psi^\varepsilon_{x1}(x, t)|^2 \, dx 

- 2C E^\varepsilon(t) + C \left[ \int_{0}^{l_0} |\theta^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l} |\psi^\varepsilon_{x1}(x, t)|^2 \, dx \right],
\]

where \( C \) is a positive constant depending on \( m_i, k_i, i = 1, 2, \) and \( C_p \).

**Proof.** Considering (3.4) and applying the Young and Poincaré inequalities, on account of (3.6) and (3.7), we have

\[
\frac{d}{dt} I_1^{u,\theta}(t) = \int_{0}^{l_0} u^\varepsilon(x, t)u_{t1}^\varepsilon(x, t) \, dx + \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx 

= -k_1 \int_{0}^{l_0} u^\varepsilon(x, t)u^\varepsilon_{xxx}(x, t) \, dx - m_1 \int_{0}^{l_0} u^\varepsilon(x, t)\theta^\varepsilon_{xxx}(x, t) \, dx + \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx 

= -k_1 \int_{0}^{l_0} |u^\varepsilon_{xx}(x, t)|^2 \, dx + m_1 \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |\theta^\varepsilon_{xx}(x, t)|^2 \, dx - [k_1 u^\varepsilon_{xxx}(l_0, t) + m_1 \theta^\varepsilon_{xx}(l_0, t)]u^\varepsilon(l_0, t) 

\leq -k_1 \int_{0}^{l_0} |u^\varepsilon_{xx}(x, t)|^2 \, dx + \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |\theta^\varepsilon_{x1}(x, t)|^2 \, dx - [k_1 u^\varepsilon_{xxx}(l_0, t) + m_1 \theta^\varepsilon_{xx}(l_0, t)]u^\varepsilon(l_0, t). 
\]

Recalling (3.9) and (3.10), we find

\[
\frac{d}{dt} I_1^{u,\theta}(t) \leq -\frac{k_1}{2} \int_{0}^{l_0} |u^\varepsilon_{xx}(x, t)|^2 \, dx + \int_{0}^{l_0} |u^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |\theta^\varepsilon_{x1}(x, t)|^2 \, dx + \int_{0}^{l_0} |\psi^\varepsilon_{x1}(x, t)|^2 \, dx + C E^\varepsilon(l_0, t) u^\varepsilon(l_0, t). 
\]
and by the same procedure
\[
\frac{d}{dt} \mathcal{I}^\epsilon_{1,\theta}(t) \leq -\frac{k_2}{2} \int_0^l \left| v^\epsilon_{xx}(x, t) \right|^2 \, dx + \int_0^l \left| v^\epsilon_{x}(x, t) \right|^2 \, dx + C \int_0^l \left| \varphi^\epsilon_x(x, t) \right|^2 \, dx - \sigma^\epsilon_x(l_0, t)v^\epsilon(l_0, t).
\]

Then, on account of (3.8), we have
\[
\frac{d}{dt} \left[ \mathcal{I}^\epsilon_{1,\theta}(t) + \mathcal{I}^{v^\epsilon,\psi}(t) \right] \leq -\frac{k_1}{2} \int_0^l \left| u^\epsilon_{xx}(x, t) \right|^2 \, dx - \frac{k_2}{2} \int_0^l \left| v^\epsilon_{xx}(x, t) \right|^2 \, dx + \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx + \int_0^l \left| v^\epsilon_{x}(x, t) \right|^2 \, dx \\
+ C \left[ \int_0^l \left| \theta^\epsilon_x(x, t) \right|^2 \, dx + \int_0^l \left| \varphi^\epsilon_x(x, t) \right|^2 \, dx \right] + \sigma^\epsilon(t) [u^\epsilon(l_0, t) - v^\epsilon(l_0, t)].
\]

Recalling (3.11), it is easy to prove that
\[
\sigma^\epsilon(t)[u^\epsilon(l_0, t) - v^\epsilon(l_0, t)] = -\frac{1}{\epsilon} \left[ \left[ u^\epsilon(l_0, t) - v^\epsilon(l_0, t) - g_1 \right]^+ - \left[ v^\epsilon(l_0, t) - u^\epsilon(l_0, t) - g_2 \right]^+ \right][u^\epsilon(l_0, t) - v^\epsilon(l_0, t) - \epsilon u^\epsilon_0(l_0, t) - v^\epsilon_0(l_0, t)]
\]
\[
\leq -2\mathcal{J}^\epsilon(t) - \epsilon [u^\epsilon_0(l_0, t) - v^\epsilon_0(l_0, t)][u^\epsilon(l_0, t) - v^\epsilon(l_0, t)],
\]

where \(\mathcal{J}^\epsilon(t)\) is defined in (3.38). Thus, we obtain
\[
\frac{d}{dt} \left[ \mathcal{I}^\epsilon_{1,\theta}(t) + \mathcal{I}^{v^\epsilon,\psi}(t) \right] \leq -\frac{k_1}{2} \int_0^l \left| u^\epsilon_{xx}(x, t) \right|^2 \, dx - \frac{k_2}{2} \int_0^l \left| v^\epsilon_{xx}(x, t) \right|^2 \, dx + \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx + \int_0^l \left| v^\epsilon_{x}(x, t) \right|^2 \, dx \\
+ C \left[ \int_0^l \left| \theta^\epsilon_x(x, t) \right|^2 \, dx + \int_0^l \left| \varphi^\epsilon_x(x, t) \right|^2 \, dx \right] - 2\mathcal{J}^\epsilon(t) - \epsilon [u^\epsilon_0(l_0, t) - v^\epsilon_0(l_0, t)][u^\epsilon(l_0, t) - v^\epsilon(l_0, t)],
\]

and inequality (4.2) follows. □

**Lemma 4.2.** Let \((u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon)\) be the solution provided by Proposition 3.2. Then
\[
\frac{d}{dt} \mathcal{I}^{u^\epsilon,\theta}(t) \leq -\frac{k_1l_0}{2} \left| u^\epsilon_{xx}(0, t) \right|^2 + \frac{1}{2} \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx + C \int_0^l \left| \theta^\epsilon_x(x, t) \right|^2 \, dx + \left[ \frac{m_1}{2} (1 + C_r) + \frac{3}{2} k_1 \right] \int_0^l \left| u^\epsilon_{xx}(x, t) \right|^2 \, dx,
\]
(4.3)
\[
\frac{d}{dt} \mathcal{I}^{v^\epsilon,\psi}(t) \leq -\frac{k_2(l-l_0)}{2} \left| v^\epsilon_{xx}(l, t) \right|^2 + \frac{1}{2} \int_0^l \left| v^\epsilon_{x}(x, t) \right|^2 \, dx + C \int_0^l \left| \varphi^\epsilon_x(x, t) \right|^2 \, dx + \left[ \frac{m_2}{2} (1 + C_r) + \frac{3}{2} k_2 \right] \int_0^l \left| v^\epsilon_{xx}(x, t) \right|^2 \, dx,
\]
(4.4)

for some positive constant \(C\) depending on \(m_i, i = 1, 2, \) and \(l, l_0.\)

**Proof.** We have
\[
\frac{d}{dt} \mathcal{I}^{u^\epsilon,\theta}(t) = - \int_0^l q(x)u^\epsilon_{x}(x, t)u^\epsilon_{tx}(x, t) \, dx - \int_0^l q(x)u^\epsilon_{xx}(x, t)u^\epsilon_{xt}(x, t) \, dx
\]
\[
= \frac{1}{2} \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx + \int_0^l q(x)u^\epsilon_{x}(x, t)[k_1u^\epsilon_{xxxx}(x, t) + m_1\theta^\epsilon_x(x, t)] \, dx
\]
\[
= \frac{1}{2} \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx - \int_0^l [q(x)u^\epsilon_{xx}(x, t) + u^\epsilon_{x}(x, t)][k_1u^\epsilon_{xxxx}(x, t) + m_1\theta^\epsilon_x(x, t)] \, dx
\]
\[
\leq \frac{1}{2} \int_0^l \left| u^\epsilon_{x}(x, t) \right|^2 \, dx + \frac{3}{2} k_1 \int_0^l \left| u^\epsilon_{xx}(x, t) \right|^2 \, dx - \frac{k_1l_0}{2} \left| u^\epsilon_{xx}(0, t) \right|^2
\]
for some positive constant $C$ depending on $m_i, k_i$, $i = 1, 2$ and $C_P, C_S, l, l_0$, and for any positive constants $\delta_1$ and $\delta_2$.

**Proof.** First, we integrate (3.4) on $[0, x)$, with $x \in [0, l_0]$, and we find

$$\int_0^x \theta_i^f(\xi, t) \, d\xi - \tau_1 \theta_i^e(x, t) - m_1 u_{xx}^e(x, t) = 0.$$ 

Subsequently, we multiply every term in the above equation by $q(x)u_i^e(x, t)$, we integrate on $[0, l_0]$, and we obtain

$$\int_0^{l_0} \int_0^x \theta_i^f(\xi, t) \, d\xi \, q(x)u_i^e(x, t) \, dx - \tau_1 \int_0^{l_0} \theta_i^e(x, t) q(x)u_i^e(x, t) \, dx - m_1 \int_0^{l_0} u_{xx}^e(x, t) q(x)u_i^e(x, t) \, dx = 0. \quad (4.7)$$

Note that the first term in the previous equation can be rewritten as

$$\int_0^{l_0} \int_0^x \theta_i^f(\xi, t) \, d\xi \, q(x)u_i^e(x, t) \, dx = \frac{d}{dt} T_{3,3}^{u, \theta}(t) - \int_0^{l_0} \int_0^x \theta_i^e(\xi, t) \, d\xi \, q(x)u_i^e(x, t) \, dx.$$ 

Substituting the previous relation in (4.7) and applying the Young inequality, we have

$$\frac{d}{dt} T_{3,3}^{u, \theta}(t) \leq \int_0^{l_0} \int_0^x \theta_i^e(\xi, t) \, d\xi \, q(x)u_i^e(x, t) \, dx + \int_0^{l_0} \int_0^x \theta_i^e(\xi, t) \, d\xi \, u_i^e(x, t) \, dx - m_1 \int_0^{l_0} u_{xx}^e(x, t) \, dx + C \int_0^{l_0} \theta_i^e(x, t) \, dx,$$

for a positive constant $C$. By an integration by parts, we obtain

$$\frac{d}{dt} T_{3,3}^{u, \theta}(t) = \int_0^{l_0} \int_0^x \theta_i^e(\xi, t) \, d\xi \, q(x)\theta_i^e(x, t) \, dx + \int_0^{l_0} \int_0^x \theta_i^e(\xi, t) \, d\xi \, u_i^e(x, t) \, dx - m_1 \int_0^{l_0} u_{xx}^e(x, t) \, dx + C \int_0^{l_0} \theta_i^e(x, t) \, dx.$$ 

By the Sobolev embedding theorem (cf. (2.11)) and by the Poincaré inequality, the term $l_{i, \theta}(t)$ becomes

$$l_{i, \theta}(t) = - \int_0^{l_0} \left[2 \theta_i^e(x, t) + q(x)\theta_i^e(x, t)\right] \left[k_1 u_{xx}^e(x, t) + m_1 \theta_i^e(x, t)\right] \, dx + l_0 \theta_i^e(0, t) \left[k_1 u_{xx}^e(0, t) + m_1 \theta_i^e(0, t)\right]$$

$$\leq l_0 k_1 \left[\frac{n_1}{2} \left|u_i^e(0, t)\right|^2 + k_1 \left(\frac{n_2}{2} \theta_i^e(x, t) \right) \left|u_i^e(x, t)\right|^2 + \left(l_0 k_1 C_P^2 C_P^{\frac{3}{2}} + \frac{2 l_0 C_P C_P^{\frac{3}{2}} + m_1 l_0 C_P C_P^{\frac{3}{2}}}{}\right) \frac{l_0}{2} \right|\theta_i^e(x, t)\right|^2 \, dx$$

$$+ \frac{k_1}{2n_3} \int_0^{l_0} \left|q(x)\right|^2 \left|\theta_i^e(x, t)\right|^2 \, dx,$$
for any $\eta_i > 0$, $i = 1, 2, 3$. Choosing $\eta_1 = \frac{\delta_1 m_1}{2}$, where the positive constant $\delta_1$ will be fixed later, and $\eta_2 = \eta_3 = \frac{m_1}{2}$, we find

$$
I_v^\theta(t) \leq l_0 k_1 \frac{\delta_1 m_1}{4} |u_{xx}(0, t)|^2 + k_1 \frac{m_1}{2} \int_0^l |u_{xx}(x, t)|^2 \, dx + C \left( \frac{1}{\delta_1} + 1 \right) \int_0^l |\theta_1^e(x, t)|^2 \, dx.
$$

and (4.5) follows.

Analogously, we integrate (3.4)$_4$ on $[x, l]$, with $x \in [l_0, l]$, and we find

$$
\int_x^l \psi_1^e(\xi, t) \, d\xi + \tau_2 \psi_2^e(x, t) + m_2 v_{xx}(x, t) = 0.
$$

Multiplying the above equation by $-q(x)\psi_1^e(x, t)$, we integrate on $[l_0, l]$, and we obtain

$$
-\int_{l_0}^l \left[ \int_x^l \psi_1^e(\xi, t) \, d\xi \right] q(x)\psi_1^e(x, t) \, dx + \tau_2 \int_{l_0}^l \psi_2^e(x, t) q(x)\psi_1^e(x, t) \, dx - m_2 \int_{l_0}^l v_{xx}(x, t) q(x)\psi_1^e(x, t) \, dx = 0.
$$

Note that the first term in the previous equation can be rewritten as

$$
-\int_{l_0}^l \left[ \int_x^l \psi_1^e(\xi, t) \, d\xi \right] q(x)\psi_1^e(x, t) \, dx = \frac{d}{dt} \int_{l_0}^l \psi_1^e(t) \, dx.
$$

Applying the Young inequality, we have

$$
d \frac{d}{dt} \int_{l_0}^l \psi_1^e(t) \, dx = -\int_{l_0}^l \left[ \int_x^l \psi_1^e(\xi, t) \, d\xi \right] q(x)\psi_1^e(t) \, dx + \tau_2 \int_{l_0}^l \psi_2^e(x, t) q(x)\psi_1^e(t) \, dx - m_2 \int_{l_0}^l |\psi_1^e(t)|^2 \, dx
$$

$$
\leq \int_{l_0}^l \left[ \int_x^l \psi_1^e(\xi, t) \, d\xi \right] q(x) \left[ k_2 v_{xx}(x, t) + m_2 \psi_1^e(x, t) \right] \, dx + C \int_{l_0}^l |\psi_2^e(x, t)|^2 \, dx - \frac{m_2}{4} \int_{l_0}^l |\psi_1^e(t)|^2 \, dx
$$

$$
= -\int_{l_0}^l \left[ \int_x^l \psi_1^e(\xi, t) \, d\xi - q(x)\psi_1^e(x, t) \right] \left[ k_2 v_{xx}(x, t) + m_2 \psi_1^e(x, t) \right] \, dx + C \int_{l_0}^l |\psi_2^e(x, t)|^2 \, dx - \frac{m_2}{4} \int_{l_0}^l |\psi_1^e(t)|^2 \, dx,
$$

for a positive constant $C$. The term $I_v^\psi(t)$ becomes

$$
I_v^\psi(t) = -\int_{l_0}^l \left[ 2\psi_2^e(t) + q(x)\psi_2^e(x, t) \right] \left[ k_2 v_{xx}(l, t) + m_2 \psi_2^e(t) \right] \, dx + (l - l_0)\psi_2^e(l, t) \left[ k_2 v_{xx}(l, t) + m_2 \psi_2^e(l, t) \right]
$$

$$
\leq (l - l_0)k_2 C_3 C_P |v_{xx}^e(l, t)| + \int_{l_0}^l \left[ |\psi_2^e(x, t)|^2 \right] \left[ \int_{l_0}^l |\psi_2^e(x, t)|^2 \, dx 
$$

$$
- 2k_2 \int_{l_0}^l \psi_2^e(x, t) v_{xx}^e(x, t) \, dx - k_2 \int_{l_0}^l q(x)\psi_2^e(x, t) v_{xx}^e(x, t) \, dx
$$

$$
+ (l - l_0)m_2 |\psi_2^e(l, t)|^2 - 2m_2 \int_{l_0}^l |\psi_2^e(x, t)|^2 \, dx - m_2 \int_{l_0}^l q(x)\psi_2^e(x, t) \psi_2^e(x, t) \, dx,
$$

and then

$$
I_v^\psi(t) \leq (l - l_0)k_2 \frac{\eta_1}{2} |v_{xx}^e(l, t)|^2 + k_2 \left( \frac{\eta_2}{2} + \frac{\eta_3}{2} \right) \int_{l_0}^l |v_{xx}^e(x, t)|^2 \, dx
$$

$$
+ \left[ \frac{(l - l_0)k_2 C_3^2 C_P^2}{2 \eta_1} + \frac{2k_2 C_3^2 C_P}{\eta_2} + \frac{m_2(l - l_0)C_3^2 C_P}{2} \right] \int_{l_0}^l |\psi_2^e(x, t)|^2 \, dx + \frac{k_2}{2 \eta_3} \int_{l_0}^l |q(x)|^2 |\psi_2^e(x, t)|^2 \, dx.
$$
for any \( \eta_i > 0, i = 1, 2, 3 \). Choosing \( \eta_1 = \frac{\delta_2 m}{2} \), where the positive constant \( \delta_2 \) will be fixed later, and \( \eta_2 = \eta_3 = \frac{m_2}{4} \), inequality (4.6) follows. \( \square \)

Now, we are ready to show that the energy \( E^\epsilon(t) \) associated to the penalized system decays exponentially.

**Theorem 4.4.** Let \((u^\epsilon, \theta^\epsilon, v^\epsilon, \varphi^\epsilon)\) be the solution provided by Proposition 3.2. Then there exist two positive constants \( M \) and \( \gamma \), independent of \( \epsilon \) and \( t \), such that

\[
E^\epsilon(t) \leq ME^\epsilon(0)e^{-\gamma t}, \quad t \geq 0.
\]

**Proof.** Let us introduce the following functional:

\[
\mathcal{E}^\epsilon(t) := NE^\epsilon(t) + \frac{1}{8} \left[ I_1^\epsilon(t) + I_2^\epsilon(t) + I_3^\epsilon(t) \right] + \frac{1}{2} \left| u^\epsilon(t) - v^\epsilon(t) \right|^2 + \frac{1}{2} \left| u_\alpha(x,t) \right|^2 dx + \frac{1}{2} \left| v_\alpha(x,t) \right|^2 dx
\]

where the positive constants \( N \) and \( \delta_i, i = 1, 2, \) will be fixed later. First, we estimate the term \( \epsilon \left| u^\epsilon(t) - v^\epsilon(t) \right|^2 \) in (4.9). Using the Sobolev embedding theorem (cf. (2.11)) and the Poincaré inequality we find

\[
\epsilon \left| u^\epsilon(t) - v^\epsilon(t) \right|^2 \leq 2 \left[ \left| u^\epsilon(t) \right|^2 + \left| v^\epsilon(t) \right|^2 \right] \leq C \left[ \int_0^l \left| u^\epsilon(x,t) \right|^2 dx + \int_0^l \left| v^\epsilon(x,t) \right|^2 dx \right] \leq CE^\epsilon(t),
\]

for a positive constant \( C \) depending on \( C_5 \) and \( C_P \), but independent of \( \epsilon \) and \( t \). Now, recalling (4.1), it is easy to see that, if \( N \) is large enough, there exist two positive constants \( C_1 \) and \( C_2 \), independent of \( \epsilon \) and \( t \), such that

\[
C_1E^\epsilon(t) \leq \mathcal{E}^\epsilon(t) \leq C_2E^\epsilon(t).
\]

On account of (3.40) and Lemmas 4.1–4.3, we have

\[
\frac{d}{dt} \mathcal{E}^\epsilon(t) \leq -\left\{ \frac{k_1}{32} - \delta_1 \left[ \frac{m_1}{2}(1 + C_P) + \frac{3}{2} k_1 \right] \right\} \int_0^l \left| u^\epsilon(x,t) \right|^2 dx - \left\{ \frac{k_2}{32} - \delta_2 \left[ \frac{m_2}{2}(1 + C_P) + \frac{3}{2} k_2 \right] \right\} \int_0^l \left| v_\alpha(x,t) \right|^2 dx
\]

\[
- \frac{1}{2} \left( \frac{1}{4} - \delta_1 \right) \int_0^l \left| u_\alpha(x,t) \right|^2 dx - \frac{1}{2} \left( \frac{1}{4} - \delta_2 \right) \int_0^l \left| v_\alpha(x,t) \right|^2 dx - N\tau_1 - C \left( \frac{1}{\delta_1} + \delta_1 + 1 \right) \int_0^l \left| \varphi^\epsilon(x,t) \right|^2 dx
\]

\[
- \left[ \tau_2 - C \left( \frac{1}{\delta_2} + \delta_2 + 1 \right) \right] \int_0^l \left| \varphi_\alpha(x,t) \right|^2 dx
\]

We choose

\[
\delta_1 \leq \min \left\{ \frac{k_1}{4 \cdot 32} \left[ \frac{m_1}{2}(1 + C_P) + \frac{3}{2} k_1 \right]^{-1} \right\},
\]

\[
\delta_2 \leq \min \left\{ \frac{k_2}{4 \cdot 32} \left[ \frac{m_2}{2}(1 + C_P) + \frac{3}{2} k_2 \right]^{-1} \right\},
\]

\[
N \geq \max\{c_1, c_2\},
\]

where

\[
c_i = \frac{C}{c_i^2} \left( \frac{1}{\delta_i} + \delta_i + 1 \right), \quad i = 1, 2.
\]

Thus, recalling also (4.10), we infer that there exists a positive constant \( C_0 \), independent of \( \epsilon \) and \( t \), such that

\[
\frac{d}{dt} \mathcal{E}^\epsilon(t) \leq -C_0 \mathcal{E}^\epsilon(t) \leq -\frac{C_0}{C_2} \mathcal{E}^\epsilon(t).
\]
Hence
\[ L^\varepsilon(t) \leq L^\varepsilon(0)e^{-\frac{C_0}{C_2}t}, \]
which, combined with (4.10), gives
\[ E^\varepsilon(t) \leq \frac{C_2}{C_1}E^\varepsilon(0)e^{-\frac{C_0}{C_2}t}, \quad t \geq 0, \]
that is (4.8), with \( M = \frac{C_2}{C_1} \) and \( \gamma = \frac{C_0}{C_2} \).

Finally, we are in position to prove that the weak solution of (1.1)–(1.7) decays exponentially.

Proof of Theorem 2.3. Thanks to the choice of the initial data (cf. (3.1)), we have \( J^\varepsilon(0) = 0 \) and, hence, from (4.8), we get
\[ E^\varepsilon(t) \leq ME^\varepsilon(0)e^{-\gamma t}, \quad t \geq 0. \]
Thus, on account of (3.34)–(3.36) and (3.43)–(3.46) (recall that \( M \) and \( \gamma \) are independent of \( \varepsilon \)), owing to weak lower semicontinuity arguments, (2.10) follows.

References