Decay property for second order hyperbolic systems of viscoelastic materials

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\begin{abstract}
We study a class of second order hyperbolic systems with dissipation which describes viscoelastic materials. The considered dissipation is given by the sum of the memory term and the damping term. When the dissipation is effective over the whole system, we show that the solution decays in $L^2$ at the rate $t^{-n/4}$ as $t \to \infty$, provided that the corresponding initial data are in $L^2 \cap L^1$, where $n$ is the space dimension. The proof is based on the energy method in the Fourier space. Also, we discuss similar systems with weaker dissipation by introducing the operator $(1 - \Delta)^{\theta/2}$ with $\theta > 0$ in front of the dissipation terms and observe that the decay structure of these systems is of the regularity-loss type.
\end{abstract}

\section{Introduction}

We consider the following second order hyperbolic systems with dissipation:

\begin{equation}
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \sum_{j,k=1}^n B^{jk} \frac{\partial u_j}{\partial x_k} + \sum_{j,k=1}^n K^{jk}(t) \ast \frac{\partial u_j}{\partial x_k} + Lu_t & = 0 \\
\end{aligned}
\end{equation}

with the initial data

\begin{equation}
\begin{aligned}
\begin{array}{ll}
u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x). \\
\end{array}
\end{aligned}
\end{equation}

Here the unknown $u$ is an $m$-vector function of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \geq 0$, $B^{jk}$ are $m \times m$ real constant matrices satisfying $(B^{jk})^T = B^{kj}$ for each $j$ and $k$, $K^{jk}(t)$ are smooth $m \times m$ real matrix functions of $t \geq 0$ satisfying $K^{jk}(t)^T = K^{kj}(t)$ for each $j, k$ and $t \geq 0$, and $L$ is an $m \times m$ real constant matrix; the symbol "\ast" denotes the convolution with respect to $t$, i.e., $(A \ast u)(t) = \int_0^t A(t - \tau) u(\tau) \, d\tau$. Note that the system (1.1) is a typical model system of viscoelasticity.

To formulate our structural conditions for (1.1), we introduce the symbols of the differential operators appearing in (1.1):

\begin{equation}
\begin{aligned}
B_\omega &= \sum_{j,k=1}^n B^{jk} \omega_j \omega_k, & K_\omega(t) &= \sum_{j,k=1}^n K^{jk}(t) \omega_j \omega_k
\end{aligned}
\end{equation}

for $\omega = (\omega_1, \ldots, \omega_n) \in S^{n-1}$ and $t \geq 0$. Notice that $B_\omega$ and $K_\omega(t)$ are real symmetric matrices. We then impose the following structural conditions.

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Consider the following two modifications of the system (1.1):

- Definite. As a simple consequence of [A4] we find that the matrix estimate of the form (1.4). Similarly, we have the decay estimate (1.4) if each \( \omega \in S^{n-1} \) and \( t \geq 0 \), and \( L \) is real symmetric and nonnegative definite.
- There are positive constants \( C_0 \) and \( c_0 \) such that \( -c_0 K_\omega(t) \leq K'_\omega(t) \leq -c_0 K_\omega(t) \) and \( -c_0 K_\omega(t) \leq K''_\omega(t) \leq C_0 K_\omega(t) \) for each \( \omega \in S^{n-1} \) and \( t \geq 0 \), where \( K'_\omega(t) = \delta_1 K_\omega(t) \) and \( K''_\omega(t) = \delta_2^2 K_\omega(t) \).

Here in [A4], for real symmetric matrices \( A \) and \( B \), \( A \geq B \) or \( B \leq A \) means that \( A - B \) is (real symmetric and) nonnegative definite. As a simple consequence of [A4] we find that the matrix \( K_\omega(t) \) decays exponentially as \( t \to \infty \).

We are interested in the decay property of the system (1.1). Under the structural conditions [A1]–[A4] we can show that the solution to the problem (1.1), (1.2) satisfies the decay estimate

\[
\|u_t(t)\|_{L^2} + \|\partial_x u(t)\|_{L^2} \leq C(1 + t)^{-n/4} \|u_1\|_{L^1} + C(1 + t)^{-n/4 - 1/2} \|u_0\|_{L^1} + Ce^{-ct} \left( \|u_1\|_{L^2} + \|\partial_x u_0\|_{L^2} \right),
\]

where \( C \) and \( c \) are positive constants (see Theorem 3.1). This decay property is of the standard type and is mainly based on the condition [A3] which shows that the dissipation is effective over every component of the solution \( u \). More precisely, we can observe that the Fourier transform of the solution satisfies

\[
\left| \hat{u}_t(\xi, t) \right| + |\xi| |\hat{u}(\xi, t)| \leq Ce^{-\rho(\xi)\xi} \left( \left| \hat{u}_1(\xi) \right| + |\xi| \left| \hat{u}_0(\xi) \right| \right)
\]

with positive constants \( C \) and \( \rho \), where \( \rho(\xi) = |\xi|^2/|\xi|^2 \) and \( \xi = (1 + |\xi|^2)^{1/2} \) (see Proposition 3.2). This pointwise estimate is obtained by applying the energy method in the Fourier space which was used in [8].

In order to investigate the decay structure based on the memory term and the damping term in more details, we also consider the following two modifications of the system (1.1):

\[
u_{tt} - \sum_{j,k=1}^n B_{jk} u_{jx_k} + (1 - \Delta)^{-\theta/2} \sum_{j,k=1}^n K_{jk} * u_{jx_k} + Lu_t = 0, \tag{1.6}
\]

\[
u_{tt} - \sum_{j,k=1}^n B_{jk} u_{jx_k} + \sum_{j,k=1}^n K_{jk} * u_{jx_k} + (1 - \Delta)^{-\theta/2} Lu_t = 0, \tag{1.7}
\]

where \( \theta > 0 \) is a parameter. The introduction of the operator \( (1 - \Delta)^{-\theta/2} \) weakens the dissipation and this gives the weaker decay estimate

\[
\|u_t(t)\|_{L^2} + \|\partial_x u(t)\|_{L^2} \leq C(1 + t)^{-n/4} \|u_1\|_{L^1} + C(1 + t)^{-n/4 - 1/2} \|u_0\|_{L^1} + C(1 + t)^{-1/\theta} \left( \|\partial_x^l u_1\|_{L^2} + \|\partial_x^{l+1} u_0\|_{L^2} \right)
\]

for both systems (1.6) and (1.7), where \( l \geq 0 \) is an integer and \( C \) is a positive constant (see Theorem 5.1). The corresponding pointwise estimate in the Fourier space is given by

\[
\left| \hat{u}_t(\xi, t) \right| + |\xi| |\hat{u}(\xi, t)| \leq Ce^{-\rho_0(\xi)\xi} \left( \left| \hat{u}_1(\xi) \right| + |\xi| \left| \hat{u}_0(\xi) \right| \right)
\]

with positive constants \( C \) and \( \rho_0 \), where \( \rho_0(\xi) = |\xi|^2/|\xi|^2 \) (see Proposition 5.2). The decay property characterized by (1.8) is of the regularity-loss type since we have the decay \( (1 + \xi)^{-1/\theta} \) only by assuming the additional \( l \)-th order regularity on the initial data. Such a decay property of the regularity-loss type (with \( \theta = 2 \)) has been observed in [2] for a hyperbolic-elliptic system of a radiating gas and in [3] for the dissipative Timoshenko system.

There are many results on the decay of solutions to viscoelastic models described by a single equation of the form

\[
u_{tt} - \sum_{j,k=1}^n b_{jk} u_{jx_k} + (1 - \Delta)^{-\theta/2} \sum_{j,k=1}^n g_{jk} * u_{jx_k} + \alpha u_t = 0. \tag{1.10}
\]

When \( g^{jk}(t) = 0 \) and the damping term is effective \( \alpha > 0 \), it is well known that the solution verifies the standard decay estimate of the form (1.4). Similarly, we have the decay estimate (1.4) if \( \alpha = 0 \) and the memory term is effective with \( \theta = 0 \) (cf. [1,5,4]). On the other hand, when \( \alpha = 0 \) and the memory term is effective with \( \theta > 0 \), the corresponding decay structure is of the regularity-loss type and the solution decays like (1.8) (cf. [6]). Our decay estimates in (1.4) and (1.8) are regarded as a generalization of the previous decay results for single equations of viscoelasticity. For the details, see [1,5,4,6,7].

The contents of this paper are as follows. In Section 2 we give some preliminary estimates for convolution type operators of matrices with vectors, which will be used to estimate the memory term. In Section 3 we show the decay estimate of solutions to the system (1.1) by using the pointwise estimate in the Fourier space. This pointwise estimate is derived in Section 4 by employing the energy method in the Fourier space. The key of the proof is to construct a Lyapunov function of the system obtained by taking the Fourier transform of (1.1). In Section 5 we discuss the decay property of the modified systems (1.6) and (1.7). We show the decay estimates of solutions for these modified systems and observe that they are of the regularity-loss type. The corresponding pointwise estimates in the Fourier space are derived in the last section.
Notations. We introduce some notations used in this paper. We denote by \( \hat{u} = \mathcal{F}[u] \) the Fourier transform of the function \( u \):
\[
\hat{u}(\xi) = \mathcal{F}[u](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)e^{-i\xi \cdot x} \, dx.
\]
For \( 1 \leq p \leq \infty \), we denote by \( L^p = L^p(\mathbb{R}^n) \) the usual Lebesgue space on \( \mathbb{R}^n \) with the norm \( \| \cdot \|_{L^p} \). For a nonnegative integer \( s \), \( H^s = H^s(\mathbb{R}^n) \) denotes the Sobolev space of \( L^2 \) functions on \( \mathbb{R}^n \), equipped with the norm \( \| \cdot \|_{H^s} \). Let \( k \) be a nonnegative integer. Then for an interval \( I \) and a Banach space \( X \), \( C^k(I; X) \) denotes the space of \( k \)-times continuously differential functions on \( I \) with values in \( X \).

Throughout the present paper, \( C \) and \( c \) denote various generic positive constants.

2. Preliminaries

In this section we give some preliminary results for convolution type operators of matrices with vectors.

Let \( \mathcal{X}^m \) be the totality of \( m \times m \) real matrices and \( (\cdot, \cdot) \) be the standard inner product in \( \mathbb{C}^m \). We introduce the operator norm of \( A \in \mathcal{X}^m \) by
\[
|A| = \sup_{\phi \neq 0} \frac{|A\phi|}{|\phi|},
\]
where the supremum is taken over all \( \phi \in \mathbb{C}^m \) with \( \phi \neq 0 \). Let \( \mathcal{S}^m \) be the totality of \( m \times m \) real symmetric matrices. For \( A \in \mathcal{S}^m \), we write \( A \succeq 0 \) if \( A \) is nonnegative definite. Also, we write \( A \succeq B \) or \( B \preceq A \) if \( A - B \succeq 0 \). It is known that if \( A \in \mathcal{S}^m \) and \( A \succeq 0 \), then
\[
|A\phi|^2 \leq |A|(A\phi, \phi)
\]
for \( \phi \in \mathbb{C}^m \) and hence
\[
|A| = \sup_{\phi \neq 0} \frac{(A\phi, \phi)}{|\phi|^2}.
\]

Let \( A(t) \in \mathcal{X}^m \) and \( \phi(t) \in \mathbb{C}^m \). We define the convolution \( A \ast \phi \) by
\[
(A \ast \phi)(t) = \int_0^t A(t - \tau)\phi(\tau) \, d\tau.
\]

Also, we introduce the related operator and the corresponding quadratic form as
\[
(A \circ \phi)(t) = \int_0^t A(t - \tau)(\phi(t) - \phi(\tau)) \, d\tau,
\]
\[
A[\phi, \phi](t) = \int_0^t \langle A(t - \tau)(\phi(t) - \phi(\tau)), \phi(t) - \phi(\tau) \rangle \, d\tau.
\]

This convolution type operator and the corresponding quadratic form were previously introduced by J.E.M. Rivera in the study of equations of viscoelasticity with memory (cf. [5,6]). The convolution \( A \ast \phi \) is related to \( A \circ \phi \) as
\[
A \ast \phi = A\phi - A \circ \phi,
\]
where \( A(t) = \int_0^t A(s) \, ds \). Also, a direct computation shows that
\[
(A \ast \phi)_t = A(0)\phi + A' \ast \phi,
\]
where \( A'(t) = dA(t)/dt \). We can rewrite \( A' \ast \phi \) by using (2.7) as \( A' \ast \phi = (A - A(0))\phi - A' \circ \phi \), which together with (2.8) gives
\[
(A \ast \phi)_t = A\phi - A' \circ \phi.
\]

The following equality plays an important role in our energy method in the Fourier space (see Section 4).

Let \( A \in \mathcal{X}^m \) and \( \phi \in \mathbb{C}^m \), then for an interval \( I \) and a Banach space \( X \), \( C^k(I; X) \) denotes the space of \( k \)-times continuously differential functions on \( I \) with values in \( X \).

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|A| = \sup_{\phi \neq 0} \frac{|A\phi|}{|\phi|},
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\[
|A\phi|^2 \leq |A|(A\phi, \phi)
\]
for \( \phi \in \mathbb{C}^m \) and hence
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|A| = \sup_{\phi \neq 0} \frac{(A\phi, \phi)}{|\phi|^2}.
\]

Let \( A(t) \in \mathcal{X}^m \) and \( \phi(t) \in \mathbb{C}^m \). We define the convolution \( A \ast \phi \) by
\[
(A \ast \phi)(t) = \int_0^t A(t - \tau)\phi(\tau) \, d\tau.
\]

Also, we introduce the related operator and the corresponding quadratic form as
\[
(A \circ \phi)(t) = \int_0^t A(t - \tau)(\phi(t) - \phi(\tau)) \, d\tau,
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\[
A[\phi, \phi](t) = \int_0^t \langle A(t - \tau)(\phi(t) - \phi(\tau)), \phi(t) - \phi(\tau) \rangle \, d\tau.
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\[
(A \ast \phi)_t = A(0)\phi + A' \ast \phi,
\]
where \( A'(t) = dA(t)/dt \). We can rewrite \( A' \ast \phi \) by using (2.7) as \( A' \ast \phi = (A - A(0))\phi - A' \circ \phi \), which together with (2.8) gives
\[
(A \ast \phi)_t = A\phi - A' \circ \phi.
\]
Lemma 2.1. (Cf. [6].) Let \( A(t) \in S^m \) and \( \phi(t) \in \mathbb{C}^m \). Then we have

\[
-2 \text{Re} \langle A \ast \phi, \phi \rangle = \frac{d}{dt} \left( A[\phi, \phi] - \langle A \phi, \phi \rangle \right) + \left( -A'[\phi, \phi] + \langle A \phi, \phi \rangle \right).
\]

(2.10)

where \( A(t) = \int_0^t A(s) \, ds \) and \( A'(t) = dA(t)/dt \).

**Proof.** Differentiate \( A[\phi, \phi] \) with respect to \( t \) to get

\[
\frac{d}{dt} A[\phi, \phi] = A'[\phi, \phi] + 2 \text{Re} (A \circ \phi, \phi) .
\]

(2.11)

We rewrite the second term on the right-hand side of (2.11) by using (2.7) as

\[
2 \text{Re} (A \circ \phi, \phi) = 2 \text{Re} \langle A \phi - A \ast \phi, \phi \rangle
\]

\[
= \frac{d}{dt} \langle A \phi, \phi \rangle - \langle A \phi, \phi \rangle - 2 \text{Re} (A \ast \phi, \phi),
\]

where we have used the fact that \( dA/dt = A \). Substituting this relation into (2.11) gives the desired equality (2.10). This completes the proof. \( \Box \)

Next we show that \( A \circ \phi \) is estimated in terms of \( A[\phi, \phi] \), provided that \( A(t) \in S^m \) and \( A(t) \geq 0 \).

Lemma 2.2. (Cf. [6].) Let \( A(t) \in S^m, A(t) \geq 0 \) and \( \phi(t) \in \mathbb{C}^m \). Then we have

\[
\left| (A \circ \phi)(t) \right|^2 \leq \int_0^t \left| A(s) \right| ds \cdot A[\phi, \phi](t).
\]

(2.12)

**Proof.** Applying (2.2) and the Hölder inequality, we obtain

\[
\left| (A \circ \phi)(t) \right| \leq \int_0^t \left| A(t - \tau) (\phi(t) - \phi(\tau)) \right| d\tau
\]

\[
\leq \int_0^t \left| A(t - \tau) \right|^{1/2} \left| (\phi(t) - \phi(\tau)) \right|^{1/2} d\tau
\]

\[
\leq \left( \int_0^t \left| A(t - \tau) \right| d\tau \right)^{1/2} \left( \int_0^t \left| (\phi(t) - \phi(\tau)) \right| d\tau \right)^{1/2}
\]

\[
= \left( \int_0^t \left| A(s) \right| ds \right)^{1/2} A[\phi, \phi](t)^{1/2}.
\]

This shows (2.12) and the proof of Lemma 2.2 is complete. \( \Box \)

Finally in this section, we derive several technical inequalities for the memory term, which will be used in this paper. For \( A(t) \in X^m \), we define

\[
\left| A \right|_1 = \int_0^\infty \left| A(s) \right| ds, \quad \left| A \right|_\infty = \sup_{s \geq 0} \left| A(s) \right|.
\]

(2.13)

Let \( A(t) \in S^m \) and \( A(t) \geq 0 \), and assume the following conditions (cf. [A4]):

[H1] \(-c_0 A(t) \leq A'(t) \leq c_0 A(t), \)

[H2] \(-c_0 A(t) \leq A''(t) \leq c_0 A(t), \)

where \( c_0 \) and \( C_0 \) are positive constants, and \( A'(t) = dA(t)/dt \) and \( A''(t) = d^2A(t)/dt^2 \).
**Lemma 2.3.** Let $A(t) \in S^m$ and $A(t) \geq 0$, and assume [H1]. Then we have

$$|A(t)| \leq |A(0)|e^{-c_0t}, \quad |A'(t)| \leq c_0|A(t)| \leq c_0|A(0)|e^{-c_0t}. \quad (2.14)$$

In particular, we get $|A|_{L^\infty} \leq |A(0)|$ and $|A|_{L^1} \leq |A(0)|/c_0$. Also, we obtain $|A'|_{L^\infty} \leq c_0|A|_{L^\infty}$ and $|A'|_{L^1} \leq c_0|A|_{L^1}$.

**Proof.** Let $\phi \in C^m$ (independent of $t$) and put $q(t) = \langle A(t)\phi, \phi \rangle$. Differentiating $q(t)$ with respect to $t$ and using $A'(t) \leq -c_0A(t)$ in [H1], we have

$$q'(t) = \langle A'(t)\phi, \phi \rangle \leq -c_0\langle A(t)\phi, \phi \rangle = -c_0q(t).$$

This differential inequality is solved as $q(t) \leq q(0)e^{-c_0t}$. Thus we get $\langle A(t)\phi, \phi \rangle \leq |A(0)|\phi, \phi)e^{-c_0t}$. We divide this inequality by $|\phi|^2$ and take the supremum over $\phi \neq 0$, which together with (2.3) gives $|A(t)| \leq |A(0)|e^{-c_0t}$.

To show the estimate for $A'(t)$, we observe that $-A'(t) \geq 0$ from [H1]. Then, applying (2.3), we see that

$$|A'(t)| = \sup_{\phi \neq 0} \frac{\langle (-A'(t))\phi, \phi \rangle}{|\phi|^2} \leq c_0 \sup_{\phi \neq 0} \frac{|A(t)\phi, \phi|}{|\phi|^2} = c_0|A(t)|,$$

where we have used the assumption $-A'(t) \leq c_0A(t)$ in [H1]. This completes the proof. □

As a simple corollary of Lemma 2.3, we have

$$|A(t)\phi|^2 \leq |A|_{L^\infty}\langle A(t)\phi, \phi \rangle,$$

$$|A'(t)\phi|^2 \leq c_0^2\langle A|_{L^\infty}\langle A(t)\phi, \phi \rangle \quad (2.15)$$

for $\phi \in C^m$, where $A(t)$ is assumed to satisfy the conditions of Lemma 2.3. The first estimate in (2.15) is an easy consequence of (2.2) and (2.14), while the second one is proved as follows:

$$|A'(t)\phi|^2 \leq |A'(t)|(|(-A'(t))\phi, \phi|$$

$$\leq c_0^2\langle A(t)\phi, \phi \rangle \leq c_0^2|A|_{L^\infty}\langle A(t)\phi, \phi \rangle,$$

where we have used $-A'(t) \geq 0$, $-A'(t) \leq c_0A(t)$, (2.2) and (2.14).

**Lemma 2.4.** Let $A(t) \in S^m$ and $A(t) \geq 0$, and assume [H1]. Let $\phi(t) \in C^m$. Then we have

$$|(A \circ \phi)(t)|^2 \leq |A|_{L^1}|A|_1|A(t)|^2,$$

$$|(A' \circ \phi)(t)|^2 \leq c_0^2|A|_{L^1}|A(t)|. \quad (2.16)$$

Moreover, under the additional condition [H2], we have

$$|(A'' \circ \phi)(t)|^2 \leq 10c_0^2|A|_{L^1}|A(t)|.$$ \hspace{1cm} (2.17)

**Proof.** The first estimate in (2.16) easily follows from (2.12). Also, using this inequality with $A(t)$ replaced by $-A'(t) \geq 0$, we obtain

$$|(A' \circ \phi)(t)|^2 \leq |A'|_{L^2}^2|\phi, \phi| \leq c_0^2|A|_{L^1}|A(t)|^2,$$

where we have used the assumption $-A'(t) \leq c_0A(t)$ in [H1] and the estimate $|A'|_{L^1} \leq c_0|A|_{L^1}$ in Lemma 2.3.

To show (2.17), we compute as

$$|A'' \circ \phi| = |(A'' + C_0A) \circ \phi - C_0A \circ \phi|$$

$$\leq |(A'' + C_0A) \circ \phi| + C_0|A \circ \phi|.$$

Since $A''(t) + C_0A(t) \geq 0$ by [H2], the first term on the right-hand side of the above inequality is estimated as

$$|(A'' + C_0A) \circ \phi|^2 \leq |(A'' + C_0A)|_{L^1}(A'' + C_0A)[\phi, \phi]$$

$$\leq 4c_0^2|A|_{L^1}|A(t)|.$$
where we have used \( A''(t) + C_0 A(t) \leq 2C_0 A(t) \) and \( |A''(t) + C_0 A(t)| \leq 2C_0 |A(t)| \). Combining all these estimates, we have
\[
|A'' \circ \phi|^2 \leq 2 |(A'' + C_0 A) \circ \phi|^2 + 2C_0^2 |A \circ \phi|^2 \\
\leq 10C_0^2 |A|_{L^1} |A| \phi, \phi|
\]
This completes the proof of Lemma 2.4. \( \square \)

3. Decay estimate

In this section, we show the decay estimate of solutions to the problem (1.1), (1.2). Under the structural conditions [A1]–[A4], the system (1.1) has the decay structure of the standard type and we have the following decay estimate.

**Theorem 3.1 (Decay estimate).** Suppose that all the conditions [A1]–[A4] are satisfied. Let \( s \) be a nonnegative integer and assume that \( u_0 \in H^{s+1} \cap L^1 \) and \( u_1 \in H^s \cap L^1 \). Then the corresponding solution \( u \) to the problem (1.1), (1.2) satisfies the decay estimate
\[
\left\| \partial_x^k u_t(t) \right\|_{L^2} + \left\| \partial_x^{k+1} u(t) \right\|_{L^2} \leq C(1 + t)^{-n/4 - k/2} \left\| u_1 \right\|_{L^1} + C(1 + t)^{-n/4 - k/2 - 1/2} \left\| u_0 \right\|_{L^1} \\
+ Ce^{-ct} \left( \left\| \partial_x^k u_1 \right\|_{L^2} + \left\| \partial_x^{k+1} u_0 \right\|_{L^2} \right)
\]
for \( k \) with \( 0 \leq k \leq s \), where \( C \) and \( c \) are positive constants.

**Remark.** This theorem is valid even in the special case where \( L = 0 \) (no damping term). In this case, instead of [A3], we may simply assume that \( K_{\omega}(0) \) is positive definite for each \( \omega \in S^{n-1} \) and obtain the same decay estimate (3.1). In another special case where \( K_{\omega}(t) = 0 \), we also have the decay estimate (3.1) if \( L \) is positive definite.

The key of the proof of this decay estimate is to derive the pointwise estimate of solutions in the Fourier space. To state our result on the pointwise estimate, we apply the Fourier transform to (1.1). This yields
\[
\hat{u}_{tt} + |\xi|^2 B_{\omega} \hat{u} - |\xi|^2 (K_{\omega} * \hat{u}) + \hat{L} u = 0.
\]
where \( B_{\omega} \) and \( K_{\omega}(t) \) are defined in (1.3) with \( \omega = \xi / |\xi| \). The corresponding initial data are given by
\[
\hat{u}(\xi, 0) = \hat{u}_0(\xi), \quad \hat{u}_t(\xi, 0) = \hat{u}_1(\xi).
\]
Our pointwise estimate for the problem (3.2), (3.3) is then given as follows.

**Proposition 3.2 (Pointwise estimate).** Assume the same conditions of Theorem 3.1. Then the solution to the problem (3.2), (3.3) satisfies the pointwise estimate
\[
|\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \leq Ce^{-c(|\xi|)^4} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right),
\]
where \( \rho(\xi) = |\xi|^2 / |\xi|^2, (\xi) = (1 + |\xi|^2)^{1/2} \), and \( C \) and \( c \) are positive constants.

**Proof of Theorem 3.1.** Here we assume that (3.4) holds true and prove the decay estimate (3.1). We apply the Plancherel theorem and use (3.4), obtaining
\[
\left\| \partial_x^k u_t(t) \right\|_{L^2} + \left\| \partial_x^{k+1} u(t) \right\|_{L^2} = \int_{\mathbb{R}^n} |\xi|^{2k} \left( |\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \right) d\xi \\
\leq C \int_{\mathbb{R}^n} |\xi|^{2k} e^{-c(|\xi|)^4} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right) d\xi.
\]
We divide the integral into two parts \( I_1 \) and \( I_2 \) corresponding to the regions \( |\xi| \leq 1 \) and \( |\xi| \geq 1 \), respectively. In the low frequency region \( |\xi| \leq 1 \), we have \( \rho(\xi) = |\xi|^2 / |\xi|^2 \geq |\xi|^2 / 2 \). Therefore we can estimate the term \( I_1 \) as
\[
I_1 \leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-c|\xi|^2} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right) d\xi =: I_{11} + I_{12},
\]
where \( I_{11} \) and \( I_{12} \) are the integrals corresponding to \( \hat{u}_1 \) and \( \hat{u}_0 \), respectively. Here we have
the real part of the resulting equation we have

$$I_{11} = C \int_{|\xi| \leq 1} |\xi|^2 e^{-|\xi|^2 t} |\hat{u}_1(\xi)|^2 d\xi$$

$$\leq C \|\hat{u}_1\|_2^2 \int_{|\xi| \leq 1} |\xi|^2 e^{-|\xi|^2 t} d\xi \leq C (1 + t)^{-n/2-k} \|u_1\|_1^2,$$

where we have used a simple fact that \( \int_{|\xi| \leq 1} |\xi|^2 e^{-|\xi|^2 t} d\xi \leq C (1 + t)^{-n/2-j} \) for each \( j \geq 0 \). Similarly, we have

$$I_{12} = C \int_{|\xi| \leq 1} |\xi|^{2(k+1)} e^{-|\xi|^2 t} |\hat{u}_0(\xi)|^2 d\xi \leq C (1 + t)^{-n/2-k-1} \|u_0\|_1^2.$$

Next we consider the term \( I_2 \) corresponding to the high frequency region \(|\xi| \geq 1\). Since \( \rho(\xi) = |\xi|^2/\langle \xi \rangle^2 \geq 1/2 \) for \(|\xi| \geq 1\), we can estimate \( I_2 \) as

$$I_2 \leq C e^{-ct} \int_{|\xi| \geq 1} |\xi|^{2k} (|\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2) d\xi$$

$$\leq C e^{-ct} \left( \|\hat{\alpha}_k u_1\|_2^2 + \|\hat{\alpha}_k^{k+1} u_0\|_2^2 \right).$$

where we again used the Plancherel theorem. Substituting all these estimates into (3.5), we obtain the desired decay estimate (3.1). This completes the proof of Theorem 3.1. \( \square \)

Finally in this section, we give the energy estimate for the problem (1.1), (1.2), which will be proved at the end of next section.

**Theorem 3.3 (Energy estimate).** Suppose that all the conditions [A1]–[A4] are satisfied. Let \( s \geq 1 \) be an integer and assume that \( u_0 \in H^{s+1} \) and \( u_1 \in H^s \). Then the solution to the problem (1.1), (1.2) satisfies the energy estimate

$$\left\| \hat{\alpha}_k u_t(t) \right\|_{H^1}^2 + \left\| \hat{\alpha}_k^{k+1} u(t) \right\|_{H^2}^2 + \int_0^t \left\| \hat{\alpha}_k^{k+1} u_t(\tau) \right\|_{L^2}^2 + \left\| \hat{\alpha}_k^{k+2} u(\tau) \right\|_{L^2}^2 d\tau \leq C \left( \left\| \hat{\alpha}_k u_1 \right\|_{H^1}^2 + \left\| \hat{\alpha}_k^{k+1} u_0 \right\|_{H^1}^2 \right)$$

(3.6)

for \( k \) with \( 0 \leq k \leq s - 1 \), where \( C \) is a positive constant.

This energy estimate is of the standard type and there is no regularity-loss in the dissipation part.

### 4. Energy method in the Fourier space

The aim of this section is to prove the pointwise estimate (3.4) and the energy estimate (3.6).

First we show the pointwise estimate (3.4) by applying the energy method in the Fourier space. For this purpose, we construct a Lyapunov function \( E \) of the system (3.2). Our Lyapunov function \( E \) is equivalent to

$$E_0 = |\hat{u}_1|^2 + |\xi|^2 |\hat{u}_0|^2 + |\xi|^2 K_\omega [\hat{u}, \hat{u}],$$

where \( K_\omega [\hat{u}, \hat{u}] \) is defined in (2.6), and satisfies the differential inequality of the form

$$\frac{d}{dt} E + F \leq 0$$

(4.2)

with the corresponding dissipation term \( F \) satisfying \( F \geq c \rho(\xi) E_0 \), where \( \rho(\xi) = |\xi|^2/\langle \xi \rangle^2 \) and \( c \) is a positive constant. Our construction of the Lyapunov function \( E \) is divided into four steps.

**Step 1.** We first derive the equality for the physical energy. To this end, we take the inner product of (3.2) with \( \hat{u}_t \). From the real part of the resulting equation we have

$$\frac{1}{2} \frac{d}{dt} \left( |\hat{u}_t|^2 + |\xi|^2 \langle B_\omega \hat{u}, \hat{u} \rangle \right) - |\xi|^2 \text{Re} (K_\omega * \hat{u}, \hat{u}_t) + \langle L \hat{u}_t, \hat{u}_t \rangle = 0.$$

(4.3)

Here the second term can be rewritten by using (2.10) as

$$-2 \text{Re} (K_\omega * \hat{u}, \hat{u}_t) = \frac{d}{dt} \left( K_\omega [\hat{u}, \hat{u}] - \langle K_\omega \hat{u}, \hat{u} \rangle + (-K'_\omega \hat{u}, \hat{u}) + \langle K_\omega \hat{u}, \hat{u} \rangle \right).$$

(4.4)
where $K_\omega(t) = \int_0^t K_\omega(s) \, ds$ and $K'_\omega(t) = \partial_t K_\omega(t)$. Substituting (4.4) into (4.3), we obtain
\[
\frac{1}{2} \frac{d}{dt} E_1 + F_1 + \langle L\dot{u}_t, \dot{u}_t \rangle = 0, \tag{4.5}
\]
where we put
\[
E_1 = |\dot{u}_t|^2 + |\xi|^2 ((B_\omega - K_\omega)\dot{u}, \dot{u}) + |\xi|^2 K_\omega[\dot{u}, \dot{u}],
\]
\[
F_1 = \frac{1}{2} |\xi|^2 (-K'_\omega[\dot{u}, \dot{u}] + \langle K_\omega \dot{u}, \dot{u} \rangle).
\tag{4.6}
\]
Since $B_\omega - K_\omega(t)$ is positive definite uniformly in $t \geq 0$ by [A2] and $-K'_\omega(t) \geq c_0 K_\omega(t)$ by [A4], we see that
\[
cE_0 \leq E_1 \leq CE_0, \quad F_1 \geq c|\xi|^2 F_0,
\tag{4.7}
\]
where $E_0$ and $F_0$ are given by (4.1) and
\[
F_0 = K_\omega[\dot{u}, \dot{u}] + \langle K_\omega \dot{u}, \dot{u} \rangle,
\tag{4.8}
\]
respectively, and $c$ and $C$ are positive constants.

**Step 2.** We make the first modification of the energy in order to take into account the dissipation comes from the memory term. For this purpose, we take the inner product of (3.2) with $-(K_\omega \ast \dot{u})_t$ and consider the real part of the resulting equality. This gives
\[
\frac{1}{2} \frac{d}{dt} |\xi|^2 |K_\omega \ast \dot{u}|^2 - \text{Re}\{\dot{u}_{tt}, (K_\omega \ast \dot{u})_t\} - |\xi|^2 \text{Re}\{B_\omega \dot{u}, (K_\omega \ast \dot{u})_t\} - \text{Re}\{L\dot{u}_t, (K_\omega \ast \dot{u})_t\} = 0. \tag{4.9}
\]
Here the second term is rewritten as
\[
-\text{Re}\{\dot{u}_{tt}, (K_\omega \ast \dot{u})_t\} = -\frac{d}{dt} \text{Re}\{\dot{u}_t, (K_\omega \ast \dot{u})_t\} + \text{Re}\{\dot{u}_t, (K_\omega \ast \dot{u})_{tt}\}
\]
\[
= -\frac{d}{dt} \text{Re}\{\dot{u}_t, (K_\omega \ast \dot{u})_t\} + \text{Re}\{\dot{u}_t, (K_\omega \ast \dot{u})_t\} + \text{Re}\{\dot{u}_t, (K'_\omega \ast \dot{u})_t\}, \tag{4.10}
\]
where we have used the relation $(K_\omega \ast \dot{u})_t = K_\omega(0)\dot{u} + K'_\omega \ast \dot{u}$ in (2.8). Substituting (4.10) into (4.9), we get
\[
\frac{1}{2} \frac{d}{dt} E_2 + \langle K_\omega(0)\dot{u}, \dot{u}_t \rangle = R_2, \tag{4.11}
\]
where we put
\[
E_2 = |\xi|^2 |K_\omega \ast \dot{u}|^2 - 2 \text{Re}\{\dot{u}_t, (K_\omega \ast \dot{u})_t\},
\]
\[
R_2 = -\text{Re}\{\dot{u}_t, (K'_\omega \ast \dot{u})_t\} + |\xi|^2 \text{Re}\{B_\omega \dot{u}, (K_\omega \ast \dot{u})_t\} + \text{Re}\{L\dot{u}_t, (K_\omega \ast \dot{u})_t\}. \tag{4.12}
\]
We need to estimate each term in (4.12). To this end, we first observe that $|K_\omega|_{L^\infty} + |K_\omega|_{L^1} \leq C$ for some positive constant $C$, which is a simple corollary of Lemma 2.3. Also, we claim that
\[
|K_\omega \ast \dot{u}|^2 \leq C(|\dot{u}|^2 + K_\omega[\dot{u}, \dot{u}]), \tag{4.13}
\]
\[
|\langle K_\omega \ast \dot{u} \rangle_t|^2 + |\langle K'_\omega \ast \dot{u} \rangle_t|^2 \leq C F_0, \tag{4.14}
\]
where $F_0$ is defined in (4.8) and $C$ is a positive constant. Once these estimates are known, we can estimate $E_2$ and $R_2$ in (4.12) as
\[
|E_2| \leq C |\dot{u}_t|^2 + C (|\dot{u}|^2 + K_\omega[\dot{u}, \dot{u}]),
\]
\[
|R_2| \leq \varepsilon |\dot{u}_t|^2 + \delta |\xi|^2 |\dot{u}|^2 + C_{\varepsilon, \delta} (|\dot{u}|^2 + K_\omega[\dot{u}, \dot{u}]) \tag{4.15}
\]
for any $\varepsilon > 0$ and $\delta > 0$, where $C$ and $C_{\varepsilon, \delta}$ are positive constants; $C_{\varepsilon, \delta}$ is a constant depending on $(\varepsilon, \delta)$. For example, by using (4.14), we can estimate the second term of $E_2$ as
\[
|\text{Re}\{\dot{u}_t, (K'_\omega \ast \dot{u})_t\}| \leq C |\dot{u}_t| F_0^{1/2} \leq C |\dot{u}_t|^2 + C (|\dot{u}|^2 + K_\omega[\dot{u}, \dot{u}]).
\]
where we have also used the fact that $F_0 \leq C(|\dot{u}|^2 + K_\omega[\dot{u}, \dot{u}])$. The other terms in $E_2$ and $R_2$ are estimated similarly.
It remains to show the estimates (4.13) and (4.14). We have from (2.7) that \( K_\omega \ast \dot{u} = K_\omega \dot{u} - K_\omega \circ \dot{u} \), where \( K_\omega (t) = \int_0^t K_\omega (s) \, ds \). Then, applying (2.16), we get
\[
|K_\omega \ast \dot{u}| \leq |K_\omega \dot{u}| + |K_\omega \circ \dot{u}|
\leq \|K_\omega\|_1 |\dot{u}| + \|K_\omega\|_2^1/2 K_\omega [\dot{u}, \dot{u}]^{1/2} \leq C (|\dot{u}|^2 + K_\omega [\dot{u}, \dot{u}]^{1/2}),
\]
which proves (4.13). Also, we have \( (K_\omega \ast \dot{u})_t = K_\omega \dot{u} - K_\omega' \circ \dot{u} \) by (2.9). Therefore, applying (2.15) and (2.16), we obtain
\[
|K_\omega \ast \dot{u}| \leq |K_\omega \dot{u}| + |K_\omega' \circ \dot{u}| \leq CF_0^{1/2}.
\]
Similarly, we have \( (K_\omega' \ast \dot{u})_t = K_\omega' \dot{u} - K_\omega'' \circ \dot{u} \) by (2.9). This together with (2.15) and (2.16) yields \( |K_\omega' \ast \dot{u}| \leq CF_0^{1/2} \). Thus we have proved (4.13) and (4.14).

**Step 3.** Next we make the second modification of the energy which corresponds to the damping term. We take the inner product of (3.2) with \( \dot{u} \) and consider its real part, obtaining
\[
\frac{1}{2} \frac{d}{dt} \left( (\dot{L} \dot{u}, \dot{u}) + 2 \Re (\dot{u} \dot{t}, \dot{u}) \right) + |\xi|^2 (B_\omega \dot{u}, \dot{u}) - |\dot{u} t|^2 - |\xi|^2 \Re (K_\omega \ast \dot{u}, \dot{u}) = 0. \tag{4.16}
\]
Using the relation \( K_\omega \ast \dot{u} = K_\omega \dot{u} - K_\omega \circ \dot{u} \) from (2.7), we can rewrite (4.16) as
\[
\frac{1}{2} \frac{d}{dt} E_3 + |\xi|^2 ((B_\omega - K_\omega) \dot{u}, \dot{u}) = R_3, \tag{4.17}
\]
where we put
\[
E_3 = (\dot{L} \dot{u}, \dot{u}) + 2 \Re (\dot{u} \dot{t}, \dot{u}), \quad R_3 = |\dot{u} t|^2 - |\xi|^2 \Re (K_\omega \circ \dot{u}, \dot{u}). \tag{4.18}
\]
Here, using (2.16), we find that
\[
|E_3| \leq C (|\dot{u} t|^2 + |\dot{u}|^2), \quad |R_3| \leq |\dot{u} t|^2 + \gamma |\xi|^2 |\dot{u}|^2 + C_{\gamma} |\xi|^2 K_\omega [\dot{u}, \dot{u}]. \tag{4.19}
\]
for any \( \gamma > 0 \), where \( C_\gamma \) and \( C_{\gamma} \) are positive constants; \( C_{\gamma} \) is a constant depending on \( \gamma \).

**Step 4.** We combine (4.5), (4.11) and (4.17) to produce our Lyapunov function \( E \). First, letting \( \alpha > 0 \) and \( \beta > 0 \), we multiply (4.11) and (4.17) by \( \alpha \) and \( \beta \), respectively, and add these two equations. The result is written as
\[
\frac{1}{2} \frac{d}{dt} \left( \alpha E_2 + \beta E_3 \right) + \alpha (K_\omega (0) + L) \dot{u} \dot{t}, \dot{u} \dot{t} + \beta |\xi|^2 ((B_\omega - K_\omega) \dot{u}, \dot{u}) - \alpha (L \dot{u}, \dot{u}) = \alpha R_2 + \beta R_3. \tag{4.20}
\]
Next, we multiply (4.20) by \( \rho (\xi) = |\xi|^2 / |\xi|^2 \) and add the result to (4.5). This yields
\[
\frac{1}{2} \frac{d}{dt} E = F + R, \tag{4.21}
\]
where we put
\[
E = E_1 + \rho (\xi) (\alpha E_2 + \beta E_3), \quad F = \rho (\xi) \left[ (K_\omega (0) + L) \dot{u} \dot{t}, \dot{u} \dot{t} + \beta |\xi|^2 ((B_\omega - K_\omega) \dot{u}, \dot{u}) \right] + F_1 + (1 - \alpha \rho (\xi)) (L \dot{u}, \dot{u}), \quad R = \rho (\xi) (\alpha R_2 + \beta R_3). \tag{4.22}
\]
This \( E \) is our desired Lyapunov function and \( F \) is the corresponding dissipation term. To see this, we recall the basic estimate (4.7) for \( E_1 \), i.e.,
\[
c E_0 \leq E_1 \leq C E_0, \tag{4.23}
\]
and claim that
\[
\rho (\xi) (\alpha E_2 + \beta E_3) \leq (\alpha + \beta) C E_0, \quad F \geq \rho (\xi) \left( \alpha \langle |\dot{u} t|^2 + \beta \langle |\xi|^2 |\dot{u}|^2 \rangle \right) + c |\xi|^2 F_0 + c (L \dot{u}, \dot{u}), \quad |R| \leq \rho (\xi) \left( (\alpha \varepsilon + \beta) |\dot{u} t|^2 + (\alpha \beta + \beta \gamma) |\xi|^2 |\dot{u}|^2 \right) + (\alpha + \beta) C E_0, \tag{4.24}
\]
and claim that
where we have assumed $\alpha \leq 1/2$ in the estimate of $F$. Here $c$ and $C$ are positive constants independent of positive parameters $(\varepsilon, \delta, \gamma, \alpha, \beta)$, while $C_{\varepsilon, \delta, \gamma}$ denotes a constant depending on $(\varepsilon, \delta, \gamma)$ but not on $(\alpha, \beta)$. The first and the third estimates in (4.24) follow from (4.15) and (4.19) together with a simple fact that $\rho(\xi) \leq 1$, $\rho(\xi) \leq |\xi|^2$ and $\rho(\xi)|\xi|^2 = |\xi|^2$.

On the other hand, the estimate of $F$ in (4.24) is a consequence of the structural conditions [A2] and [A3] together with the requirement $\alpha \leq 1/2$.

Now, in view of the estimates (4.23) and (4.24), we choose the positive parameters such that

$$\alpha \varepsilon + \beta \leq \alpha c/2, \quad \alpha \delta + \beta \gamma \leq \beta c/2,$$

$$\alpha \leq 1/2, \quad (\alpha + \beta)C \leq c/2, \quad (\alpha + \beta)C_{\varepsilon, \delta, \gamma} \leq c/2.$$  

This choice is possible. In fact, we first determine $\varepsilon, \delta$ and $\gamma$ such that $\varepsilon = \gamma = c/4$ and $\delta = (c/4)^2$. Then we take $\beta$ in terms of $\alpha$ as $\beta = \alpha c/4$. Finally, we can choose $\alpha > 0$ so small that

$$\alpha \leq 1/2, \quad \alpha (1 + c/4)C \leq c/2, \quad \alpha (1 + c/4)C_{\varepsilon, \delta, \gamma} \leq c/2.$$  

For this choice of the parameters, we see that $|R| \leq F/2$ and hence the energy equality (4.21) is reduced to the differential inequality (4.2). Moreover, we see that our $E$ and $F$ satisfy the required property. Namely, we have

$$cE_0 \leq E \leq CE_0,$$

$$F \geq \rho(\xi)\left(|\hat{u}_t|^2 + |\xi|^2|\hat{u}|^2\right) + c|\xi|^2F_0 + cL_\omega \hat{u}_t \hat{u}$$  

(4.25)

with positive constants $c$ and $C$. In particular, we have $F \geq c\rho(\xi)E_0$ for a positive constant $c$.

All these observations are summarized as follows.

**Proposition 4.1** (Lyapunov function). Under the conditions [A1]–[A4], the system (3.2) admits a Lyapunov function $E$ which satisfies the differential inequality (4.2). This Lyapunov function $E$ and the corresponding dissipation term $F$ verify the estimates in (4.25) together with $F \geq c\rho(\xi)E_0$.

The rest of this section gives the proof of Proposition 3.2 for pointwise estimates and Theorem 3.3 for energy estimates.

**Proof of Proposition 3.2.** Consider the Lyapunov function $E$ and the corresponding dissipation term $F$ constructed in Proposition 4.1. Since $F \geq c\rho(\xi)E_0$ and $E$ is equivalent to $E_0$ by (4.25), we have $F \geq c\rho(\xi)E$. Substituting this estimate into (4.2), we obtain

$$\frac{d}{dt}E + c\rho(\xi)E \leq 0.$$  

Solving this differential inequality, we have $E(\xi, t) \leq e^{-c\rho(\xi)t}E(\xi, 0)$ and hence $E_0(\xi, t) \leq Ce^{-c\rho(\xi)t}E_0(\xi, 0)$. Thus, recalling the definition (4.1) of $E_0$, we arrive at the estimate

$$|\hat{u}_t(\xi, t)|^2 + |\xi|^2|\hat{u}(\xi, t)|^2 + |\xi|^2K_\omega|\hat{u}, \hat{u}|(\xi, t) \leq Ce^{-c\rho(\xi)t}\left(|\hat{u}_t(\xi)|^2 + |\xi|^2|\hat{u}_0(\xi)|^2\right),$$  

(4.26)

where $\rho(\xi) = |\xi|^2/|\xi|^2$, and $C$ and $c$ are positive constants. Here we have used that fact that the term $K_\omega|\hat{u}, \hat{u}|$ vanishes at $t = 0$. Thus we have shown the desired pointwise estimate (3.4). This completes the proof of Proposition 3.2. □

**Proof of Theorem 3.3.** Consider the differential inequality (4.2) for the Lyapunov function $E$. We integrate (4.2) with respect to $t$ to get

$$E(\xi, t) + \int_0^t F(\xi, \tau) \, d\tau \leq E(\xi, 0).$$  

We multiply this inequality by $|\xi|^2|\xi|^{2k}$, where $k$ is a nonnegative integer. Since $F \geq c\rho(\xi)E_0$ and $E$ is equivalent to $E_0$, we obtain

$$|\xi|^2|\xi|^{2k}E_0(\xi, t) + \int_0^t |\xi|^{2(k+1)}E_0(\xi, \tau) \, d\tau \leq C|\xi|^2|\xi|^{2k}E_0(\xi, 0),$$  

where we have used the relation $\rho(\xi)|\xi|^2 = |\xi|^2$. Now, integrating this inequality with respect to $\xi \in \mathbb{R}^n$ and recalling the definition (4.1) of $E_0$, we arrive at the estimate
\[ \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \, d\xi + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2(k+1)} \left( |\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \right) \, d\xi \, dt \]
\[ \leq C \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \, d\xi. \]  
(4.27)

Here we have used the inequality \( E_0 \geq |\hat{u}|^2 + |\xi|^2 |\hat{u}|^2 \) and the fact that the term \( K_{\omega}[\hat{u}, \hat{u}] \) vanishes at \( t = 0 \). The estimate (4.27) together with the Plancherel theorem gives the desired energy estimate (3.6). Thus the proof of Theorem 3.3 is complete. □

5. Decay estimates of regularity-loss type

We discuss the decay property of the modified systems (1.6) and (1.7) with a parameter \( \theta > 0 \). It is shown that the dissipative structure of these modified systems is of the regularity-loss type and we have the following decay estimate.

**Theorem 5.1 (Decay estimate).** Suppose that all the conditions [A1]–[A4] are satisfied. Let \( \theta > 0 \). Let \( s \) be a nonnegative integer and assume that \( u_0 \in H^{s+1} \cap L^1 \) and \( u_1 \in H^s \cap L^1 \). Then the corresponding solution \( u \) to the problem (1.6) (or (1.7)), (1.2) satisfies the decay estimate

\[ \|\partial_x^ku_t(t)\|_{L^2} + \|\partial_x^{k+1}u(t)\|_{L^2} \leq C(1 + t)^{-n/4-k/2} \|u_1\|_{L^1} + C(1 + t)^{-n/4-k/2-1/2} \|u_0\|_{L^1} \]
\[ + C(1 + t)^{-l/\theta} \left( \|\partial_x^{k+1}u_1\|_{L^2} + \|\partial_x^{k+l+1}u_0\|_{L^2} \right) \]  
(5.1)

for nonnegative integers \( k \) and \( l \) with \( k + l \leq s \), where \( C \) is a positive constant.

The decay rate \( (1 + t)^{-l/\theta} \) in (5.1), which comes from the high frequency part, is obtained only by assuming the additional \( l \)-th order regularity on the initial data. Therefore the above decay estimate (5.1) for the modified systems (1.6) and (1.7) is of the regularity-loss type.

As in Section 3, the pointwise estimate in the Fourier space is crucial in the proof of Theorem 5.1. To see this, we consider the systems (1.6) and (1.7) in the Fourier space:

\[ \hat{u}_{tt} + |\xi|^2 B_{\omega} \hat{u} - |\xi|^2 \langle \xi \rangle^{-\theta} (K_{\omega} * \hat{u}) + L\hat{u}_t = 0, \]  
(5.2)

\[ \hat{u}_{tt} + |\xi|^2 B_{\omega} \hat{u} - |\xi|^2 (K_{\omega} * \hat{u}) + \langle \xi \rangle^{-\theta} L\hat{u}_t = 0, \]  
(5.3)

where \( B_{\omega} \) and \( K_{\omega}(t) \) are defined in (1.3) with \( \omega = \xi/|\xi| \). Our pointwise estimate for (5.2) and (5.3) is given as follows.

**Proposition 5.2 (Pointwise estimate).** Assume the same conditions of Theorem 5.1. Then the solution to the problem (5.2) (or (5.3)), (3.3) satisfies the pointwise estimate

\[ |\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \leq Ce^{-c_{\rho_0}(\xi)\tau} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right), \]  
(5.4)

where \( \rho_0(\xi) = |\xi|^2/|\xi|^{2+\theta} \), and \( C \) and \( c \) are positive constants.

**Proof of Theorem 5.1.** As in the proof of Theorem 3.1, by applying the Plancherel theorem and using (5.4), we have

\[ \|\partial_x^ku_t(t)\|_{L^2} + \|\partial_x^{k+1}u(t)\|_{L^2} \leq C \int_{\mathbb{R}^n} |\xi|^{2k} e^{-c_{\rho_0}(\xi)\tau} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right) \, d\xi. \]  
(5.5)

We divide the integral into two parts \( I_1 \) and \( I_2 \) corresponding to the regions \( |\xi| \leq 1 \) and \( |\xi| \geq 1 \), respectively. In the low frequency region \( |\xi| \leq 1 \), we have \( \rho_0(\xi) = |\xi|^2/|\xi|^{2+\theta} \geq c|\xi|^{2} \) with a positive constant \( c \). Therefore the term \( I_1 \) can be estimated just in the same way as in the proof of Theorem 3.1 and we obtain

\[ I_1 = C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-c|\xi|^{2+\theta}} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right) \, d\xi \]
\[ \leq C(1 + t)^{-n/2-k} \|u_1\|_{L^2}^2 + C(1 + t)^{-n/2-k-1} \|u_0\|_{L^1}^2. \]

On the other hand, in the high frequency region \( |\xi| \geq 1 \), we have \( \rho_0(\xi) = |\xi|^2/|\xi|^{2+\theta} \geq c|\xi|^{-\theta} \) with a positive constant \( c \). Therefore we can estimate the term \( I_2 \) as
We obtain the desired decay estimate (5.1). Therefore the proof of Theorem 5.1 is complete.

Proof of Proposition 5.2 for (5.2).

The proof is parallel to that for the system (3.2). We first treat the system (5.2).

6. Energy method in the Fourier space: regularity-loss case

The energy estimates for the modified systems (1.6) and (1.7) are given as follows.

Theorem 5.3 (Energy estimate). Suppose that all the conditions [A1]–[A4] are satisfied. Let $\theta > 0$ and $s \geq 1 + \theta/2$, and assume that $u_0 \in H^{s+1}$ and $u_1 \in H^s$. Then the solution to the problem (1.6) or (1.7), (1.2) satisfies the energy estimate

\[
\left\| \frac{\partial^k u(t)}{\partial \tau^k} \right\|^2_{L^2} + \left\| \frac{\partial^{k+1} u(t)}{\partial \tau^{k+1}} \right\|^2_{L^2} + \frac{1}{\tau} \left\| \frac{\partial^{k+1} u(t)}{\partial \tau^{k+1}} \right\|^2_{L^2} d\tau 
\]

for $k$ with $0 \leq k \leq s - 1 - \theta/2$, where $C$ is a positive constant.

Note that in this energy estimate (5.6), we have $\theta/2$-th order regularity-loss in the dissipation part. The dissipative property stated in Theorems 5.1 and 5.3 for the modified systems (1.6) and (1.7) is of the regularity-loss type and completely different from the one for the original system (1.1).

6. Energy method in the Fourier space: regularity-loss case

We prove the pointwise estimate (5.4) for the systems (5.2) and (5.3) by applying the energy method in the Fourier space. The proof is parallel to that for the system (3.2). We first treat the system (5.2).

Proof of Proposition 5.2 for (5.2). We prove the pointwise estimate (5.4) for the system (5.2) which is a modification of (3.2) by replacing $K_0$ by $(\xi)^{-\theta}K_\omega$. In this case we modify $E_0$ in (4.1) by $E_0^\theta$:

\[
E_0^\theta = |\hat{u}_t|^2 + |\xi|^2 |\hat{u}|^2 + |\xi|^2 \langle \xi \rangle^{-\theta} K_\omega [\hat{u}, \hat{u}].
\]

We construct a Lyapunov function for (5.2). First, as a counterpart of (4.5), we have

\[
\frac{1}{2} \frac{d}{dt} E_1^\theta + \langle \xi \rangle^{-\theta} F_1 + (L \hat{u}_t, \hat{u}_t) = 0,
\]

where

\[
E_1^\theta = |\hat{u}_t|^2 + |\xi|^2 (B_{\omega} - K_\omega) \hat{u}, \hat{u}) + |\xi|^2 \langle \xi \rangle^{-\theta} K_\omega [\hat{u}, \hat{u}] + |\xi|^2 (1 - \langle \xi \rangle^{-\theta}) (K_\omega \hat{u}, \hat{u}),
\]

and $F_1$ is the same as in (4.6). Here we have used the relation $-\langle \xi \rangle^{-\theta} K_\omega = -K_\omega + (1 - \langle \xi \rangle^{-\theta}) K_{\omega}$. Second, as a counterpart of (4.11), we have

\[
\frac{1}{2} \frac{d}{dt} E_2^\theta + (K_\omega (0) \hat{u}_t, \hat{u}_t) = R_2,
\]

where

\[
E_2^\theta = |\xi|^2 \langle \xi \rangle^{-\theta} |K_\omega \ast \hat{u}|^2 - 2 \text{Re} (\hat{u}_t, (K_\omega \ast \hat{u})_t),
\]

and $R_2$ is the same as in (4.12). Also, we have

\[
\frac{1}{2} \frac{d}{dt} E_3 + |\xi|^2 (B_{\omega} - K_\omega) \hat{u}, \hat{u}) + |\xi|^2 (1 - \langle \xi \rangle^{-\theta}) (K_\omega \hat{u}, \hat{u}) = R_3^\theta,
\]

which is a counterpart of (4.17). Here $E_3$ is given in (4.18) and

\[
R_3^\theta = |\hat{u}_t|^2 - |\xi|^2 \langle \xi \rangle^{-\theta} \text{Re} (K_\omega \circ \hat{u}, \hat{u}).
\]
Now we combine the above three equalities (6.2), (6.3) and (6.4) similarly as in Section 4 by using \( \rho_0(\xi) = |\xi|^2/\langle \xi \rangle^{2+\theta} \) instead of \( \rho(\xi) = |\xi|^2/\langle \xi \rangle^{2+\theta} \). Then, as a counterpart of (4.21), we have

\[
\frac{1}{2} \frac{d}{dt} E^0 + F^0 = R^0, \tag{6.5}
\]

where we put

\[
E^0 = E_1^0 + \rho_0(\xi) (\alpha E_2^0 + \beta E_3^0),
\]

\[
F^0 = \rho_0(\xi) (|\xi|^2(K_{\omega}(0) + \hat{L}\tilde{u}_t, \hat{u}_t) + |\xi|^2(B_{\omega} - K_{\omega})\hat{u}, \hat{u}) + \langle \xi \rangle^{-\theta} F_1 + (1 - \alpha \rho_0(\xi)) (L\tilde{u}_t, \hat{u}_t) + \beta \rho_0(\xi) |\xi|^2 (1 - \langle \xi \rangle^{-\theta}) (K_{\omega}\hat{u}, \hat{u}),
\]

\[
R^0 = \rho_0(\xi) (\alpha R_2 + \beta R_3^0). \tag{6.6}
\]

This \( E^0 \) is our desired Lyapunov function of (5.2) and \( F^0 \) is the corresponding dissipative term. This can be verified as follows. It is obvious that \( E^0 \) is equivalent to \( E_0^0 \) in (6.1):

\[
cE_0^0 \leq E^0 \leq C E_0^0.
\]

Also, estimating each term in (6.6) similarly as in Section 4, we find that

\[
\rho_0(\xi) |\xi|^2 (K_{\omega}(0) + \hat{L}\tilde{u}_t, \hat{u}_t) + |\xi|^2(B_{\omega} - K_{\omega})\hat{u}, \hat{u}) + \langle \xi \rangle^{-\theta} F_1 + (1 - \alpha \rho_0(\xi)) (L\tilde{u}_t, \hat{u}_t) + \beta \rho_0(\xi) |\xi|^2 (1 - \langle \xi \rangle^{-\theta}) (K_{\omega}\hat{u}, \hat{u})
\]

\[
\leq \rho_0(\xi) (|\xi|^2(\alpha + \beta)\tilde{u}_t^2 + c|\xi|^2 F_0 + c(L\tilde{u}_t, \hat{u}_t)),
\]

\[
|R^0| \leq \rho_0(\xi) (|\xi|^2(\alpha + \beta)\tilde{u}_t^2 + c|\xi|^2 F_0 + c(L\tilde{u}_t, \hat{u}_t)), \tag{6.7}
\]

which are essentially the same as in (4.24), where \( \alpha \leq 1/2 \) was assumed in the estimate of \( F^0 \). Therefore, by choosing all the parameters just in the same way as in Section 4, we find that \( |R^0| \leq F^0/2 \) and hence the energy equality (6.8) is reduced to

\[
\frac{d}{dt} E^0 + F^0 \leq 0. \tag{6.8}
\]

Moreover, we see that our \( E^0 \) and \( F^0 \) satisfy

\[
cE_0^0 \leq E^0 \leq C E_0^0,
\]

\[
F^0 \geq c \rho_0(\xi) (\tilde{u}_t^2 + |\xi|^2 \hat{u}_t^2) + c|\xi|^2 F_0 + c(L\tilde{u}_t, \hat{u}_t) \tag{6.9}
\]

with positive constants \( c \) and \( C \). These observations show that \( E^0 \) is the desired Lyapunov function of (5.2).

It follows from (6.9) that \( F^0 \geq c \rho_0(\xi) E_0^0 \) and hence \( F^0 \geq c \rho_0(\xi) E_0^0 \) with a positive constant \( c \). Substituting this estimate into (6.8), we obtain

\[
\frac{d}{dt} E^0 + c \rho_0(\xi) E^0 \leq 0.
\]

Solving this differential inequality, we arrive at

\[
\begin{aligned}
|\tilde{u}(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 + |\xi|^2 \langle \xi \rangle^{-\theta} K_{\omega}[\hat{u}, \hat{u}](\xi, t) & \leq C e^{-c \rho_0(\xi)t} \left( |\hat{u}_1(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right).
\end{aligned}
\]

This gives the desired pointwise estimate (5.4) and hence the proof of Proposition 5.2 is complete for the system (5.2). \( \square \)

**Proof of Proposition 5.2 for (5.3).** We prove the pointwise estimate (5.4) for the system (5.3) which is a modification of (3.2) by replacing \( L \) by \( \langle \xi \rangle^{-\theta} L \). We construct a Lyapunov function for (5.3). First, as a counterpart of (4.5), we have

\[
\frac{1}{2} \frac{d}{dt} E_1 + F_1 + \langle \xi \rangle^{-\theta} (L\tilde{u}_t, \hat{u}_t) = 0, \tag{6.10}
\]

where \( E_1 \) and \( F_1 \) are the same as in (4.6). Second, as a counterpart of (4.11), we have

\[
\frac{1}{2} \frac{d}{dt} E_2 + |K_{\omega}(0)\tilde{u}_t, \hat{u}_t| = R_2^0, \tag{6.11}
\]

where \( E_2 \) is given in (4.12) and

\[
R_2^0 = - \text{Re} [\tilde{u}_t, (K_{\omega} \ast \hat{u})_t] + |\xi|^2 \text{Re} [B_{\omega}\hat{u}, (K_{\omega} \ast \hat{u})_t] + \langle \xi \rangle^{-\theta} \text{Re} [L\tilde{u}_t, (K_{\omega} \ast \hat{u})_t].
\]
Also, we have
\[ \frac{1}{2} \frac{d}{dt} E_3^0 + |\xi|^2 (B_{\omega} - \mathcal{K}_0) \hat{u}, \hat{u} \rangle = R_3, \]  
(6.12)
which is a counterpart of (4.17). Here
\[ E_3^0 = \langle \xi \rangle^{-\theta} (L\hat{u}, \hat{u}) + 2 \Re (\hat{u}_t, \hat{u}), \]
and \( R_3 \) is the same as in (4.18).

Combining the equalities (6.10), (6.11) and (6.12) just in the same way as for the system (5.2), we obtain
\[ \frac{1}{2} \frac{d}{dt} \bar{E}^\theta + \bar{F}^\theta = \bar{R}^\theta, \]  
(6.13)
where we put
\[ \bar{E}^\theta = E_1 + \rho_0 (\xi) (\alpha E_2 + \beta E_3^0), \]
\[ \bar{F}^\theta = \rho_0 (\xi) \left\{ \alpha \left[ (K_0 (0) + L) \hat{u}_t, \hat{u}_t \right] + \beta |\xi|^2 (B_{\omega} - \mathcal{K}_0) \hat{u}, \hat{u} \right\} + F_1 + \left( 1 - \alpha \rho (\xi) \right) \langle \xi \rangle^{-\theta} (L\hat{u}_t, \hat{u}_t), \]
\[ \bar{R}^\theta = \rho_0 (\xi) (\alpha R_2^0 + \beta R_3). \]  
(6.14)
This \( \bar{E}^\theta \) is our desired Lyapunov function of (5.3) and \( \bar{F}^\theta \) is the corresponding dissipation term. In fact, estimating each term in (6.14) similarly as in Section 4, we find that
\[ \rho_0 (\xi) |\alpha E_2 + \beta E_3^0 | \leq (\alpha + \beta) CE_0, \]
\[ \bar{F}^\theta \geq \rho_0 (\xi) \{ |c| \hat{u}_t|^2 + \beta c |\xi|^2 |\hat{u}|^2 \} + c |\xi|^2 F_0 + c (\xi)^{\theta} \langle \xi \rangle^{-\theta} (L\hat{u}_t, \hat{u}_t), \]
\[ |\bar{R}^\theta | \leq \rho_0 (\xi) \{ (\alpha \varepsilon + \beta) \hat{u}_t^2 + (\alpha \delta + \beta \gamma) |\xi|^2 |\hat{u}|^2 \} + (\alpha + \beta) CE_{\delta, \gamma} |\xi|^2 F_0, \]  
(6.15)
which are essentially the same as in (4.24), where \( \alpha \leq 1/2 \) was assumed in the estimate of \( \bar{E}^\theta \). Therefore, by choosing all the parameters as before, we conclude that \( |\bar{R}^\theta | \leq \bar{E}^\theta / 2, \)
\[ CE_0 \leq \bar{E}^\theta \leq CE_0, \]
\[ \bar{F}^\theta \geq c \rho_0 (\xi) \{ |\hat{u}_t|^2 + |\xi|^2 |\hat{u}|^2 \} + c |\xi|^2 F_0 + c (\xi)^{\theta} \langle \xi \rangle^{-\theta} (L\hat{u}_t, \hat{u}_t), \]  
(6.16)
and the energy equality (6.13) is reduced to
\[ \frac{d}{dt} \bar{E}^\theta + \bar{F}^\theta \leq 0. \]  
(6.17)
This shows that \( \bar{E}^\theta \) is the desired Lyapunov function of (5.3).

It follows from (6.16) that \( \bar{F}^\theta \geq c \rho_0 (\xi) \bar{E}^\theta \). Therefore (6.17) can be reduced to \( \frac{d}{dt} \bar{E}^\theta + c \rho_0 (\xi) \bar{E}^\theta \leq 0 \). Solving this differential inequality, we obtain
\[ |\hat{u}_t (\xi, t)\|^2 + |\xi|^2 |\hat{u}_t (\xi, t)|^2 + |\xi|^2 K_0 [\hat{u}, \hat{u}] (t) \leq Ce^{-c \rho_0 (\xi) t} \left\{ |\hat{u}_1 (\xi)|^2 + |\xi|^2 |\hat{u}_0 (\xi)|^2 \right\}, \]
which gives the desired pointwise estimate (5.4). This completes the proof of Proposition 5.2 for the system (5.3). □

**Proof of Theorem 5.3.** The proof is similar to that of Theorem 3.3. We prove the energy estimate (5.6) only for the system (1.6) and omit the arguments for (1.7). We integrate the energy inequality (6.8) with respect to \( t \) to get
\[ E^0 (\xi, t) + \int_0^t F^0 (\xi, \tau) d\tau \leq E^0 (\xi, 0). \]
We multiply this inequality by \( \langle |\xi|^{2+\theta}|\xi|^{2k} \rangle \). Since \( F^0 \geq c \rho_0 (\xi) E_0^0 \) and \( E^0 \) is equivalent to \( E_0^0 \) in (6.1), we get
\[ \langle |\xi|^{2+\theta}|\xi|^{2k} E_0^0 (\xi, t) \rangle + \int_0^t |\xi|^{2(k+1)} E_0^0 (\xi, \tau) d\tau \leq C \langle |\xi|^{2+\theta}|\xi|^{2k} E_0^0 (\xi, 0) \rangle. \]
Moreover, integrating this inequality with respect to \( \xi \in \mathbb{R}^n \), we have as a counterpart of (4.27) that

\[
\int_{\mathbb{R}^n} \langle \xi \rangle^{2+\theta} |\xi|^{2k} \left( |\hat{u}_t(\xi, t)|^2 + |\hat{u}(\xi, t)|^2 \right) d\xi + \int_0^t \int_{\mathbb{R}^n} |\xi|^{2k+1} \left( |\hat{u}_t(\xi, t)|^2 + |\xi|^2 |\hat{u}(\xi, t)|^2 \right) d\xi \, d\tau \\
\leq C \int_{\mathbb{R}^n} \langle \xi \rangle^{2+\theta} |\xi|^{2k} \left( |\hat{u}_t(\xi)|^2 + |\xi|^2 |\hat{u}_0(\xi)|^2 \right) d\xi,
\]

which together with the Plancherel theorem gives the desired energy estimate (5.6). Thus the proof of Theorem 5.3 is complete. \( \Box \)

References